Instabilité liée au laser

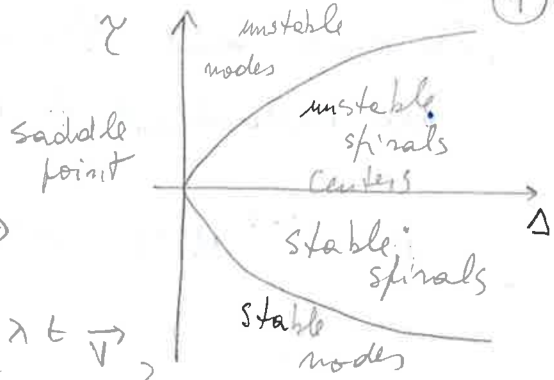
$$\frac{du}{dt} = -K u + G u v$$

$$\frac{dv}{dt} = -f v + P - G u v$$

$$t = \tilde{t}\, T \rightarrow \tilde{t} = \frac{t}{T}$$

$$\frac{d}{dt} = \frac{1}{T}\frac{d}{d\tilde{t}}$$

$$\dot{u} = \frac{du}{d\tilde{t}} \quad \text{et} \quad \dot{v} = \frac{dv}{d\tilde{t}}$$



Rappel :

$$\dot{\vec{x}} = A\vec{x}$$

$$\rightarrow x(t) = e^{\lambda t}\vec{v}$$

$$\Rightarrow \lambda \vec{v} = A\vec{v}$$

Si $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \Delta = 0$$

avec $\tau = Tr(A)$ et $\Delta = \det(A)$

$$\Delta = \lambda_1 \lambda_2 \qquad \tau = \lambda_1 + \lambda_2$$

(*) $\rightarrow$

$$\begin{cases} \frac{1}{T}\dot{u} = -Ku + G u v \\ \frac{1}{T}\dot{v} = -f v + P - G u v \end{cases} \longrightarrow \begin{array}{l} \dot{u} = -KTu + GTuv \\ \dot{v} = -fTv + TP - TGuv \end{array}$$

$$\rightarrow \text{si } KT = 1 \Rightarrow T = \frac{1}{K} \rightarrow \begin{cases} \dot{u} = -u + \underbrace{\left(\frac{G}{K}\right)}_{b} uv \\ \dot{v} = -\underbrace{\left(\frac{f}{K}\right)}_{\varepsilon} v + \underbrace{\left(\frac{P}{K}\right)}_{a} - buv \end{cases}$$

$$\Rightarrow \begin{cases} \dot{u} = -u + buv \\ \dot{v} = -\varepsilon v + a - buv \end{cases}$$

a) États stationnaires :

$$0 = -u + buv \longrightarrow u = buv \longrightarrow \text{soit } u = 0 \text{ ou } 1 = bv$$

$$0 = -\varepsilon v + a - buv \rightarrow \varepsilon v = a - buv$$

$u = 0 \Rightarrow v = \frac{a}{\varepsilon} \longrightarrow (u_0, v_0) = \left(0, \frac{a}{\varepsilon}\right)$

Si $1 = bv \Rightarrow \varepsilon v = a - u \Rightarrow u = a - \varepsilon v$

$\downarrow$

$v = \frac{1}{b}$ $\qquad u = a - \frac{\varepsilon}{b}$

$\longrightarrow (u_1, v_1) = \left(a - \frac{\varepsilon}{b}, \frac{1}{b}\right)$

b) $\begin{cases} \dot{u} = f(u,v) \\ \dot{v} = g(u,v) \end{cases}$ $\qquad J = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \rightarrow$ matrice Jacobienne

$\downarrow$

$J = \begin{pmatrix} -1 + bv & bu \\ -bv & -\varepsilon - bu \end{pmatrix}$

c) $J_0 = J(u_0, v_0)$

$\downarrow$

$J_0 = \begin{pmatrix} -1 + b\frac{a}{\varepsilon} & 0 \\ -b\frac{a}{\varepsilon} & -\varepsilon \end{pmatrix}$

d) $\tau = Tr(J_0) = -1 + b\frac{a}{\varepsilon} - \varepsilon$

$\Delta = \left(-1 + b\frac{a}{\varepsilon}\right)(-\varepsilon) = \varepsilon - ba$

$\frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = $

$= \frac{\tau}{2}\left(1 \pm \sqrt{1 - \frac{4\Delta}{\tau^2}}\right)$

Par hypothèse

Si $\boxed{\varepsilon - ba < 0} \Rightarrow \underline{\Delta < 0} \longrightarrow \lambda_+ \lambda_- < 0$

$\Rightarrow \lambda_+ > 0$ et $\lambda_- < 0$

$\Rightarrow (u_0, v_0)$ est un <u>noeud-col</u> !

$$J_1 = \begin{pmatrix} -1 + b/b & b\left(a - \frac{\varepsilon}{b}\right) \\ -b/b & -\varepsilon - b\left(a - \frac{\varepsilon}{b}\right) \end{pmatrix}$$

$$J_1 = \begin{pmatrix} 0 & ba - \varepsilon \\ -1 & -\varepsilon - ba + \varepsilon \end{pmatrix} = \begin{pmatrix} 0 & ba - \varepsilon \\ -1 & -ba \end{pmatrix}$$

f

$$\mathrm{Tr}\, J_1 = \tau = -ba < 0$$

$\det J_1 = \Delta = -\varepsilon + ba \longrightarrow$ comme $\varepsilon - ba < 0$

$-\varepsilon + ba > 0$

Si λ_+ et λ_- réels :

$\lambda_+ + \lambda_- < 0$ et $\lambda_+ \lambda_- > 0$

soit $\lambda_+ > 0$ et $\lambda_- > 0$

soit $\lambda_- < 0$ et $\lambda_+ < 0$

$\Rightarrow \lambda_- < 0$ et $\lambda_+ < 0 \Rightarrow$ point stable.

Si λ_+ et λ_- sont complexes (conjugués)

$\Rightarrow \lambda_+ + \lambda_- = 2\,\mathrm{Re}\,\lambda_+ = 2\,\mathrm{Re}\,\lambda_- < 0$.

$\hookrightarrow$ on tombe sur le point si on fait une perturbation.

g) L'état stationnaire (u_1, v_1) est un foyer stable si les racines son complexes !

$\rightarrow$ Pour $\overline{\overline{J_1}} \rightarrow \det(\overline{\overline{J_1}} - \lambda I) = +\lambda(ab + \lambda) + ba - \varepsilon = 0$

équation caractéristique

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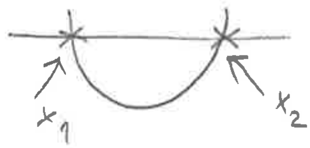
$$\Rightarrow \; +\lambda^2 + \lambda ab + ba - \varepsilon = 0$$

$$\frac{-ab \pm \sqrt{(ab)^2 - 4(ba-\varepsilon)}}{2} = \lambda_\pm$$

$$(ab)^2 < 4(ab - \varepsilon)$$

$$\underbrace{(ab)^2 - 4ab + 4\varepsilon < 0}$$

h) poser $ab = x \;\rightarrow\; x^2 - 4x + 4\varepsilon < 0$



$\rightarrow$ on cherche $x_1 < x < x_2$

$$x_{1,2} = \frac{4 \pm \sqrt{4^2 - 4^2\varepsilon}}{2} = 2 \pm 2\sqrt{1-\varepsilon}$$
$$= 2\left(1 \pm \sqrt{1-\varepsilon}\right)$$

si $\varepsilon < 1$ (en supposant ε).

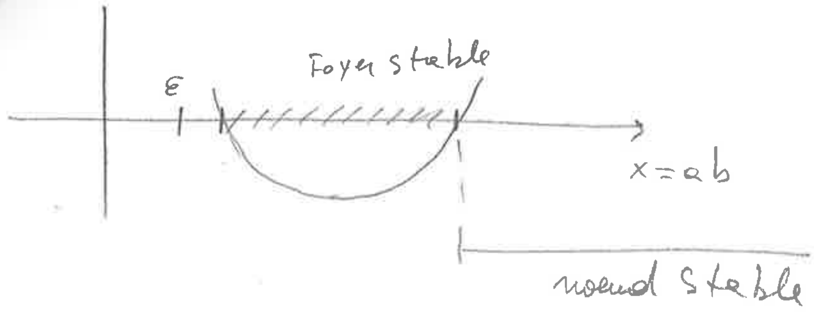
$$2\left(1-\sqrt{1-\varepsilon}\right) < ab < 2\left(1+\sqrt{1-\varepsilon}\right) \longrightarrow \text{Foyer stable.}$$

et $\boxed{\varepsilon - ba < 0}$ $\rightarrow$ hypothèse d'avant $\Rightarrow \varepsilon < ba$

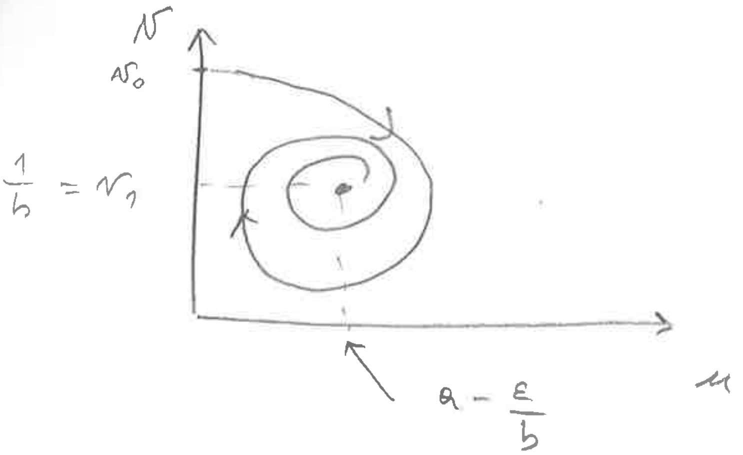
$\Big($ $\Rightarrow$ on montre que $\varepsilon < 2\left(1-\sqrt{1-\varepsilon}\right)$

car $\sqrt{1-\varepsilon} = 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} \ldots$ $\Big)$

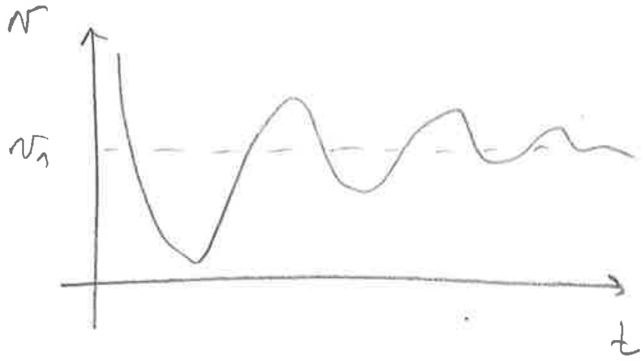
$\rightarrow$ si l'inégalité est pas satisfaite, (u_1, v_1) est un noeud stable



Foyer stable
ε
x = ab
noeud stable



v
v_0
$\frac{1}{b} = v_1$
$a - \frac{\varepsilon}{b}$
u



v
v_1
t