Statistical Mechanics (M1)

Statistical Mechanics using the perspective of stochastic processes

1 A colloidal particle in the over-damped limit

Consider a (1D) colloidal particle whose position x obeys in the over-damped limit the following equation :

$$\dot{x} = -\partial_x U + \sqrt{2D}\xi(t)\,, \tag{1}$$

where U is a conservative potential, D a constant, and $\xi(t)$ a delta correlated noise such that $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$.

a) Write down the Fokker-Planck equations corresponding to Eq. (1).

b) Find the probability of finding the particle at position x in the steady state.

c) Assume that $U = \frac{k}{2}x^2$ and write the steady-state probability for this specific potential. Compute $\langle x \rangle$ and $\langle x^2 \rangle$.

d) Now, write the canonical partition function for a single particle subject to a potential energy $U = \frac{k}{2}x^2$ in a system at temperature T. Write first the probability of finding the particle at position x with velocity v, and compute the probability of finding the particle at position x. Compute also $\langle x \rangle$ and $\langle x^2 \rangle$.

e) Compare the obtained results with the one obtained with the Fokker-Planck. Discuss how D should be for both approaches to coincide.

2 Beyond the over-damped limit

As in the previous exercise, but considering inertial effect, so that Eq. (1) is replaced with :

$$m\dot{v} = -\partial_x U + \zeta\, v + \sqrt{2D}\xi(t)\,, \tag{2}$$

where $v = \dot{x}$ and m is the mass of the particle.
Discuss similarities and differences with the previous exercise, and the relation between ζ and D.

3 A long transient

Consider a (1D) colloidal particle that obeys :

$$\dot{x} = \sqrt{2D}\xi(t)\,, \tag{3}$$

Assume that at $t = 0$, the particle is located at $x = 0$.

a) Write down the Fokker-Planck equations corresponding to Eq. (3).

b) Discuss the temporal evolution of $p(x, t)$ considering that above indicated initial condition.

c) Comment on $\langle x \rangle$ and $\langle x^2 \rangle$ as function of time t.