Dynamical Systems

Numerical Projects :
The problems combine numerics and analytical calculations.

1 Hopf and nomal form

Consider the following dynamical system :

$$\begin{aligned} \dot{x} &= rx(\alpha + x)(1 - x) - cxy & (1) \\ \dot{y} &= -\alpha\delta y + (c - \delta)xy & (2) \end{aligned}$$

where r, c, δ are real, positive constants and $c > \delta$. Use α as control parameter.

1. There is a fixed point that undergoes a Hopf bifurcation. Make a change of variable to place that point at the origin. Verify the transversality condition.
2. Perform a formal form reduction and eliminate "non-resonance" non-linear terms.
3. Obtain a phase portrait of the system numerically, before and after the bifurcation.

2 A closed manifold

Consider the following dynamical system :

$$\begin{aligned} \dot{x} &= y & (3) \\ \dot{y} &= x - \gamma y - x^3 + xy + \epsilon_1 + \epsilon_2 x^2 \,, & (4) \end{aligned}$$

where ϵ_1, ϵ_2 and γ are real and positive constants.

1. Compute fixed points. Using $\gamma = 1$, find an analytical expression for the curve that separates region in parameter (ϵ_1, ϵ_2) with one and three fixed points.
2. Integrate numerically the system for different initial condition in the region with one fixed point. Describe the observed behavior.
3. Study the behavior of the system when $\epsilon_2 = 0.45$ and ϵ_1 varies between 0.25 and 0.28. Similarly, for $\epsilon_1 = 0.25$ when ϵ_2 varies between 0.28 and 0.36. Discuss the change of behavior observed. Compute the stable and unstable manifold associated to the saddle point in the last case.
4. Verify numerically that in the limit $\gamma \to \infty$ all orbits collapse to a closed manifold. Discuss whether the behavior of the manifold is analogous to $\dot{\theta} = \mu - \cos(\theta)$.

3 Poincaré map

From the following dynamical system :

$$\begin{aligned} \dot{\rho} &= \rho(\alpha - \rho^2) & (5) \\ \dot{\phi} &= 1 \,, & (6) \end{aligned}$$

where α is a positive constant, build a Poincaré map.

1. Show that the system has the following solution for $\alpha > 0$:

$$\begin{aligned} \rho(t) &= \left(\frac{1}{\alpha} + (\frac{1}{\rho_0^2} - \frac{1}{\alpha}) e^{-2\alpha t} \right)^{-1/2} \\ \phi &= t + \phi_0 \,, \end{aligned}$$

2. Define a section Π orthogonal to the flux given by Eqs. (5) and (6) and build the map $\Pi \to \Pi$.
3. Compute the fixed points and stability of the map.

4 The logistic map

Consider the following dynamical system :

$$x_{n+1} = r\,x_n\,(1 - x_n) \tag{7}$$

where $0 \leq r \leq 4$ and $0 \leq x \leq 1$.

1. Construct the cobweb associated to Eq. (7) for a given value of r and initial condition.
2. Compute the value of r at which orbits of period 2, 4, 8, and 16 emerge.
3. Build the bifurcation diagram (x^* vs r) including orbits of all periods.
4. Discuss what happens with the Lyapunov exponent.