Nonlinear synchronization in long-range coupled systems

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The synchronization transition is studied in a lattice of chaotic maps with power-law coupling in one dimension. By increasing the coupling a transition from a spatio-temporal chaotic evolution to a fully (chaotic) synchronized state is observed. Recently, C. Anteneodo et al. [Phys. Rev. E 68 (2003) 045202R] have found theoretically (within a linear approach) the transition line in the corresponding parameter space. This synchronization line has been determined by the vanishing of the second Lyapunov exponent. We have shown that the linear analysis is insufficient to describe the synchronization transition when the nonlinear effects prevail on the linear ones. In particular, the nonlinear evolution becomes predominant for systems of Bernoulli Maps, while for systems of Logistic Maps the description in terms of the Lyapunov exponents fully captures all the aspects of the transition itself. Our results extend and confirm the results found by L. Baroni et al. [Phys. Rev. E 63 (2001) 036226] suggesting that the transverse Lyapunov exponent is not an appropriate indicator to characterize the synchronization transition in spatially extended systems whenever nonlinear mechanisms predominate on linear ones.

The Model

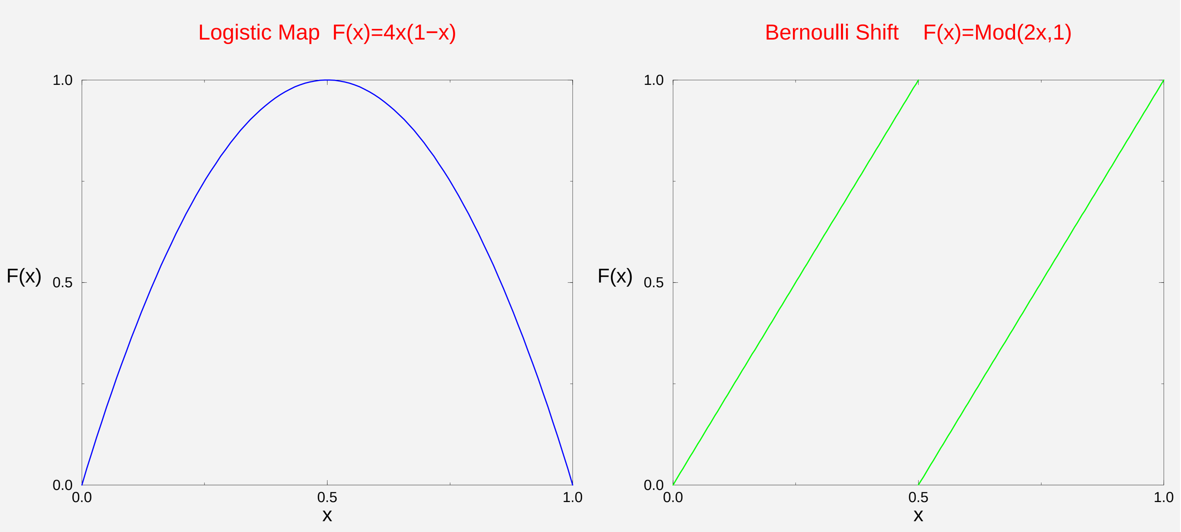
The following model, introduced in [1], is examined:

$$x_i(t+1) = (1-\varepsilon)F(x_i(t)) + \frac{\varepsilon}{\eta(\alpha)}\sum_{k=1}^{L'}\frac{F(x_{i-k}(t))+F(x_{i+k}(t))}{k^\alpha} \qquad (1)$$

where t and i are the discrete temporal and spatial indices, L is the lattice size ($i = 1,\dots,L$), $x_i(t)$ the state variable, $\varepsilon \in [0:1]$ measures the strength of the diffusive coupling, α the power-law decay of the coupling, and $\eta(\alpha) = 2\sum_{k=1}^{L'} k^{-\alpha}$, where $L' = (L-1)/2$ for odd L-values. Periodic boundary conditions and random initial conditions are assumed.

The model (1)

- in the limit $\alpha \to 0$ reduces to globally coupled maps (GCM's) [2];
- while for $\alpha \to \infty$ becomes the usual coupled map lattices (CML's) with nearest neighbour coupling [3].



$F(x)$ is a chaotic map of the interval ruling the local dynamics. As local map we have considered the continuous Logistic Map at the crisis:

$$F(x) = 4 * x * (1-x)$$

and the discontinuous Bernoulli Map

$$F(x) = Mod(rx,1) \qquad \text{with} \qquad r > 1 \quad .$$

Linear Analysis

The evolution in tangent space is obtained by linearizing Eq. (1), i.e.

$$\delta x_i(t+1) = (1-\varepsilon)F'_i\delta x_i(t) + \frac{\varepsilon}{\eta(\alpha)}\sum_{k=1}^{L'}\frac{F'_{i-k}\delta x_{i-k}(t)+F'_{i+k}\delta x_{i+k}(t)}{k^\alpha}$$

where F'_j is the first derivative of F estimated at site j and time t.

The Lyapunov spectrum for the fully syncronized case reads as [4]:

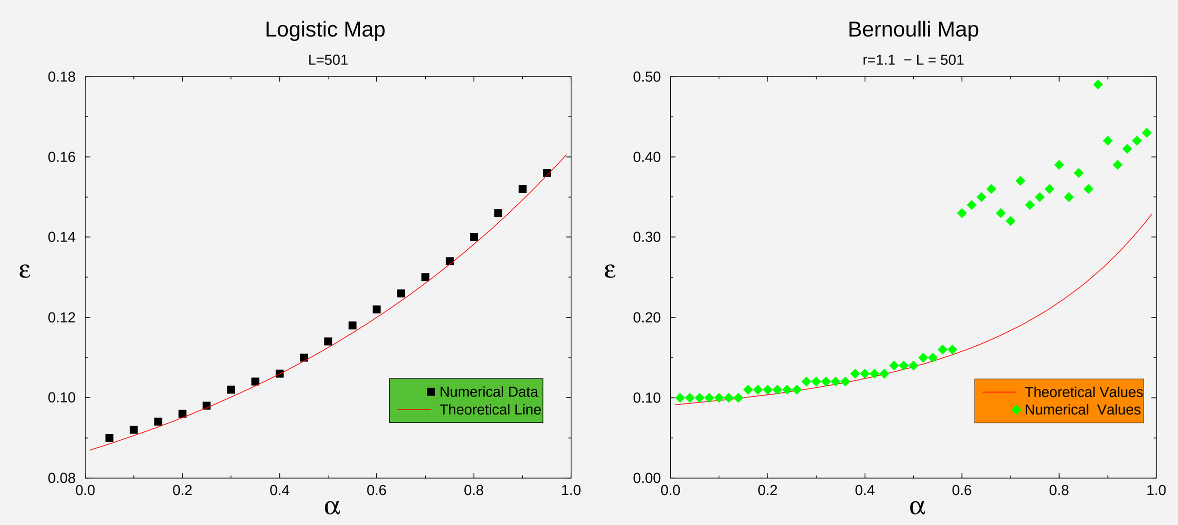
$$\lambda^{(k)} = \lambda^U + \left|1-\varepsilon+\frac{\varepsilon}{\eta(\alpha)}2\sum_{m=a}^{L'}\frac{\cos(2\pi(k-1)m/L)}{m^\alpha}\right| \qquad k = 1,\dots,L$$

where λ^U is the Lyapunov of the uncoupled map.

From a linear point of view the synchronization occurs when the transverse Lyapunov exponent $\lambda^{(2)}$ vanishes, this gives the following expression for the critical line in the (α,ε)-plane:

$$\varepsilon_c = \left(1-\mathrm{e}^{-\lambda_U}\right)\left(1-\frac{2}{\eta(\alpha)}\sum_{m=1}^{L'}\frac{\cos(2\pi m/L)}{m^\alpha}\right)^{-1}$$

in the limit $L \to \infty$ synchronization can be achieved only for $\alpha < 1$.



The disagreement between the linear estimate and the numerical data, observed for the Coupled Bernoulli Maps also depends on the slope r of the map, on the exponent α and on the chain length L.

Nonlinear Analysis

In order to characterize the synchronization transition the following indicator is introduced:

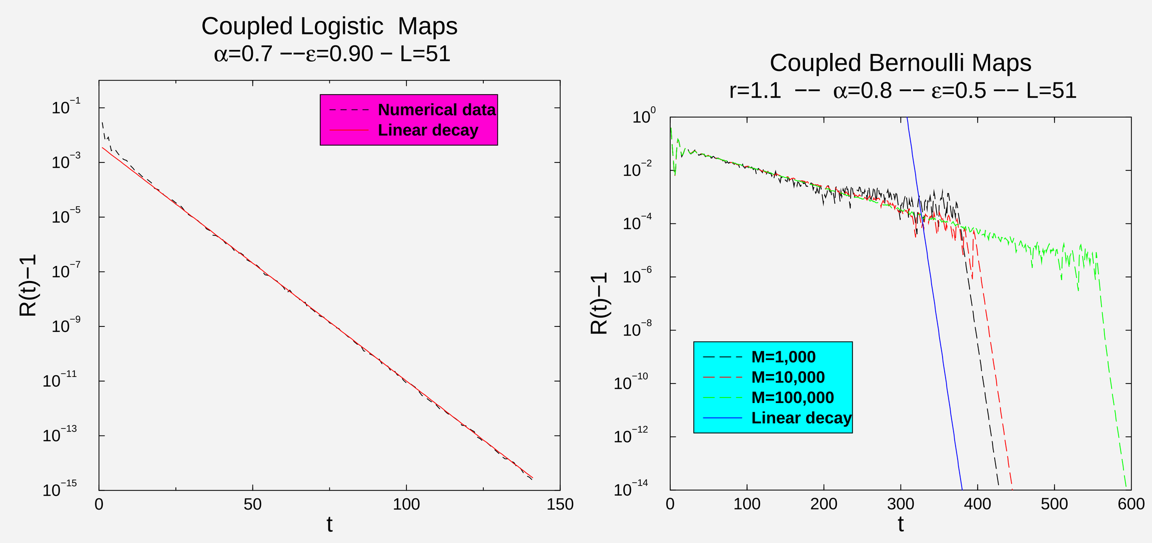
$$R(t) = \left|\frac{1}{L}\sum_{j=1}^{L}\mathrm{e}^{2\pi i x_j(t)}\right|$$

a time average $< R >$ of the indicator over a time window T_w , obtained after discarding an initial transient T_r, is typically considered.

- in the completely synchronized case: $< R > \equiv 1$;
- in the case of unsynchronized dynamics: $< R > \sim O(L^{-1/2})$.

For a synchronized state, it can be easily shown thah, within a linear approximation, $R(t)$ has the following time decay ruled by the transverse Lyapunov exponent $\lambda(2) < 0$:

$$R(t) \sim 1 + 2\pi^2(\delta_0^t)^2\mathrm{e}^{2\lambda^{(2)}t} \qquad .$$

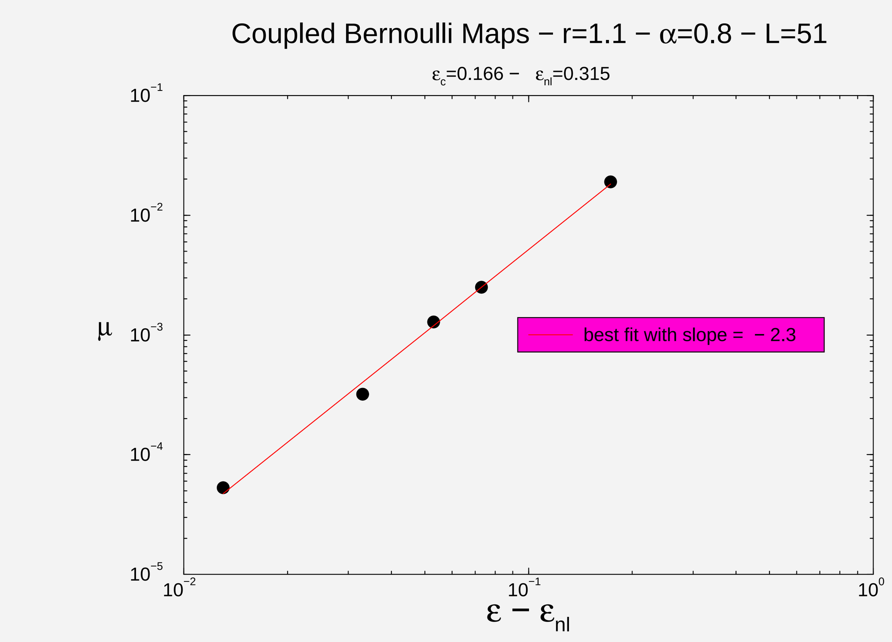


The linear expression is indeed correct for the Logistic map as shown in the Figure above, while for the Bernoulli map a transition from a nonlinear behaviour to a linear decay is clearly observable.

The nonlinear decay can be characterized in terms of the exponent μ:

$$R(t) \sim \mathrm{e}^{-\mu t} \qquad (2)$$

this nonlinear exponent vanishes at the nonlinear synchronization transition ε_{nl}, as shown in the following figure.



Main Results

- two different kinds of synchronization transitions have been observed : linear or nonlinear;
- when nonlinear effects prevail on linear ones the transverse Lyapunov exponent is not anymore able to characterize the transition as already shown for usual CML's in [5];
- a new nonlinear indicator μ has been introduced able to indicate the occurence of the nonlinear synchronization transition;
- in the case of the nonlinear transition the synchronization times diverge exponentially for $\varepsilon_c < \varepsilon < \varepsilon_{nl}$ and as a power law for $\varepsilon_{nl} < \varepsilon$.

Open Problems

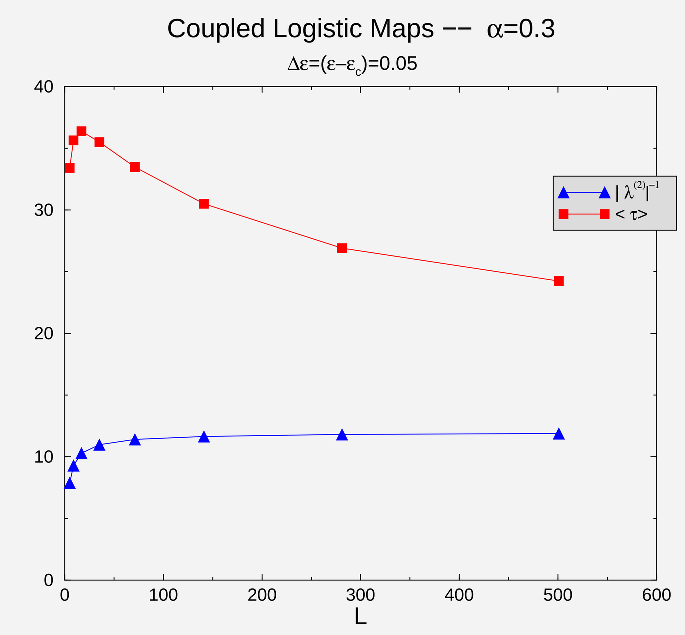
- Lack of a theoretical approach able to go beyond the linear analysis and therefore to predict the occurence of nonlinear transitions.
- Lack of experimental evidences of nonlinear synchronization transitions.

Synchronization Times

In order to characterize the different regimes associated to the synchronization transition the average synchronization times $< \tau >$ have been estimated. In particular, the system above the linear transition line is considered and the first passage time needed to the indicator $R(t) - 1$ to decrease below a given threshold (typically $\sim 10^{-6}$) is estimated by averaging over many ($\sim 100,000$) different initial conditions.

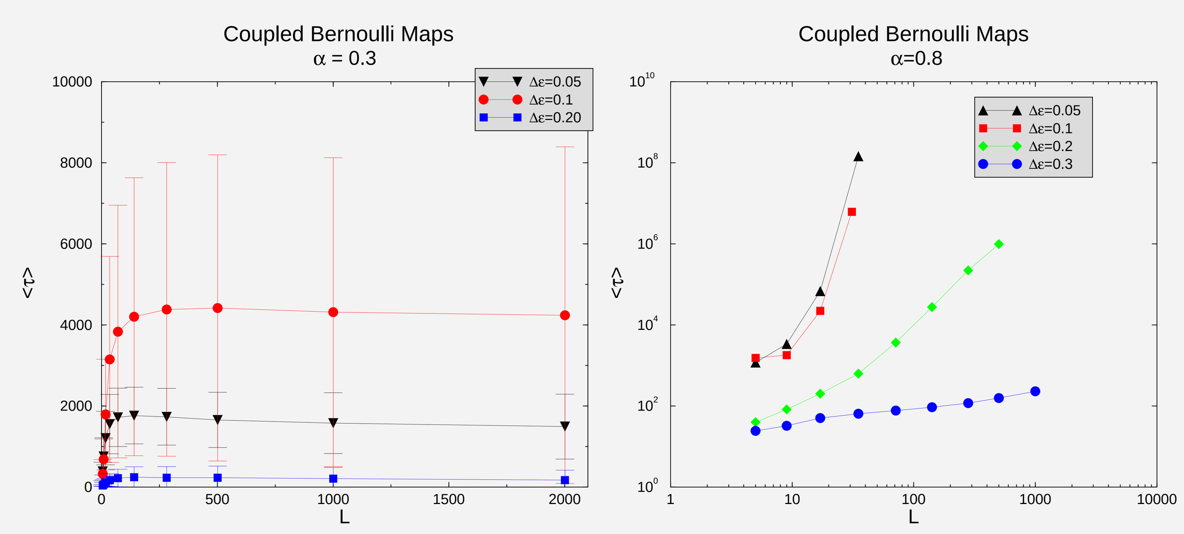
The dependence of $< \tau >$ on the length L of the examined system has been studied. Since the critical value ε_c will depend on L, the $< \tau >$ dependence on L is examined at a fixed distance $\Delta\varepsilon = \varepsilon - \varepsilon_c(L)$ from the linear transition line.

Coupled Logistic Maps



From the results reported in the above Figure we observe a saturation of $< \tau >$ for increasing L, and that the synchronization time is roughly $\sim 2/|\lambda^{(2)}|$ This suggests that the critical "linear" line is indeed the "true" synchronization line for this system.

Coupled Bernoulli Maps



- $\alpha = 0.3$ - a behaviour of $< \tau >$ vs L similar to the logistic map is observed: $< \tau >$ saturates;
- $\alpha = 0.8$ - in proximity of the linear transition line ε_c an exponential divergence is observed, while a power law divergence is found for $\varepsilon > \varepsilon_{nl}$. The power law exponent decreases for increasing $\Delta\varepsilon$.

A nonlinear synchronization transition is observed whenever nonlinear effects prevail on linear ones.

This happens for discontinuous (or almost discontinuous) coupled maps when the number of maps in the chain L is sufficiently high and for not too strong coupling (i.e. for $\alpha > \alpha_0(L)$).

References

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