# Statistical Mechanics (M1)

## The Canonical Ensemble : continuous approach

# 1 The ideal monatomic gas (classic approximation)

Consider a gas composed of $N \gg 1$ monatomic molecules without interaction, contained in a volume $V$ at equilibrium at temperature $T$.

a) Obtain the density of states $\rho(E)$ of a single particle and derive the partition function $Z$ of the system.

b) Obtain the free energy and internal energy of this gas. Derive the equation of state of the ideal gas.

# 2 Density of states of a two dimensional ideal gas

Consider an ideal gas in a square box of side $L$. Show that density of states $\rho(E)$ of a single particle of such a gas is a constant.

# 3 The ideal gas in a rotating cylinder

Let us consider a monatomic perfect gas, contained in a cylinder of height $H$ and radius $R$, composed of $N$ particles of mass $m$ (see Fig. 1). The system is in thermodynamic equilibrium at temperature $T$. The cylinder rotates around its axis at constant angular velocity $\omega$. Assume that in stationary regime a molecule at distance $r$ of the axis has an energy :

$$E = \frac{p^2}{2m} - \frac{m\omega^2 r^2}{2} \tag{1}$$

a) Write the number of particles $dN$ with an impulsion comprised between $(p_x, p_y, p_z)$ and $(p_x + dp_x, p_y + dp_y, p_z + dp_z)$ and located in a cylinder of height $H$ and radius comprised between $r$ and $r + dr$.

b) Show that the partition function $z$ of one particle is

$$z = \frac{2\pi H}{h^3}(2\pi mkT)^{2/3}\frac{kT}{m\omega^2}\left[e^{\frac{m\omega^2 R_0^2}{2kT}} - 1\right], \tag{2}$$

where $k$ is the Boltzmann constant and $h$ the Planck constant.

c) Obtain the number $dN_p$ of particles with impulsion comprised between $(p_x, p_y, p_z)$ and $(p_x + dp_x, p_y + dp_y, p_z + dp_z)$.

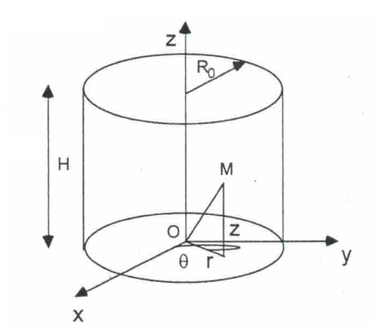


FIGURE 1 – The cylinder contains an ideal gas.

d) Obtain the number $dN_v$ of particles with a speed (the modulus of their velocity) comprised between $v$ and $v + dv$. Show that it can be written as $dN_v = N f(v) dv$. Trace the $f(v)$ curve.

e) Derive the most probable speed, $v_p$, and the average value of the speed $\langle v \rangle$.

f) Obtain the number of particles, $dN_r$, located at a distance from the rotation axis comprised between $r$ and $r + dr$.

g) Derive the density, $n(r)$, (number of particles per unit volume) and the pressure $p(r)$ at a distance $r$ of the rotation axis.

h) Application. Calculate $v_p$, and $p(R_0)/p(0)$ for a cylinder of $R_0 = 25$cm, rotating at 3000rpm, full of oxygen of molar mass $M = 32$g/mol, at $T = 300K$.

# 4 The ideal gas subject to gravitational force

Let us consider an ideal monatomic gas of $N$ particles of mass $m$, in equilibrium at temperature $T$, contained in a cylinder of radius $R$ and infinite height. In this problem we will not neglect the gravitation field, $g$, we consider it is uniform and constant, in the direction of the cylinder's axis (vertical). Obtain the average energy and the heat capacity of the system.

# 5 Dielectric polarization

The charge distribution in certain non-symmetric molecules when placed in an electrical field makes it possible to consider them as permanent electrical dipoles.

Let us assume that all the dipoles have a dipolar moment $\mathbf{p}$, of the same modulus $p$, but with a different orientation from one molecule to the other. The direction of the vector $\mathbf{p}$ is given in spherical coordinates by the angles $\theta$ and $\Phi$, with the axis $Oz$ as a reference. We assume that there is a uniform electric field $\mathbf{E}$, parallel to the $Oz$ axis. The potential energy of such a dipole in the electrical field $\mathbf{E}$ is $\varepsilon_p = -\mathbf{p} \cdot \mathbf{E}$ (where the dot represents the scalar product of the two vectors). This energy is minimum when $\mathbf{p} \parallel \mathbf{E}$. Nevertheless, the thermal agitation opposes to a perfect orientation of all the dipoles parallel to the field. Let us assume that the number of dipoles oriented inside an elementary solid angle $d\Omega$ around the direction $(\theta, \Phi)$ is :

$$dN_{\theta\Phi} = A \exp\left(-\frac{\varepsilon_p}{k_B T}\right) d\Omega \tag{3}$$

where $k_B$ is Boltzmann's constant and $T$, the temperature.

a) Obtain the partition function $Z$ of the system. We will call $x = pE/k_B T$.

b) Show that the total dipolar moment $\mathbf{P}$ is parallel to $Oz$ and calculate the average value $\langle p_z \rangle$ of the projection of $\mathbf{P}$ along $Oz$.

c) When the field is not so high or the temperature is high enough, one has $x \ll 1$. Show that in such a case the molecules of the dielectric have an average dipolar moment proportional to the electric field and obtain the expression of $\alpha = \langle p_z \rangle / E$ (polarizability).