Statistical Mechanics (M1)

The Canonical Ensemble : discrete systems

1 A minimalistic solid model

A solid is composed of N identical and independent atoms, each of which carries a magnetic moment. Each magnetic moment may be oriented in either two directions (up or down). In the up state the corresponding magnetic energy is ε and in the down state it is $-\varepsilon$. The solid is in equilibrium with a heat reservoir at temperature T. Calculate the average energy of the solid and discuss the limit of high and low temperature.

2 A system of identical and independent particles with 3 energy levels

Consider a system of $N >> 1$ identical, distinguishable and independent particles that can be placed in three energy levels of energies 0, ε and 2ε respectively. Only the level of energy ε is degenerate, of degeneracy $g = 2$. This system is in equilibrium with a heat reservoir at temperature T.

a) Obtain the partition function of the system.

b) Which is the probability of finding each particle in each energy level ?

c) Calculate the average energy $\langle E \rangle$, the specific heat at constant volume, c_v, and the entropy S, of the system.

d) Define the high and low temperature limits. Give the mean energy and the entropy of the system in these limits. Justify qualitatively these results. Trace qualitatively $c_v(T)$ and $S(T)$.

3 The 1D Ising model

Consider the 1D Ising model in equilibrium with a heat reservoir at temperature T.

a) Find the partition function of this system.

b) Calculate the mean energy and the average number $\langle x \rangle$ of broken links (links between sites of antiparallel spin).

c) Study the high and low temperature limits.

d) Study the thermodynamic limit. Compare with previous results.

4 A model of a paramagnet

Let us assume that the energy of a magnetic moment $\mathbf{m}$ in an external magnetic field B, is $E_m = -\mathbf{m} \cdot \mathbf{B}$. The component $\hat{z}$ of $\mathbf{m}$, m_z, is expressed as $m_z = \mu m_J$ with μ a constant and $m_J \in [-J, J]$. Consider $N \gg 1$ identical and distinguishable magnetic moments without interaction among them, subject to a magnetic field $\mathbf{B} = B\hat{z}$ and in contact with a heat bath at temperature T.

a) Compute the average magnetization per particle.

b) Obtain the magnetic susceptibility per particle in zero field.

c) Discuss $J = 1/2$ and $J \to \infty$ (such that $\mu \to 0$ with $\mu J \to \mu_0$).

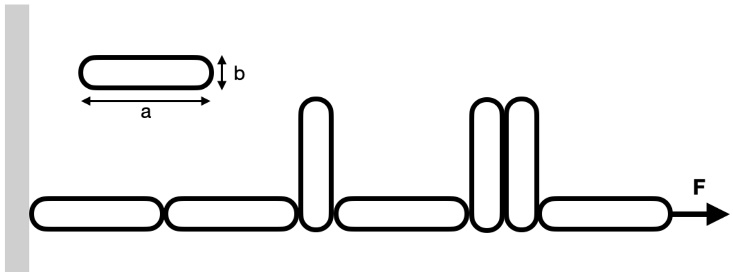

FIGURE 1 – The scheme illustrates the possible orientations of the macromolecules on the fiber.

5 Model of an elastic fiber

Consider a chain made of $N >> 1$ identical non-spherical macro molecules attached one after the other as shown in figure 1.

Each molecule may have one of the two possible orientations α of energy ε_α and β of energy ε_β. The chain is fixed by one extreme to a wall and a constant force F (see figure) is applied to the other extreme, located at abscise x. The contribution to the energy of the work done by the force is : $E_p = -FL$, where L is the total length of the chain. We assume the chain in thermal equilibrium with a heat bath at temperature T.

a) Obtain the partition function of the chain and calculate $\langle L \rangle$.

b) Find the relationship between L and T assuming that the energy is fixed (microcanonical ensemble). Compare with previous result.

c) Assume $\varepsilon_\alpha = \varepsilon_\beta$, show that Hooke's law is verified : $F = f(T)(L - L_0)$ if $kT >> aF$. Discuss this T dependence.

d) Obtain the statistical entropy of the chain as a function of x. Find the relationship between the temperature and the force needed to maintain the length of the chain equal to x.