Statistical Mechanics (M1)

The microcanonical ensemble

1 Throwing dices

Consider one normal dice of 6 faces :

— when we throw the dice, what is the probability P_{n_1} of each outcome $n_1 \in [1,6]$?

Let's now take two dices, dice 1 and dice 2 :

— Compute all possible outcomes and the associated entropy.
— We throw the two dices, what are the possible values of the output $n = n_1 + n_2$, where n_i is the output of the dice i ?
— When we throw the two dices, what is the probability P_n of each outcome n ? What is the number of configurations (micro-states) of the two dices that give the output n ?
— Re-formulate the answer above for 2 dices which are indistinguishable.

2 Cartoon of a thermal contact

Let us consider an isolated system consisting of two identical cubic boxes in contact through a wall. The quantum states of the particle in each box are characterized by the quantum numbers n_x, n_y, n_z, and the corresponding energy is given by :

$$\varepsilon_{n_x,n_y,n_z} = \varepsilon_0(n_x^2 + n_y^2 + n_z^2), \text{with } \ n_x, n_y, n_z \geq 1. \tag{1}$$

I. At t=0 each box contains two identical and distinguishable particles. The energy of the first box is $E_1 = 9\varepsilon_0$ and that of the second box is $E_2 = 12\varepsilon_0$. Consider that the wall separating the boxes is adiabatic. Obtain the number of microstates accessible to box 1, to box 2 and to the global system.

II. Consider now that the wall separating both boxes allows heat transfer (but no particle transfer). In this condition, the system will evolve to a new equilibrium state :
   a) Which physical quantity is conserved during the transformation ?
   b) Which and how many microstates are now accessible to the system ?

III. Assuming that the system has reached an equilibrium state :
   a) Obtain the probability of a particular microstate.
   b) Which is the probability of box 1 to have energies of $6\varepsilon_0$, $9\varepsilon_0$, $12\varepsilon_0$.
   c) Calculate the average energies of box 1, $\langle E_1 \rangle$, and of box 2, $\langle E_2 \rangle$. Which is the most probable energy of each box ?
   d) Which is the energy lost in average by box 2.

3 A two level system

Consider $N >> 1$ identical and distinguishable particles that can be placed in two non-degenerate energy levels.

a) Prove that the population of each level can be completely determined as a function of N and of the total energy U. Prove the same for the entropy.

b) Obtain the internal energy U as a function of the temperature T. Obtain the population of each level as a function of T.

4 Simple statistics of a polymer

Consider a one-dimensional polymer of $N >> 1$ segments, where each segment can be in one of two state : oriented toward the right or the left. One extreme of the polymer is attached to a fixed wall and the other is located at position x. The chain is fixed by one extreme to a wall and the other extreme is located at x. Obtain the statistical entropy of the polymer as function of x. Find the relationship between the temperature and the force needed to maintain the length of the polymber equal to x.

5 Joule expansion

Consider the expansion of an ideal gas that occupies a compartment A of volume V_0 in the initial state. In the final state, compartment $A + B$ and their volume is $2V_0$.

a) Find the final state of the system.

b) Which is the probability for a particle to be in compartment A in the final state ? Which is the probability that all the particles are found in the compartment A in the final state ?

c) Compute the entropy increase during the expansion.

6 The 1D Ising model

Consider a chain of spins variables with periodic boundary conditions (a ring). Each spin may be either of two states $s_i = \pm 1$ and the energy of the system is :

$$H = -\sum_i J\, s_i\, s_{i+1}, \tag{2}$$

where $J > 0$ is the interaction constant.

a) Find the energy of the ground state and its degeneracy.

b) Which is the degeneracy of the first excited state ?

c) Find all the energy levels of the system and indicate their degeneracy. Indicate the level of maximum energy.

d) For a given energy level n of E_n ($n >> 1$), give the corresponding temperature and entropy. Obtain the proportion x of broken links (antiparallel spins). Discuss the limits of low and high temperature.