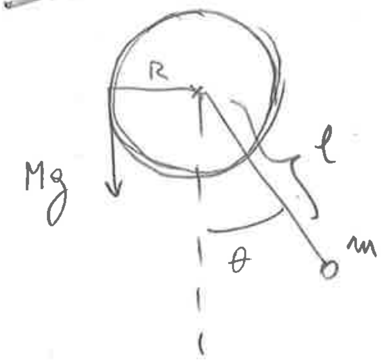


$$I\ddot{\theta} = MgR - l\,mg\sin\theta - \gamma l^2\dot{\theta}$$

On regarde un syst plus simple :

$I_z = \frac{1}{2}MR^2$

$I_z = \int \rho x^2 dv$

$$I\ddot{\theta} = l\cdot mg\sin\theta \rightarrow \text{le pendule}$$

$$\ddot{\theta} = \frac{l\,mg}{I}\sin\theta$$

Si $t = T\tilde{t} \Rightarrow \frac{t}{T} = \tilde{t}$ $\qquad \frac{d}{dt} = \frac{\partial}{\partial\tilde{t}}\underbrace{\frac{\partial\tilde{t}}{\partial t}}_{\frac{1}{T}} = \frac{1}{T}\frac{\partial}{\partial\tilde{t}}$

$$\Rightarrow \frac{d^2}{dt} = \frac{1}{T^2}\frac{\partial^2}{\partial\tilde{t}}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = \frac{1}{T^2}\frac{\partial^2\theta}{\partial\tilde{t}^2} = \frac{l\,mg}{I}\sin\theta$$

$$\frac{\partial^2\theta}{\partial\tilde{t}^2} = T^2\frac{l\,mg}{I}\sin\theta$$

$\rightarrow$ Si $T^2\frac{l\,mg}{I} = 1 \Rightarrow T = \sqrt{\frac{I}{l\,mg}}$

On fait la même chose pour notre système :

$$\frac{1}{T^2}\frac{\partial^2\theta}{\partial\tilde{t}^2} = \frac{MgR}{I} - \frac{l\,mg}{I}\sin\theta - \frac{\gamma l^2}{I}\frac{1}{T}\frac{\partial\theta}{\partial\tilde{t}}$$

$$\downarrow$$

$$\ddot{\theta} = \frac{MgRT^2}{I} - l\frac{mg}{I}T^2\sin\theta - \frac{\gamma l^2}{I}\frac{T^2}{T}\overbrace{\frac{\partial\theta}{\partial\tilde{t}}}^{\dot{\theta}}$$

si $T = \sqrt{\frac{I}{lmg}}$

$$\Rightarrow \ddot{\theta} = \frac{M\cancel{g}R}{\cancel{I}}\frac{\cancel{I}}{lmg} - \frac{lmg}{I}\frac{I}{lmg}\sin\theta - \frac{\gamma l^2}{I}\sqrt{\frac{I}{lmg}}\,\dot{\theta}$$

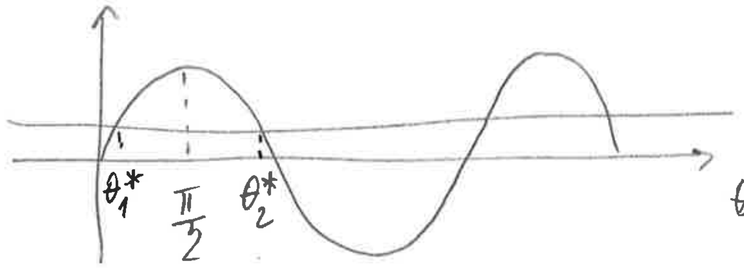
$$\ddot{\theta} = \overset{\Gamma}{\frac{MR}{lm}} - \sin\theta - \underset{D}{\frac{\gamma l^2}{\sqrt{Ilmg}}}\,\dot{\theta}$$

(a)

$$\Rightarrow \boxed{\ddot{\theta} = \Gamma - \sin\theta - D\dot{\theta}}$$

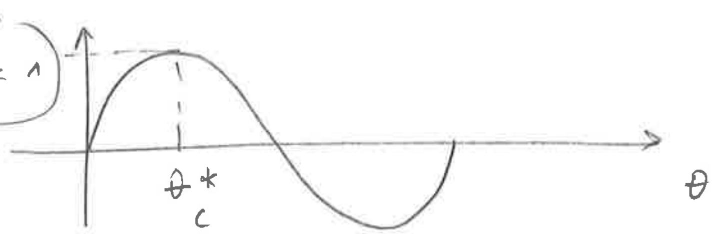
(b) $\ddot{\theta} = \dot{\theta} = 0 \longrightarrow 0 = \Gamma - \sin\theta \Rightarrow \Gamma = \sin\theta$

$\Gamma < 1$



pour $\Gamma < 1$
2 états
equilibre

$\Gamma = 1$



→ 1 état
équilibre

$\Gamma > 1$ ⟶ 0 état d'équilibre.

→ $\Gamma = 1$ est un point de bifurcation : bifurcation nœud-col.

2 solutions disparaissent par coalescence en 1 point de tangence donc $\Gamma_c = 1$ et $\theta_c^* = \frac{\pi}{2}$

$$\ddot{\theta} + D\dot{\theta} = \Gamma - \sin\theta$$

$$\rightarrow \quad \theta = \theta_c^* + a\psi = \frac{\pi}{2} + a\psi$$

→ on suppose ψ petit et a un facteur à déterminer.

$$\theta - \theta_c^* = a\psi$$

$$a\ddot{\psi} + Da\dot{\psi} = \Gamma - \sin\left(\frac{\pi}{2} + a\psi\right)$$

$$a\ddot{\psi} + Da\dot{\psi} = \Gamma - \underbrace{\cos(a\psi)}$$

$$\cos(a\psi) = 1 - \frac{a^2\psi^2}{2}$$

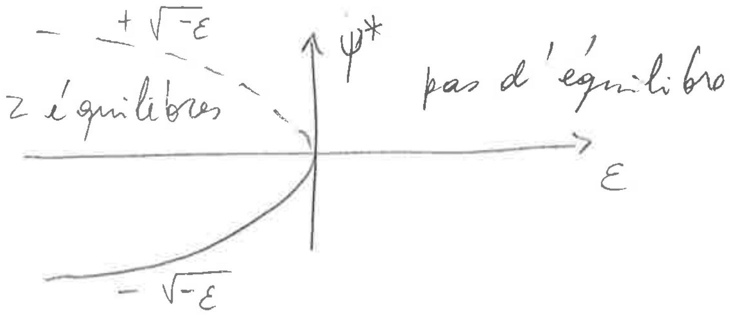
$$\Rightarrow a\ddot{\psi} + Da\dot{\psi} = \Gamma - 1 + \frac{a^2}{2}\psi^2$$

$$\ddot{\psi} + D\dot{\psi} = \frac{\Gamma - 1}{a} + \frac{a}{2}\psi^2$$

avec $a = 2$ $\longrightarrow$ $\ddot{\psi} + D\dot{\psi} = \underbrace{\frac{\Gamma - 1}{2}}_{\varepsilon} + \psi^2$

$$\boxed{\ddot{\psi} + D\dot{\psi} = \varepsilon + \psi^2}$$

On reconnaît la forme normale d'une bifurcation noeud-col dans le membre de droite !



$\psi = \psi_s + \eta$ → on veut étudier le système autour de la solution stationnaire stable

$\psi_s = -\sqrt{-\varepsilon} = -\sqrt{|\varepsilon|}$ (dans la limite $\varepsilon \to 0$ mais $\varepsilon < 0$)

$\Rightarrow \quad \psi = -\sqrt{|\varepsilon|} + \eta$ → petit

$$\ddot{\psi} + D\dot{\psi} = \varepsilon + \psi^2$$

$$\ddot{\eta} + D\dot{\eta} = -|\varepsilon| + \left(-\sqrt{|\varepsilon|} + \eta\right)^2$$

$$= -\cancel{|\varepsilon|} + \cancel{|\varepsilon|} - 2\sqrt{|\varepsilon|}\,\eta + O(\eta^2)$$

$$\ddot{\eta} + D\dot{\eta} = -2\sqrt{|\varepsilon|}\,\eta$$

$$\Rightarrow \quad \boxed{\ddot{\eta} + D\dot{\eta} + 2\sqrt{|\varepsilon|}\,\eta = 0}$$

en divisant par $2\sqrt{|\varepsilon|}$:

$$\frac{1}{2\sqrt{|\varepsilon|}}\ddot{\eta} + \frac{D}{2\sqrt{|\varepsilon|}}\dot{\eta} + \eta = 0$$

$$\downarrow \qquad \downarrow$$

$$\tau_i^2 \ddot{\eta} + \tau_v \dot{\eta} + \eta = 0$$

$$\tau_i = \frac{1}{\sqrt{2\sqrt{\varepsilon}}} \qquad \tau_v = \frac{D}{2\sqrt{|\varepsilon|}} \rightarrow \frac{\tau_i}{\tau_v} = \frac{2\sqrt{|\varepsilon|}}{D}\frac{1}{\sqrt{2}\,|\varepsilon|^{1/4}}$$

$$\Rightarrow \frac{\tau_i}{\tau_v} = \frac{2}{\sqrt{2}D}|\varepsilon|^{1/3} \longrightarrow 0$$

si $\varepsilon \to 0$

→ Donc ici c'est τ_v qui domine :

on redéfinit le temps comme $\hat{t} = \dfrac{\tilde{t}}{\tau_v} \longrightarrow \hat{t}\,\tau_v = \tilde{t}$

$$\frac{d}{d\tilde{t}}\cdot\frac{d\tilde{t}}{d\hat{t}} = \tau_v\frac{d}{d\tilde{t}} = \frac{d}{d\hat{t}}$$

$$\rightarrow \frac{d^2}{d\hat{t}^2} = \tau_v^2\frac{d^2}{d\tilde{t}^2}$$

→ avec la nouvelle définition du temps.

$$\frac{\tau_i^2}{\tau_v^2}\underbrace{\tau_v^2\frac{d^2\eta}{d\tilde{t}^2}} + \tau_v\frac{d\eta}{d\tilde{t}} + \eta = 0$$

$$\frac{\tau_i^2}{\tau_v^2}\frac{d^2\eta}{d\hat{t}^2} + \frac{d\eta}{d\hat{t}} + \eta = 0$$

$$\frac{2}{\sqrt{2}D}|\varepsilon|^{1/4}\ddot{\eta} + \dot{\eta} + \eta = 0$$

si $\varepsilon \to 0$

$$\dot{\eta} + \eta = 0$$

$$\eta(t) = \eta(0)\,e^{-\hat{t}}$$

$$\boxed{\eta(t) = \eta(0)\,e^{-\tilde{t}/\tau_v}}$$

encore $\eta(t) = \eta(0)\, e^{-\frac{t}{T} \to \frac{t}{\tau_v}}$

Donc le temps de retour caractéristique de retour

à l'équilibre est $T\tau_v = \sqrt{\frac{I}{mgl}}\ \frac{1}{2\sqrt{|\epsilon|}} = \sqrt{\frac{I}{mgl}}\ \frac{\sqrt{2}}{2}\ \frac{1}{\sqrt{\Gamma - 1}}$

$$= \sqrt{\frac{I}{2mgl(\Gamma - 1)}}$$

En général :

Proche d'une bifurcation noeud - col $\to \dot{x} = \epsilon + x^2$, le temps de retour à l'équilibre diverge comme $\frac{1}{\sqrt{|\epsilon|}}$ lorsque $\epsilon \to 0$. ($\epsilon < 0$)

(f) Si $\nu = 0$, $\tau_v = 0$ quel que soit ϵ on a un oscillateur harmonique :

$$\tau_i^2\, \ddot{\eta} + \eta = 0 \iff \ddot{\eta} = -\omega_0^2\, \eta$$

avec $\omega_0 = \frac{1}{\tau_i}$

$\to$ avec une période $T = 2\pi\, \tau_i = 2\pi\, \frac{1}{\sqrt{2\sqrt{|\epsilon|}}}$

$\to$ le temps caractéristique diverge comme $|\epsilon|^{-1/4}$

si $\varepsilon > 0$, il n'y a plus d'équilibre, mais lorsque $\varepsilon \to 0$ ($\varepsilon < 0$), il y a un "ralentissement critique" lorsque le pendule passe dans le voisinage de l'équilibre "fantôme" $\to \theta_c^* = \frac{\pi}{2}$ (en $\psi = 0$)

$$\to \quad \ddot{\psi} + \nu \dot{\psi} = \varepsilon + \psi^2$$

supposons $0 < \varepsilon << 1$ et on s'intéresse à des amplitudes de ψ d'ordre $\sqrt{\varepsilon}$:

$\to$ poser $\psi = \sqrt{\varepsilon}\, \Phi$ (ou $\Phi \simeq O(1)$)

$$\sqrt{\varepsilon}\, \ddot{\Phi} + \nu \sqrt{\varepsilon}\, \dot{\Phi} = \varepsilon + \varepsilon \Phi^2$$

$$\frac{1}{\sqrt{\varepsilon}} \ddot{\Phi} + \frac{\nu}{\sqrt{\varepsilon}} \dot{\Phi} = 1 + \Phi^2$$

$$\tau_i^2 \ddot{\Phi} + \tau_v \dot{\Phi} = 1 + \Phi^2$$

Comme précédemment on rescale le temps : $\Phi' = \frac{d\Phi}{d\hat{t}}$ avec $\hat{t} = \frac{\tilde{t}}{\tau_v}$

$$\frac{\tau_i^2}{\tau_v^2} \Phi'' + \Phi' = 1 + \Phi^2.$$

$$\underbrace{|\varepsilon|^{1/4}}_{\to 0 \text{ si } \varepsilon \to 0} \Phi'' + \Phi' = 1 + \Phi^2$$

$\Phi(0) = 0$, $\Phi' \simeq 1$ (tant que $\Phi \ll 1$)

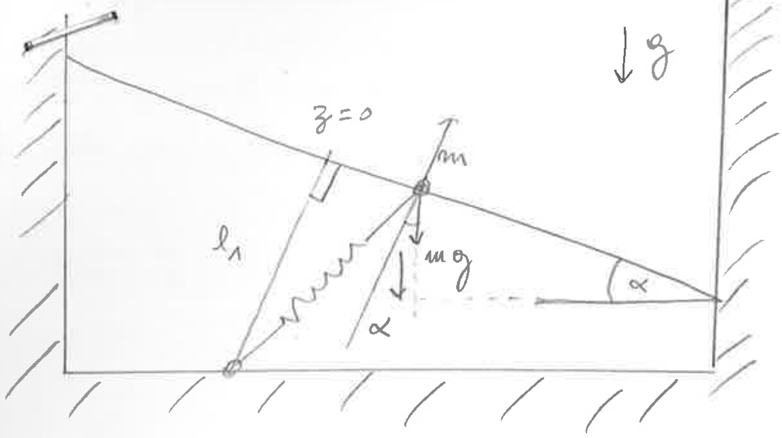
$$\rightarrow \quad \Phi(\hat{t}) \simeq \hat{t}$$

$$\rightarrow \quad \Phi(\tilde{t}) = \frac{\tilde{t}}{\tilde{\tau}_V}$$

$$\rightarrow \quad \Phi(t) \simeq \frac{t}{\sqrt{\frac{I}{mgl}}\frac{\partial}{\sqrt{2|\varepsilon|}}}$$

→ on voit que pour que $\Phi(t)$ varie d'une quantité d'ordre 1 (c'est à dire ψ varie d'ordre $O(\sqrt{\varepsilon})$) il faut attendre un temps d'ordre $\sqrt{\frac{I}{mgl}}\frac{\partial}{\sqrt{2|\varepsilon|}} \sim |\varepsilon|^{-1/2}$

1



$$+ \; mg \sin\alpha$$

$$+ \; F_e = -\frac{\partial U}{\partial z} = -\frac{\partial}{\partial z}\left\{ \left[\left(z^2 + l_1^2\right)^{1/2} - l_0 \right]^2 \frac{k}{2} \right\}$$

$$+ \; F_{Frot} = -\gamma \dot{z}$$

$$\downarrow$$

$$-k\left\{ \left(z^2+l_1^2\right)^{1/2} - l_0 \right\} \cdot \frac{z}{\sqrt{z^2+l_1^2}}$$

(a) $$\boxed{m\ddot{z} = -\gamma\dot{z} + mg\sin\alpha - k\left\{ \left(z^2+l_1^2\right)^{1/2} - l_0 \right\} \frac{z}{\sqrt{z^2+l_1^2}}}$$

Dimless version

(b) $$\ddot{z} = -\frac{\gamma}{m}\dot{z} + g\sin\alpha - \frac{k}{m}\left\{ \left(z^2+l_1^2\right)^{1/2} - l_0 \right\} \frac{z}{\sqrt{z^2+l_1^2}}$$

$u = \frac{z}{a}$

$$\ddot{u} = -\frac{\gamma}{m}\dot{u} + g\frac{\sin\alpha}{a} - \frac{k}{m}\underbrace{\left\{ \left(z^2+l_1^2\right)^{1/2} - l_0 \right\} \frac{z/a}{\sqrt{z^2+l_1^2}}}$$

$$\left\{ \left(a^2u^2+l_1^2\right)^{1/2} - l_0 \right\} \frac{\overset{u}{z/a}}{\sqrt{a^2u^2+l_1^2}}$$

Si $a = l_1 \longrightarrow \left\{ \left(u^2+1\right)^{1/2} l_1 - l_0 \right\} \frac{u}{l_1\sqrt{u^2+1}} =$

$$= \left\{ \left(u^2+1\right)^{1/2} - \frac{l_0}{l_1} \right\} \frac{u}{\sqrt{u^2+1}}$$

$$\Rightarrow \ddot{u} = -\frac{\gamma}{m}\dot{u} + g\frac{\sin\alpha}{l_1} - \frac{k}{m}\left\{ \left(u^2+1\right)^{1/2} - \underbrace{\frac{l_0}{l_1}}_{R} \right\} \frac{u}{\sqrt{u^2+1}}$$

Si $t = \tilde{t}T \rightarrow \frac{t}{T} = \tilde{t}$

$$\frac{1}{T^2}\frac{d^2u}{d\tilde{t}^2} = -\frac{\gamma}{m}\frac{1}{T}\frac{du}{d\tilde{t}} + g\sin\alpha - \frac{k}{m}\left\{ \left(u^2+1\right)^{1/2} - R \right\} \frac{u}{\sqrt{u^2+1}}$$

(2)

$$\frac{d^2u}{d\tilde{t}^2} = -\frac{\gamma}{m} T \frac{du}{d\tilde{t}} + \frac{g \sin\alpha}{l_1} T^2 - \frac{k T^2}{m} \left\{ (u^2+1)^{1/2} - R \right\} \frac{u}{\sqrt{u^2+1}}$$

$$\underbrace{T = \tau_i = \sqrt{\frac{m}{k}}}_{Si}$$

$$\frac{d^2u}{d\tilde{t}^2} = -\frac{\gamma}{\sqrt{km}} \dot{u} + \frac{mg}{kl_1}\sin\alpha - \left\{ (u^2+1)^{1/2} - R \right\} \frac{u}{\sqrt{u^2+1}}$$

$$\downarrow \qquad \tau_v = \frac{\gamma}{k}, \quad \tau_i = \sqrt{\frac{m}{k}}$$

$$\frac{\tau_v}{\tau_i} = \frac{\gamma}{k} \frac{\sqrt{k}}{\sqrt{m}} = \frac{\gamma}{\sqrt{mk}} \qquad D = \frac{\tau_i}{\tau_v}$$

$$\ddot{u} = -\frac{1}{D}\dot{u} + b - \left[(u^2+1)^{1/2} - R \right] u \left(1 - \frac{u^2}{2}\right)$$

$$\downarrow$$

$$\left(1 + \frac{1}{2}u^2 - R\right) u \left(1 - \frac{u^2}{2}\right)$$

$$= \left(u + \frac{1}{2}u^3 - Ru\right)\left(1 - \frac{u^2}{2}\right) \simeq u + \cancel{\frac{u^3}{2}} - Ru - \cancel{\frac{u^3}{2}} + \frac{Ru^3}{2}$$

$$\underbrace{\simeq (1-R)u + \frac{R}{2}u^3}_{f(u)}$$

$$\text{Si } T = \tau_v = \frac{\gamma}{k} \qquad \swarrow \quad \dot{u} = \frac{du}{d\tilde{t}} \qquad \tau_i = \sqrt{\frac{m}{k}}$$

$$\frac{1}{T^2}\ddot{u} = -\frac{\gamma}{m}\frac{1}{T}\dot{u} + \frac{g \sin\alpha}{l_1} - \frac{k}{m} f(u)$$

$$\frac{1}{\tau_v^2}\ddot{u} = \underbrace{-\frac{\gamma}{m}\overset{\tau_v^{-1}}{\left(\frac{k}{\gamma}\right)}}\dot{u} + \frac{g \sin\alpha}{l_1} - \tau_i^{-2} f(u)$$

$$\frac{1}{\tau_v^2}\ddot{u} = -\tau_i^{2}\dot{u} + g\sin\alpha / l_1 - \tau_i^{2} f(u)$$

$$\lim \frac{\tau_i}{\tau_v} \to 0 \qquad \text{(b)}$$

$$\ddot{u}\left(\frac{\tau_i}{\tau_v}\right)^2 = -\dot{u} + \overset{B}{\frac{mg\sin\alpha}{l_1 k}} - f(u)$$

$$\boxed{\dot{u} = B - (1-R)u - \frac{R}{2}u^3}$$

$$\dot{u} = b + (R-1)\,u - \frac{R}{2}u^3$$

$$u = \rho x \Rightarrow \rho\dot{x} = b + (R-1)\rho x - \frac{R}{2}\rho^3 x^3$$

$$\dot{x} = \frac{b}{\rho} + (R-1)\,x - \left(\frac{R}{2}\rho^2\right)x^3$$

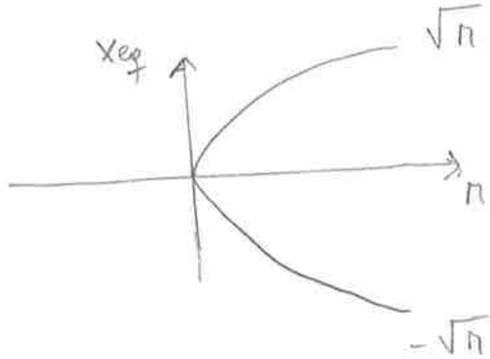
$$\frac{R}{2}\rho^2 = 1 \Rightarrow \rho = \sqrt{\frac{2}{R}}$$

$$\dot{x} = \underbrace{b\sqrt{\frac{R}{2}}}_{h} + \underbrace{(R-1)}_{r}\,x - x^3$$

$$\Rightarrow \boxed{\dot{x} = h + r x - x^3}$$

(d) Dans le cas $\alpha = 0 \longrightarrow b = 0 \Rightarrow h = 0$

$$\Rightarrow \quad \dot{x} = (r - x^2)\,x$$



<u>Étude de stabilité linéaire :</u>

$\underline{x_{eq} = 0} \longrightarrow \dot{\delta} = r\delta$ $\longrightarrow r < 0 \longrightarrow$ stable

$\longrightarrow r > 0 \longrightarrow$ pas stable $\rightarrow$ instable.

$\underline{x_{eq} = \sqrt{r}} \longrightarrow x = x_{eq} + \delta \longrightarrow \dot{\delta} = \left(r - (x_{eq}+\delta)^2\right)(x_{eq}+\delta)$

$$\Rightarrow \dot{\delta} = \left(r - \left(x_{eq}^2 + 2x_{eq}\delta + \delta^2\right)\right)(x_{eq}+\delta)$$

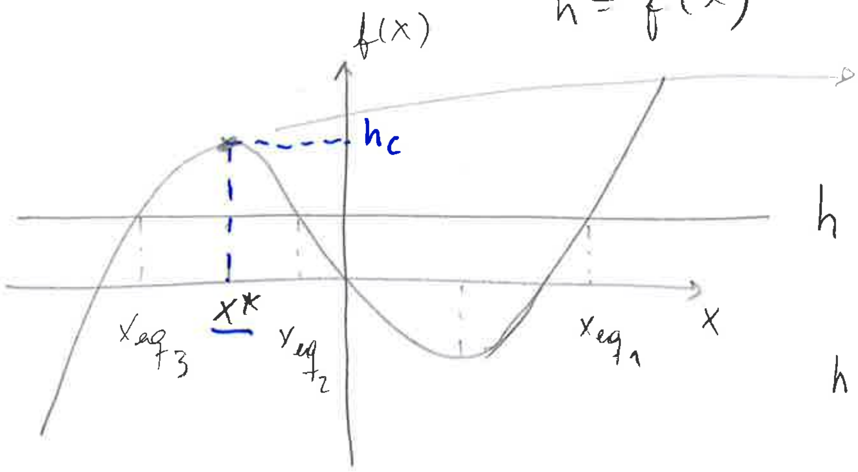
$$\dot{\delta} = \left(\cancel{r} - \cancel{x_{eq}^2} - 2x_{eq}\delta - \delta^2\right)x_{eq} + \left(r - x_{eq}^2 - 2x_{eq}\delta - \delta^2\right)\delta$$

$\dot{\delta} = -2x_{eq}^2\,\delta \longrightarrow$ Toujours <u>stable</u> ! (mais pour $r > 0$!).

$\to \dot{x} = h + rx - x^3$

$0 = h + rx - x^3 \Rightarrow h = x^3 - rx$

$h = f(x)$



ici on voit une bif. noeud-col!

h>0 toujours

$$\frac{df}{dx} = 0 \longrightarrow 3x^2 - r = 0 \Rightarrow r = 3x^2$$

$$\sqrt{\frac{r}{3}} = |x|$$

$$\to x^* = -\sqrt{\frac{r}{3}} \Rightarrow h_c = -\left(\frac{r}{3}\right)^{\frac{3}{2}} + r\left(\frac{r}{3}\right)^{1/2}$$

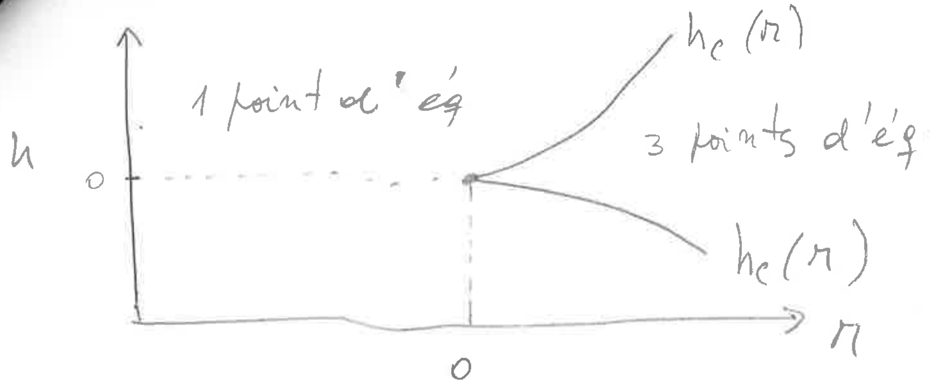
$$= r^{3/2}\frac{1}{\sqrt{3}} - \underbrace{r^{2/3}\frac{1}{3^{3/2}}}_{\frac{1}{3\sqrt{3}}} = \frac{r^{3/2}}{\sqrt{3}}\underbrace{\left(1 - \frac{1}{3}\right)}_{\frac{2}{3}}$$

$\Rightarrow \boxed{h_c = \frac{2}{3}\frac{r^{3/2}}{\sqrt{3}}} \longrightarrow$
- Si $h > h_c \to$ un seul point d'éq.
- si $h = h_c \to$ 2 points
- si $h < h_c \to$ 3 points

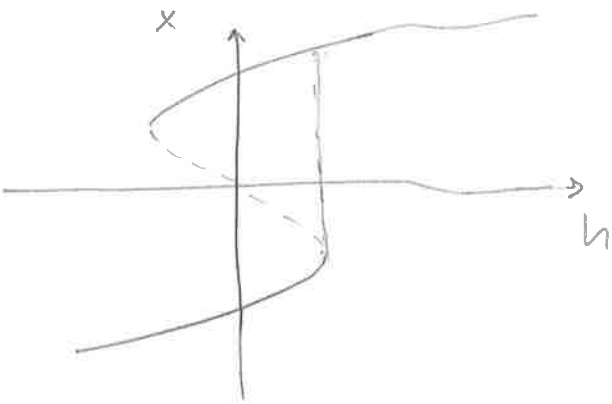
$\to h = b\sqrt{\frac{R}{2}}$ $\qquad r = (R-1)$

$$h_c = \frac{mg\sin\alpha_c}{l_1 k}\sqrt{\frac{R}{2}} = \frac{2}{3}\frac{(R-1)^{3/2}}{\sqrt{3}}$$

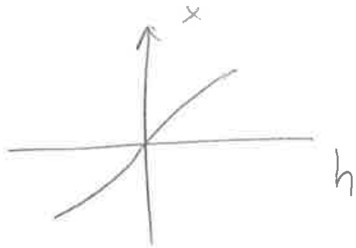
$$\sin\alpha_c = \frac{l_1 k}{mg}\sqrt{\frac{2}{R}}\,\frac{2}{3}\,\frac{(R-1)^{3/2}}{\sqrt{3}}$$



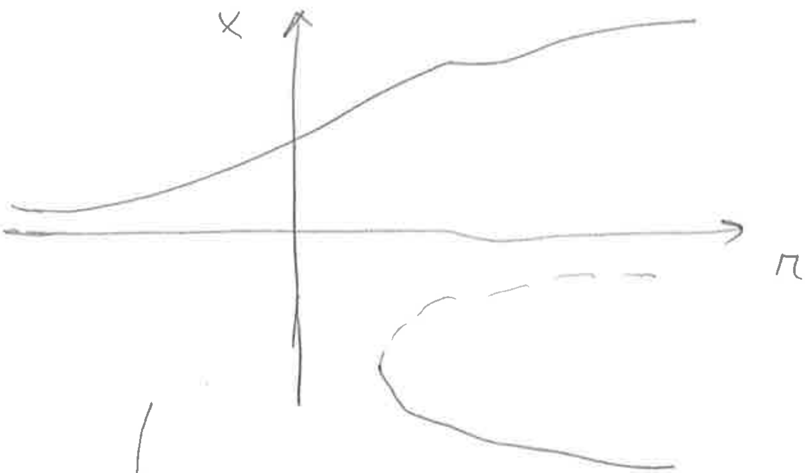
$r > 0$ fixé



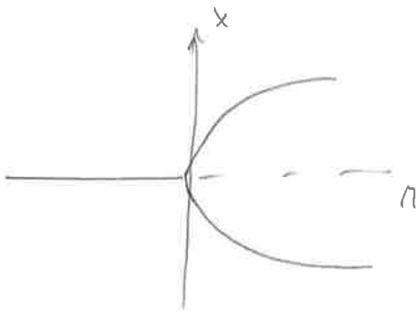
$r \leq 0$



$h \neq 0$ fixé



$h = 0$



$$0 = h + r x - x^3$$

$$x^3 - h = r x$$

$$\frac{x^3 - h}{x} = r$$

$$x^2 - \frac{h}{x} = r$$

$$r = g(x)$$