Coherence Resonance

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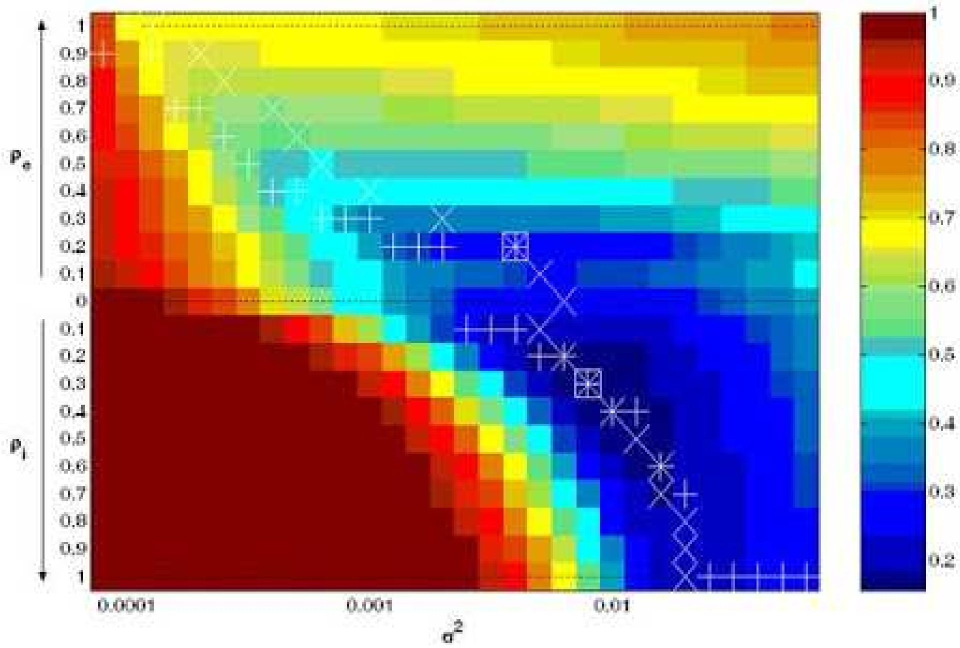
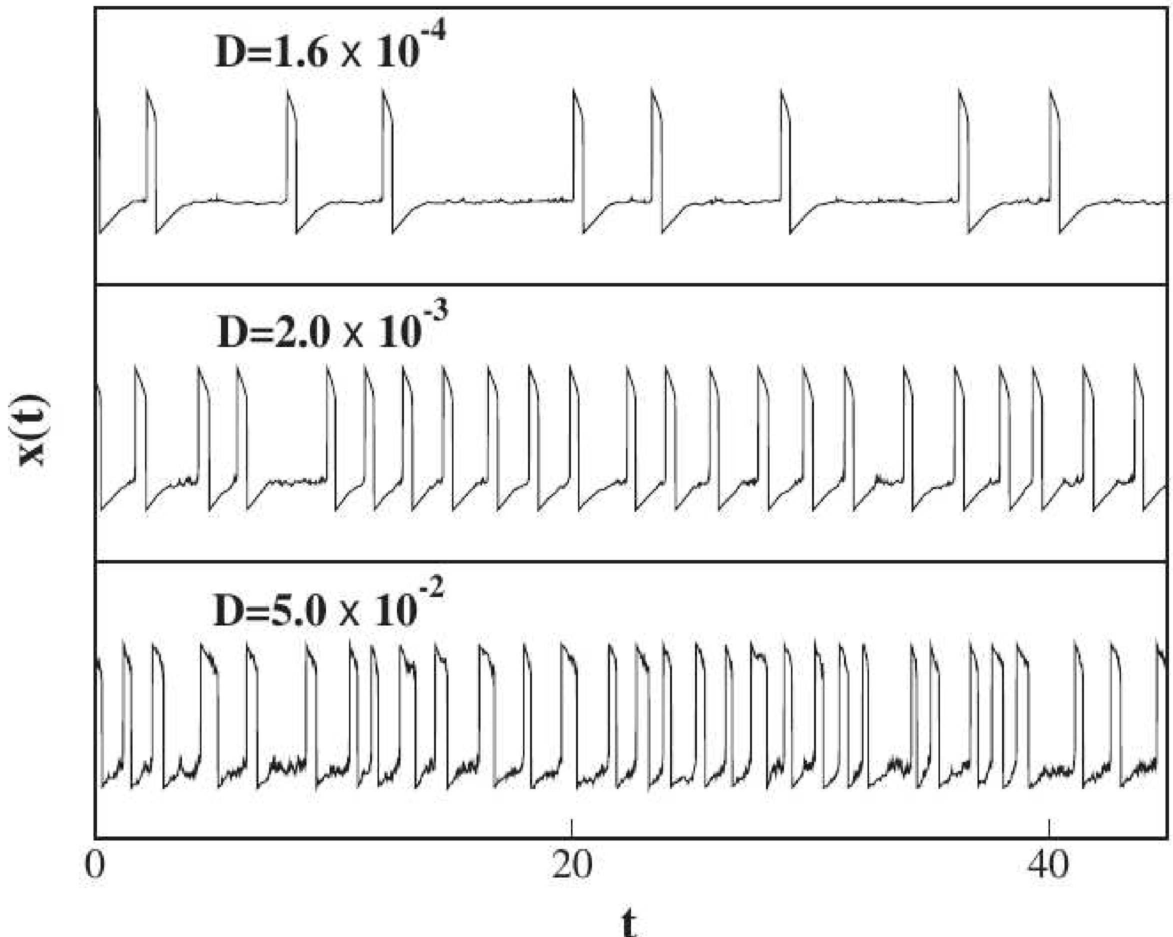


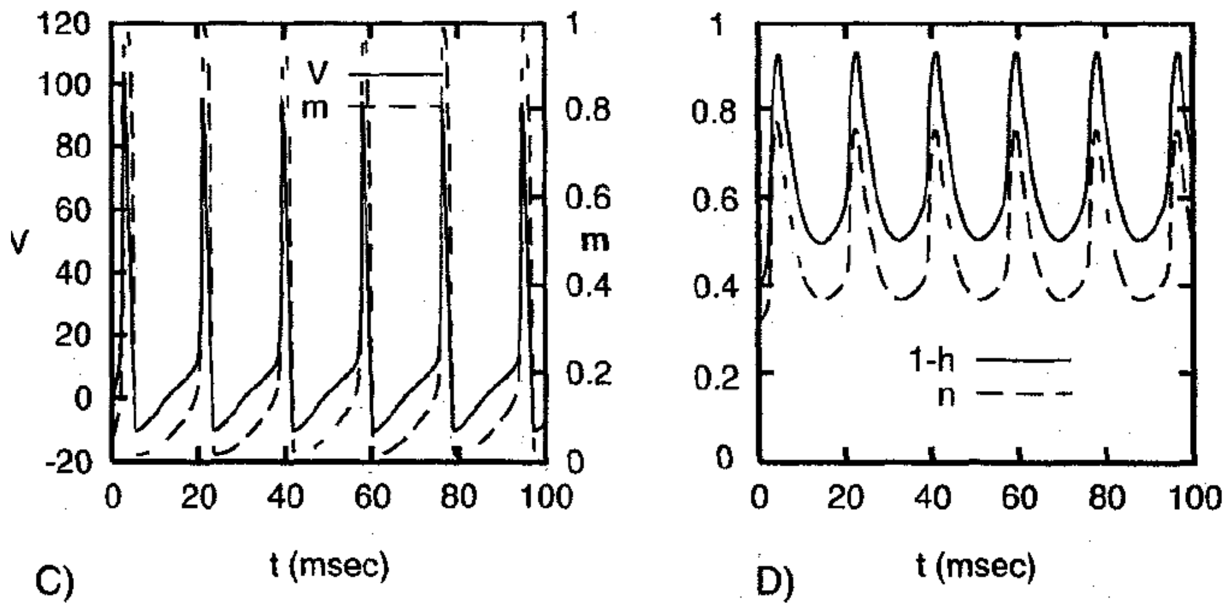
Introduction

The term coherence resonance (CR) refers to a phenomenon whereby addition of certain amount of noise in excitable system (neurons) makes its oscillatory responses most coherent. Thus a coherence measure of stochastic oscillations attains an extremum at optimal noise intensity, hence the word “resonance”.



A simplified model

The HH model is too complex may we simplify it ?



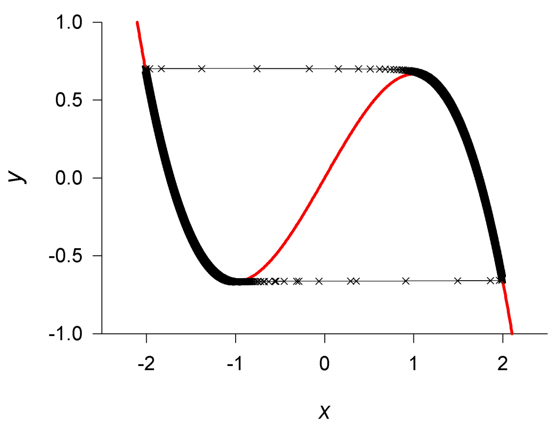
- the variables V and m evolve similarly on a time scale $\tau_m \simeq 0.4$ ms;
- n and $1-h$ are also evolving similarly on a slower time scale $\tau_n \simeq 5$ ms.

FHN model

FitHugh (1961) and Nagumo, Arimoto, Yoshizawa (1962) introduced a model for an excitable neuron with only two variables

$$\varepsilon \frac{dx}{dt} = x - \frac{x^3}{3} - y$$
$$\frac{dy}{dt} = x + a + D\xi(t)$$

- $\varepsilon = 0.01 \rightarrow x$ is fast – y is slow
- $\xi(t)$ is a white Gaussian noise with zero mean and correlation $< \xi(t)\xi(0) >= \delta(t)$
- D is the noise amplitude

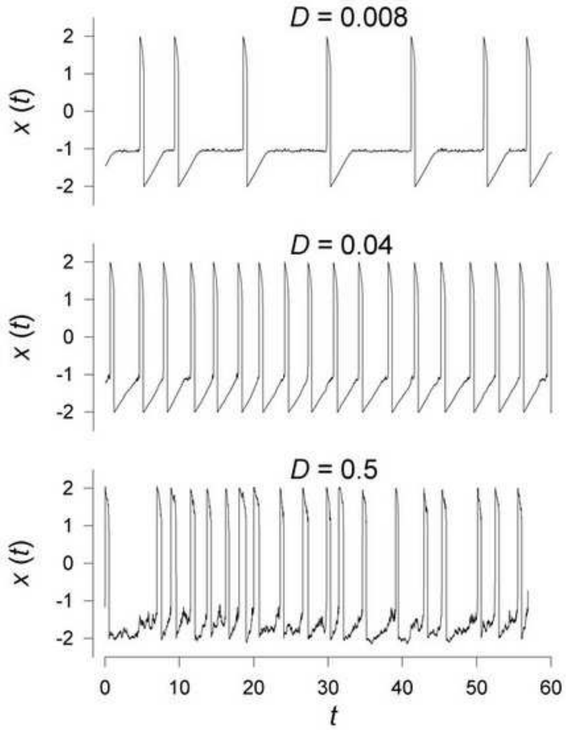


Equilibrium Point – $D = 0$

- $x_0 = -a \quad y_0 = a^3/3 - a$
- $a > 1$ Stable Fixed Point
- $a = 1$ Hopf Bifurcation $\rightarrow$ Oscillations
- Figure refers to $a = 0.99$

FHN model with noise

We fix $a = 1.05$ nearby the bifurcation, but where the deterministic system has a stable fixed point, then we increase the noise amplitude D



- At low noise amplitude we have few spikes
- At high noise we have irregular spiking
- We have an optimal value of the amplitude of the noise $D = 0.04$ where the train of spikes is regular: Coherence Resonance

Gaussian white noise

The random variables ξ are distributed as a Gaussian (Normal) distribution of zero average and standard deviation $\sigma = 1$

$$P_G(\xi) = \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{\xi^2}{2\sigma^2}}$$

The values $\{\xi\}$ at any time are identically distributed and statistically independent (and hence uncorrelated).

A Gaussian random generator in Python

numpy.random.normal(μ, σ,1000)

This generates 1000 random numbers ξ with average μ accordingly to a Gaussian distribution of standard deviation σ

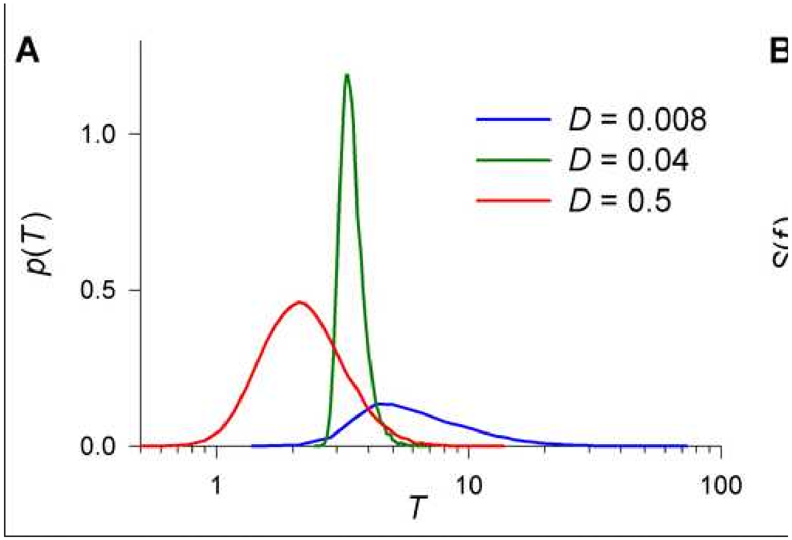
How to measure the CR ?

The ISI PDF

Usually one measures the spike times $\{t_i\}$ at which a spike is emitted, then one estimates the Inter-Spike Intervals (ISIs) between two spikes

$$T_i = t_{i+1} - t_i$$

and the corresponding probability distribution function (PDF) $p(T)$, where $p(T)dT$ is the probability to have a value T of the ISI n the interval $(T, T + dT)$.



How to measure the CR ?

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The coefficient of variation CV

To give a measure of the level of coherence in the model one usually uses the Coefficient of Variation (CV) :

$$CV = \frac{\sqrt{<T^2> - <T>^2}}{<T>}$$

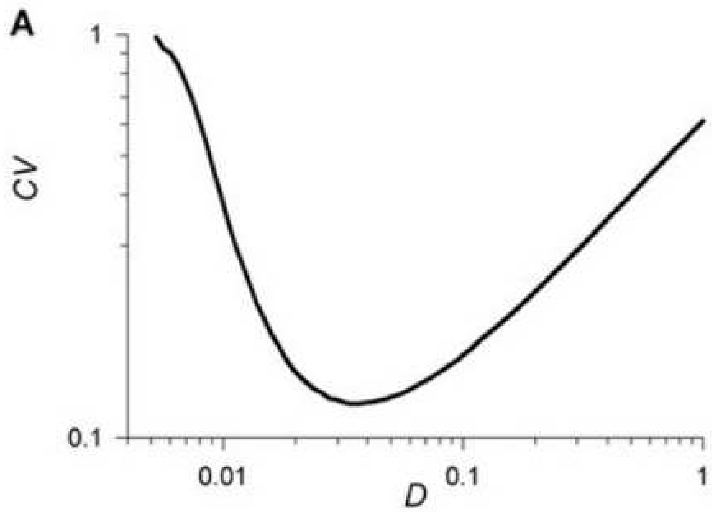
- $< \cdot >$ is an average over time
- $Var(T) = \sigma^2 =< T^2 > - < T >^2$ is the Variance of T
- CV is the ratio of the standard deviation σ over the average

- For a periodic signal $\sigma = 0$ therefore $CV = 0$
- For a random Poisson process $\sigma =< T >$ therefore $CV = 1$

$$0 \leq CV \leq 1$$

How to measure the CR ?

The Coherence Resonance corresponds to a minimum value of the CV for some finite value of the noise amplitude D



Poisson Process

It is the simplest random process with no memory of previous events and characterized by a single parameter : the rate of occurrence λ.

The probability density of a Poisson process to occur within a time interval t is given simply by

$$p(t) = \lambda e^{-\lambda t}$$

Therefore the probability P that one event occurs in the interval $\Delta t << 1$ is given by

$$P_1(\Delta T) = \Delta t \times p(\Delta t) = \lambda \Delta t e^{-\lambda \Delta t} \simeq \lambda \Delta t$$

and that it does not occur is

$$P_0(\Delta T) = 1 - P_1 \simeq 1 - \lambda \Delta t \simeq e^{-\lambda \Delta t}$$

Poissonian ISI PDF

- our neuron generates spikes following a Poisson process
- a spike has been emitted at time t_i
- the probability to have no spike in the time interval $(t_i, t_i + T)$ is $\boxed{P_0(T) = \mathrm{e}^{-\lambda T}}$
- Therefore the probability that the next spike is emitted at $t_{i+1} = t_i + T$ is $\boxed{P_1(T) = 1 - \mathrm{e}^{-\lambda T}}$

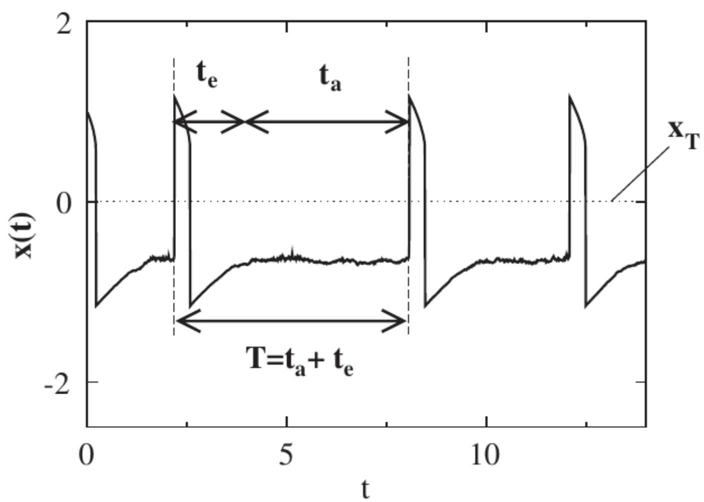
The probability density function is the derivative of P_1 then the Poissonian ISI PDF is

$$p(T) = \lambda \mathrm{e}^{-\lambda T}$$

It is easy to demonstrate that

- the average ISI is $\boxed{< T >= \lambda^{-1}}$
- the variance $\boxed{Var(T) = \sigma^2 =< T^2 > - < T >^2 = \lambda^{-2}}$
- the coefficient of variation is $\boxed{CV = \frac{\sigma}{<T>} = 1}$

The physical mechanism



Each ISI $T = t_a + t_e$ can be divided in two time sub-intervals:

- the activation time t_a is the time needed to escape from the stable fixed point due to noise fluctuations;
- the excursion time t_e is the time to return to the fixed point, the duration of the spike itself.

The physical mechanism

- t_a follows a Poissonian statistics and it strongly reduces by increasing the noise amplitude D ;
- t_e is a relaxation time and therefore it depends much weaker on the noise intensity D.

Let us write

$$CV^2 = \frac{Var(T \equiv t_a + t_e)}{< T >^2} = \frac{Var(t_a)}{< t_a >^2}\frac{< t_a >^2}{< T >^2} + \frac{Var(t_e)}{< t_e >^2}\frac{< t_e >^2}{< T >^2}$$

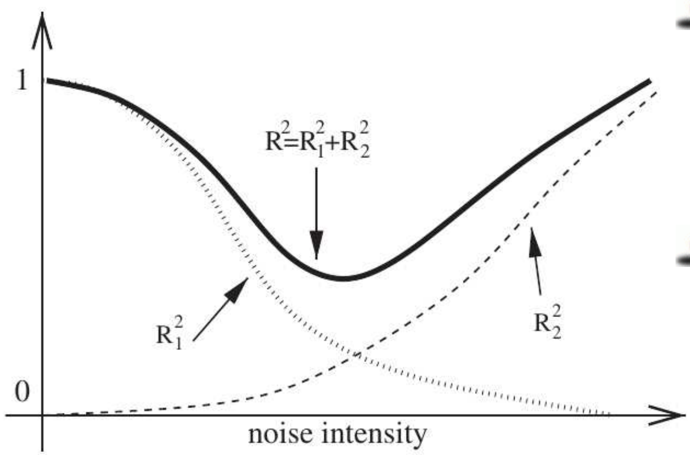
where we have used the fact that the two processes (activation and excursion) are independent therefore $Var(T \equiv t_a + t_e) = Var(t_a) + Var(t_e)$.
Then we have

$$CV^2 = CV_a^2 \left(\frac{< t_a >}{< T >}\right)^2 + CV_e^2 \left(\frac{< t_e >}{< T >}\right)^2$$

where $CV_a = 1$ is the coefficient of varaton of the activation process and CV_e of the excursion time.

The physical mechanism

$$CV^2 = \left(\frac{<t_a>}{<T>}\right)^2 + CV_e^2\left(\frac{<t_e>}{<T>}\right)^2 = R_1^2 + R_2^2$$



- at low noise intensity D
  - $T \simeq t_a$ therefore $R_1^2 \simeq 1$
  - $CV_e^2 \simeq 0$ therefore $R_2^2 \simeq 0$
- at high noise D
  - t_a decreases towards zero $\rightarrow R_1^2 \rightarrow 0$
  - $T \rightarrow t_e$ and fluctuations become important for the excursion time therefore R_2^2 increases with D

The minimum in the CV appears at a noise intensity where both terms are just small, i.e. when the noise strength suffices to generate a small t_a compared to t_e, while it is still weak enough to introduce only a small jitter in the excursion time.
In this case, the noise induced oscillations are mainly determined by the rather regular excursion time and may thus look quite regular.

References

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