Dynamical Systems

Bifurcations

1 Loss of balance in the forced pendulum

1. Show that the differential equation describing the dynamics of the pendulum, represented in Fig. 1, reads : $I\frac{d^2\Theta}{dt^2} = Mgr - mgl\sin(\Theta) - \gamma l^2\frac{d\Theta}{dt}$.

   Write the equation in dimensionless form with two dimensionless numbers (ν, Γ) : $\nu = \frac{\gamma l^2}{\sqrt{mglI}}$ and $\Gamma = \frac{Mr}{ml}$.
2. Recall how the stationary states of the dimensionless equation can be solved graphically. Deduce that this system has a saddle-node bifurcation for a critical value Γ_c. This point corresponds to a critical equilibrium Θ_c^*.
3. Perform a change of variable in the neighborhood of the equilibrium corresponding to the pendulum forced by the critical moment of force (torque) Γ_c. Show that the dynamical system can be written in the normal form of a saddle-node bifurcation :

$$\ddot{\Psi} + \nu\dot{\Psi} = \epsilon + \Psi^2\,,$$

   where $\epsilon = (\Gamma - \Gamma_c)/2$ et $\Psi = (\Theta - \Theta_c^*)/2$
4. We want to study the behavior of the pendulum around its stable equilibrium, $\Psi = \Psi_s + \eta$, within the limit where $|\epsilon| \ll 1$. Show that we obtain the linear equation : $\ddot{\eta} + \nu\dot{\eta} + 2\sqrt{|\epsilon|}\eta = 0$.
5. Show that when $|\epsilon|$ is small enough the return to stable equilibrium occurs exponentially with a characteristic time proportional to $|\epsilon|^{-\frac{1}{2}}$.
6. (*) Discuss how the situation changes from the previous point, in the absence of viscous friction, $\nu = 0$.

2 The "catastrophe" of a ball attached by a spring on an inclined rail

We consider a ball of mass m which can slide on an inclined rail with an angle θ (see figure 2). The ball is subjected to a viscous friction, with viscosity constant γ. The ball is also attached to a spring of constant k and length at rest l_0. The other end of the spring is attached to to the surface in such a way that length of the spring is equal to l_1 when the latter is perpendicular to the rail. This position corresponds to $z = 0$. Depending on whether the ratio $R = \frac{l_0}{l_1}$ is smaller or greater than 1, the position $z = 0$ will correspond to a stretched or compressed spring.

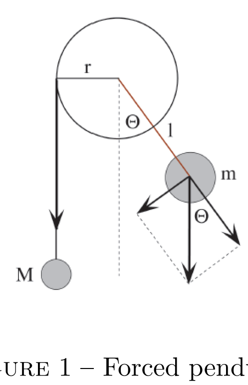

Figure 1 – Forced pendulum

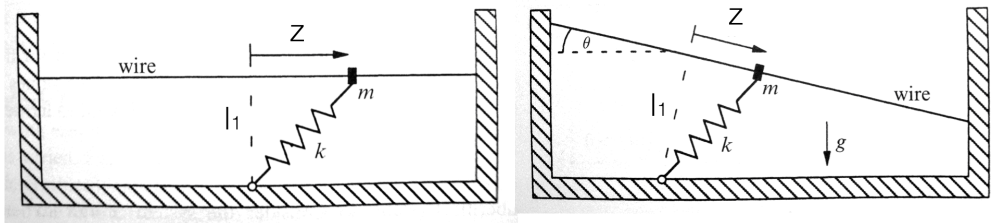

FIGURE 2 – A ball attached by a spring on an inclined rail

1. Show that the equation of motion of the ball is :

$$m\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} = mg\sin(\theta) - k\left(\sqrt{l_1^2+z^2} - l_0\right)\frac{z}{\sqrt{l_1^2+z^2}}$$

2. To adimensionalize the equation, we consider the variable $u = z/a$ and the dimensionless parameters $R = l_0/l_1$, $b = mg\sin(\theta)/kl_1$ and $\nu = \tau_i/\tau_v$, where $\tau_i = \sqrt{m/k}$ and $\tau_v = \gamma/k$. Let us assume that the viscous friction is quite high ($\nu \gg 1$) and $z \ll l_1$. With these assumptions, show that we can choose a to simplify the equation of the dynamical system in the form :

$$\dot{u} = b + (R-1)\,u - \frac{R}{2}u^3\,,$$

where $\dot{u}$ is the derivative of u with respect to the dimensionless time t/τ_v. Indication : $(1+u^2)^{-1/2} \sim 1 - \frac{u^2}{2} + \cdots$ as $u \to 0$.

3. Find a change in variable $u = q\,x$ (determine q) allowing to further simplify the previous equation in order to put it in the normal form of the imperfect pitchfork bifurcation (often called catastrophe) :

$$\dot{x} = h + r\,x - x^3 \tag{1}$$

4. What happens to the equation (1) in the case of the horizontal rail ? Represent the stationary solutions of the system in a bifurcation diagram and provide a physical interpretation. Perform a linear stability study of the obtained equilibria.

5. How do the stationary states of the system move when the rail is tilted slightly ? Using equation (1), show that there exists a critical value $h_c(r) > 0$ such that equilibrium positions disappear when $h > h_c(r)$. What type of bifurcation is it ? Determine an analytical expression (according to all the parameters of the system) of the critical angle of inclination θ_c beyond which the ball can no longer have an equilibrium position with $z < 0$.

6. Draw a phase diagram in parameter space (r, h) and indicate the regions of this space where there are, respectively, 1, 2, or 3 equilibrium positions of the ball on the rail. Next, represent a bifurcation diagram of stationary solutions as a function of h with $r > 0$ fixed, then a bifurcation diagram as a function of r, with $h > 0$ fixed. Use the former to describe how it is possible to observe a phenomenon of hysteresis in this dynamic system.

3 Nonlinear study of a dynamical system

For the following dynamical systems :

$$\dot{x} = a + x - \ln(1+x) \tag{2}$$

$$\dot{x} = -a\,x + \frac{x^2}{1+x^2} \tag{3}$$

$$\dot{x} = a\,x - \sinh x \tag{4}$$

1. Determine the fixed points (x^*) as a function of a and identify the critical point (or critical parameter) a_c.
2. Assume that $x = x^* + \delta(t)$ and $a = a_c + \epsilon$ and find the equation for the time evolution of $\delta(t)$ and determine the type of bifurcation.