Dynamical Systems

Fixed points and their stability [for ODEs]

1 One-dimensional systems

For the following systems, find the fixed points and evaluate their stability :

1. $\dot{x} = \sin(x)$
2. $\dot{x} = \sin(x + \pi)$
3. $\dot{x} = (x - 1)(x - 3)$
4. $\dot{x} = (x - 1)(3 - x)$
5. $\dot{x} = \sinh(-x)$ [reminder : $\sinh(w) = \frac{e^w - e^{-w}}{2}$]
6. $\dot{x} = x^2 - \frac{x}{2} + \frac{1}{16}$

2 Two-dimensional systems

For the following systems, find the fixed points and evaluate their stability :

1. $\dot{x} = x - y$ and $\dot{y} = x^2 - 4$
2. $\dot{x} = 1 + y - e^{-x}$ and $\dot{y} = x^3 - y$
3. $\dot{x} = \sin(x)$ and $\dot{y} = \cos(x)$
4. $\dot{x} = \sin(y)$ and $\dot{y} = x - x^3$
5. $\dot{x} = y + x - x^3$ and $\dot{y} = -y$
6. $\dot{x} = y$ and $\dot{y} = -(x + y)$
7. $\ddot{x} = -\dot{x} - x$
8. $\ddot{x} = -\dot{x} - \sin(x)$
9. $\ddot{x} = \dot{x} - x$

3 A classical problem from physics : the LASER

Under certain assumptions, the dynamics of a laser can be described by the following nonlinear system :

$$\frac{du}{dt} = -ku + Guv \tag{1}$$

$$\frac{dv}{dt} = -fv + P - Guv \tag{2}$$

where the variables (u, v) are respectively the number of photons and the number of atoms excited in the optical cavity of the laser. These variables are therefore with positive or zero real values. The system parameters are as follows : G is the gain parameter linked to the stimulated emission, f the spontaneous emission rate, k the loss coefficient of the cavity and P the pump parameter making it possible to excite the atoms.

To reduce the number of parameters we work with an adimensional time. Show that this can be chosen so that the equations are written :

$$\dot{u} = -u + buv \tag{3}$$

$$\dot{v} = -\epsilon v + a - buv \tag{4}$$

Specify the new parameters a, b, ϵ.

1. Calculate the possible stationary states of this system according to the parameters a, b, ϵ. In the case where there are two stationary states, we will denote these (u_0, v_0) and (u_1, v_1).
2. Write the Jacobian matrix $J(u, v)$ of the system as a function of the variables u, v.
3. Calculate the Jacobian matrix $J_0 = J(u_0, v_0)$ according to the parameters a, b, ϵ.
4. Discuss the stability of the steady state (u_0, v_0) assuming that $\epsilon < b\,a$.
5. Calculate the Jacobian matrix $J_1 = J(u_1, v_1)$ according to the parameters.
6. Discuss the stability of the steady state (u_1, v_1) under the previous assumption.