Collective Synchronization

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Foreword and Summary

Populations of biological oscillators can spontaneously synchronize to a common frequency, despite a distribution of natural frequencies among the population

- swarms of fireflies flash in synchrony;
- crickets chirp in unison;
- groups of women whose menstrual cycles synchronize.

Summary

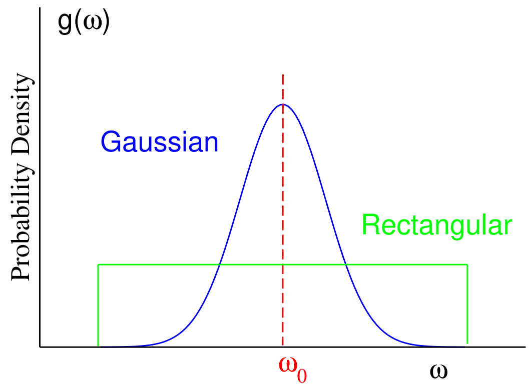
- The Kuramoto Model
- The Millennium Bridge
- Collective Behaviour of Limit-Cycle Oscillators

Kuramoto Model

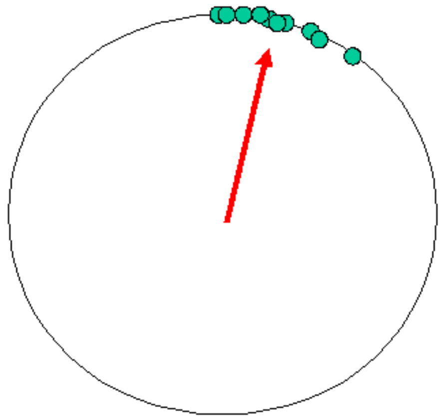
- N coupled phase oscillators with different frequencies ω_k
- Frequencies distributed according to $g(\omega)$

$$\frac{d\varphi_k}{dt} = \omega_k + \frac{k}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k)$$

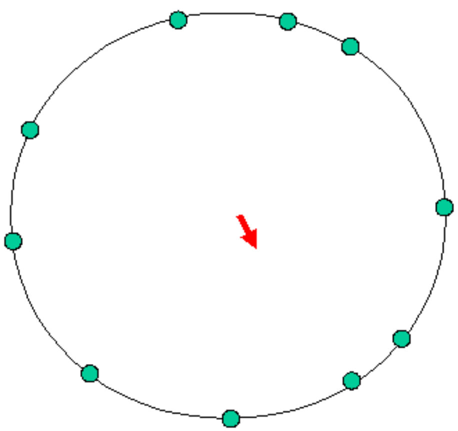
- The coupling is rescaled by N to avoid divergence of the forcing term in the thermodynamic limit ($N \to \infty$)



Order Parameter



$r \approx 1$



$r \approx 0$

Mean Field Variable Amplitude R and Phase Θ

$$Z = R\cos\Theta + iR\sin\Theta = Re^{i\Theta} = \frac{1}{N}\sum_{k=1}^{N} e^{i\varphi_k}$$

$$R\cos\Theta = \frac{1}{N}\sum_{k=1}^{N}\cos(\varphi_k) \qquad R\sin\Theta = \frac{1}{N}\sum_{k=1}^{N}\sin(\varphi_k)$$

Z is a Coherence Indicator

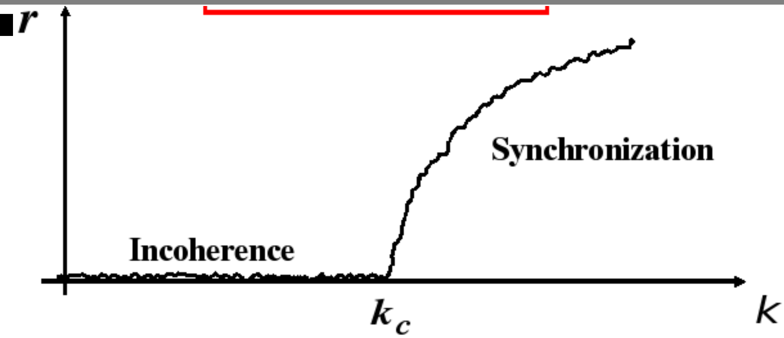
- if $\varphi_k = \varphi_j \quad \forall j, k$ System Fully Synchronized $R \equiv 1$
- if φ_k are equally distributed on the circle
  Desynchronized System $R \simeq \frac{1}{\sqrt{N}}$
- If some oscillator are frequency locked $R \neq 0$

The model can be rewritten as

$$\frac{d\varphi_k}{dt} = \omega_k + kR\sin(\Theta - \varphi_k)$$

each oscillator is forced by the Self-Consistent Mean Field

Synchronization Transition



By increasing the coupling k a coherent behaviour emerges in the system characterized by a non zero order parameter R

This effect can be explained in terms of self-consistency: a non zero R forces some oscillator to synchronize, in turn these oscillators form a coherent group which generate a finite R

To estimate k_c analytically, we make the following hypothesis:

- the frequency distribution is peaked around ω_0
- $g(\omega)$ is an even function $g(\omega - \omega_0) = g(-\omega + \omega_0)$

On average the mean field will oscillate with the peak frequency and the amplitude will be almost constant, therefore we can assume

$$\Theta = \omega_0 t \qquad R = const.$$

Synchronization Transition

By setting $\psi_j = \varphi_j - \omega_0 t$ and $\Omega_j = \omega_j - \omega_0$, the equation for each oscillator is again the Adler equation

$$\frac{d\psi_j}{dt} = \Omega_j - kR\sin(\psi_j)$$

- The synchronized state, where φ_j is locked to Θ corresponds to $\frac{d\psi_j}{dt} = 0$ therefore

$$\sin\psi_j = \frac{\Omega_j}{kR} \qquad \text{for} \qquad \Omega_j < kR$$

- The asynchronous regime is found for $\Omega_j > kR$

Now we impose the self-consistency by estimating the order parameter in the synchronized phase found before

Now a few integrals . . .

Synchronization Transition

$$\sin\psi = \frac{\Omega}{kR} \qquad d\Omega = kR\cos\psi d\psi$$

$$R = \left|\frac{1}{N}\sum_{k=1}^{N} \mathrm{e}^{i\psi_k}\right| = \left|\int_{-\infty}^{+\infty} d\Omega \;\; g(\Omega) \;\; \mathrm{e}^{i\psi(\Omega)}\right| = \left|\int_{-\infty}^{+\infty} d\Omega \;\; g(\Omega) \;\; [\cos(\psi(\Omega)) + i\sin(\psi(\Omega))]\right|$$

Since $G(\Omega)$ is even we remain with the real part only, since g is peaked we can limit to consider the second order Taylor expansion around ω_0, then

$$R = \int d\Omega \;\; [g(\omega_0) - \frac{g^{"}(\omega_0)\Omega^2}{2}] \;\; \cos(\psi(\Omega)) \qquad d\Omega = kR\cos\psi d\psi$$

$$1 = k\int_{-\pi/2}^{\pi/2} d\psi \;\; \cos^2\psi[g(\omega_0) - \frac{g^{"}(\omega_0)k^2R^2}{2}\sin^2\psi] \;\; = \frac{k\pi}{2}\left[g(\omega_0) - \frac{g^{"}(\omega_0)k^2R^2}{8}\right]$$

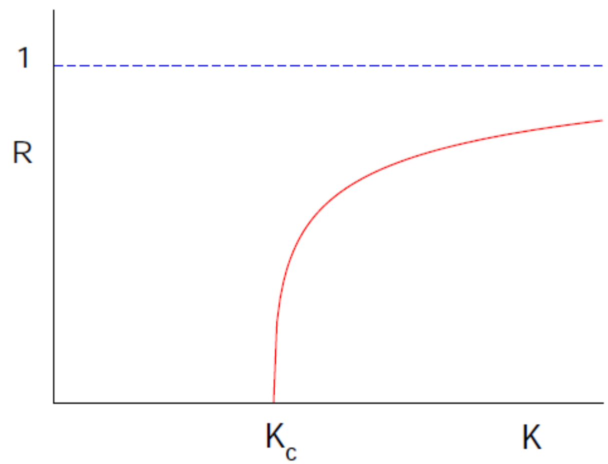
Synchronization Transition

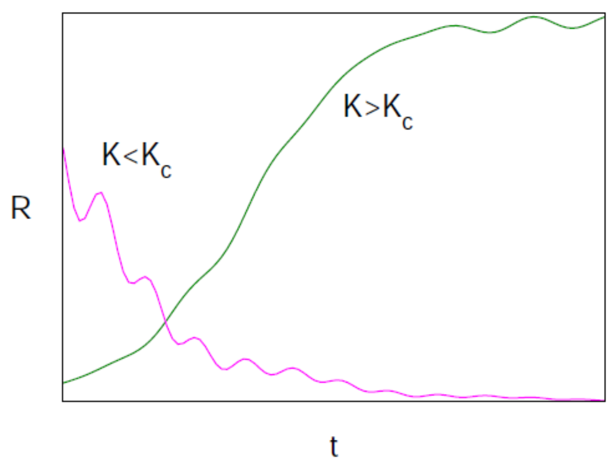
Therefore

$$R = \sqrt{\frac{8[k\pi g(\omega_0) - 2]}{g''(\omega_0) k^3}} \qquad k_c = \frac{2}{\pi g(\omega_0)}$$

We observe a continuous transition from the incoherent to the coherent regime at k_c, In proximity of the transition the order parameter increases as

$$R \propto \sqrt{k - k_c} \qquad \beta = 1/2$$





Kuramoto Summary

The Kuramoto model for a population of fully coupled oscillators with different natural frequency exhibit three different collective behaviours :

- Incoherence: all the oscillators run at their natural frequencies ($k < k_c$ and $R \sim 0$)
- Partial Locking: some of the oscillators are locked, while the others drifts at different frequencies ($k > k_c$ and $R \neq 0$)
- Complete Locking : all the oscillators are locked , the phase difference between any two oscillators is constant in time ($k >> k_c$ and $R \sim 1$)

All these states are characterized by a constant R value, no time evolution for R, for sufficiently large number of oscillators N (eventually $N \to \infty$)

The single oscillator has 1 degree of freedom, even with external forcing cannot display chaotic behaviour (at least 2 degrees of freedom + forcing)

Millennium Bridge

Crowd Synchrony on the Millennium Bridge

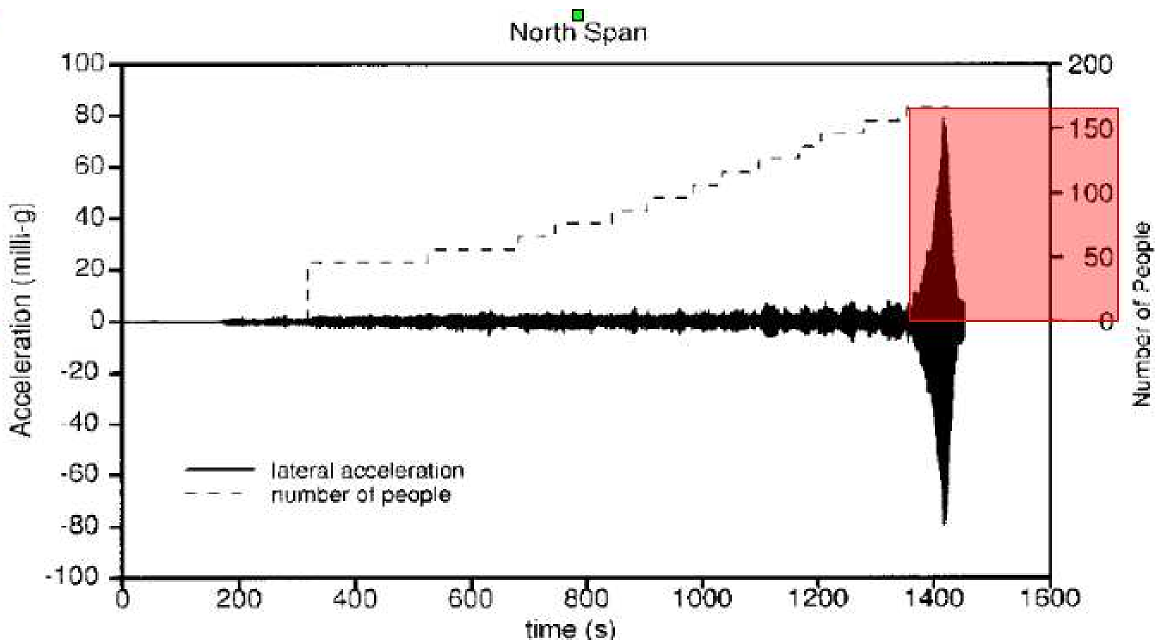
Strogatz, Abrams, Mc Robie, Eckhardt, Ott Nature, 438 (2005) 43

When the London Millennium Bridge opened on June 10, 2000, soon after the crowd streamed on the bridge, the bridge begins to oscillate from side to side (to wobble): many pedestrians synchronize spontaneously their steps with the bridge's vibrations, amplyfying them. The synchronized steps of the people caused such heavy oscillations that the bridge had to close down until dampers were put in 2 years later.

Collective synchronization was responsible for the wobbling of the bridge

Millennium Bridge

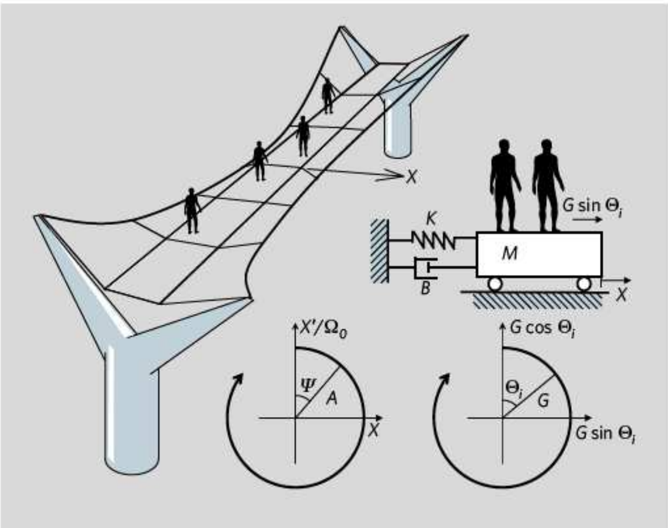
The experiment by Arup



Groups of people of increasing number walk together along the bridge until it begins to wobble, there is a critical number of people

Millennium Bridge

The Model



- The left/rigth walking cycle of each pedestrian is seen as an oscillator θ_i with his own frequency ω_i forced by the bridge oscillations;
- The lateral motion X of the bridge is schematized as a weakly damped harmonic oscillator driven by the collective motion of the pedestrians.

$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N} \sin\theta_i \quad \text{where} \quad X = A\sin\psi$$

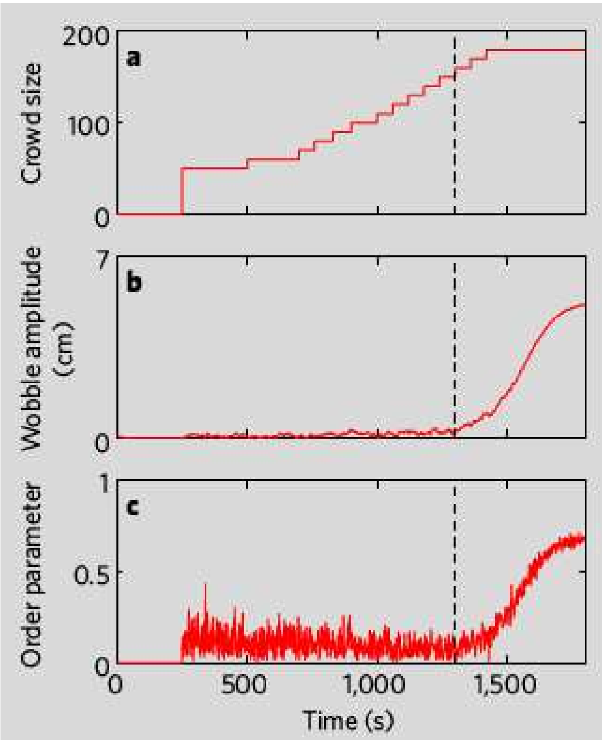
where M, B and K are the mass, the damping and the stifness associated to the lateral motion of the bridge and G is the maximal force exterted by a pedestrian

$$\frac{d\theta_i}{dt} = \omega_i + CA\sin(\psi - \theta_i + \alpha)$$

where C is the sensitivity of the pedestrian to bridge vibration, to be fitted

Millennium Bridge

The Simulation Result

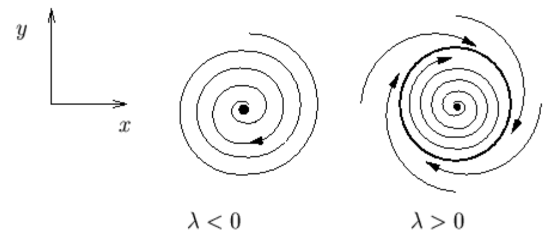


- The simulations have been performed by employing realistic values for the parameters, apart C which has been fitted to the experimental data by Arup
- The simulation start with bridge at rest and $N = 50$ pedestrians on the bridge
- The number of pedestians is increased by 10 at each step

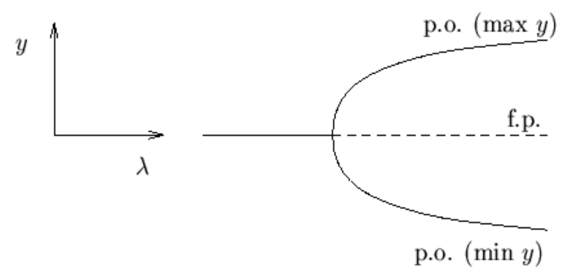
Bernard Feldman, a writer for Physics Today, however, believes Strogatz is wrong: since the frequency of the lateral oscillation of bridges is around 0.5 Hz whereas the average frequency of walking is 1.0 Hz (2 steps per second). Therefore it is unlikely that synchronized footsteps could have intensified the wobbling of the Millennium Bridge.

Limit-Cycle Oscillator

Phase Space:



Bifurcation Diagram:



In proximity of a Hopf bifurcation from a fixed point solution to a periodic limit-cycle oscillation all the non-linear dynamical systems in 2d can be rewritten in a general Normal Form

$$\frac{dz}{dt} = i\omega_0 z + z(1 - |z|^2) \qquad z = x + iy$$

This is a weakly nonlinear oscillator characterized by an amplitude and a phase $z(t) = A(t)\mathrm{e}^{i\theta}$ where the oscillator frequency is independent from the amplitude (Isochronous Oscillator)

The Stuart-Landau equation with an external forcing can become chaotic at variance with the phase oscillator, therefore we expect a much richer behaviour for a population of these oscillators

Population of amplitude oscillators

A model of linearly coupled amplitude oscillators is the following

$$\frac{dz_j}{dt} = i\omega_j z_j + z_j(1 - |z_j|^2) + \frac{K}{N}\sum_{i=1}^{N}(z_i - z_j) = i\omega_j z_j + z_j(1 - |z_j|^2) + K(\bar{z} - z_j)$$

where K is the coupling strength and

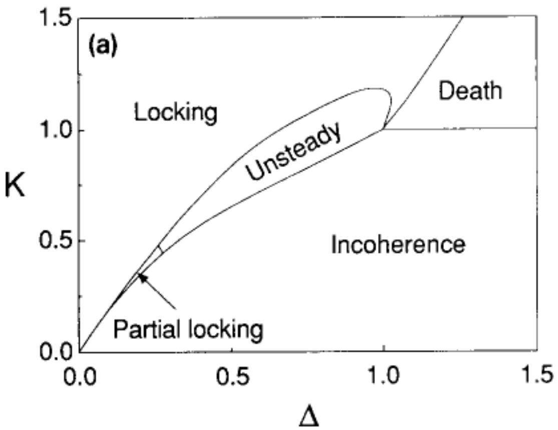
$$\bar{z} = Re^{i\theta} = \frac{1}{N}\sum_{i=1}^{N} z_i$$

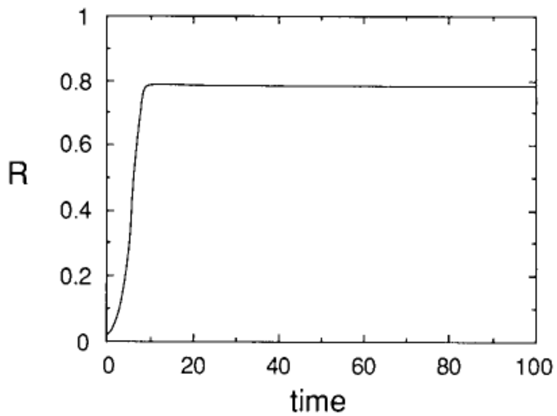
is a macroscopic order parameter (mean field variable) which gives a measure of the degree of synchronization within the system.

The frequencies ω_j are randomly chosen from a distribution $g(\omega)$ of zero mean and width Δ

PC Matthews and SH Strogatz, Physical Review Letters 65 (1990) 1701

Steady Collective Behaviours

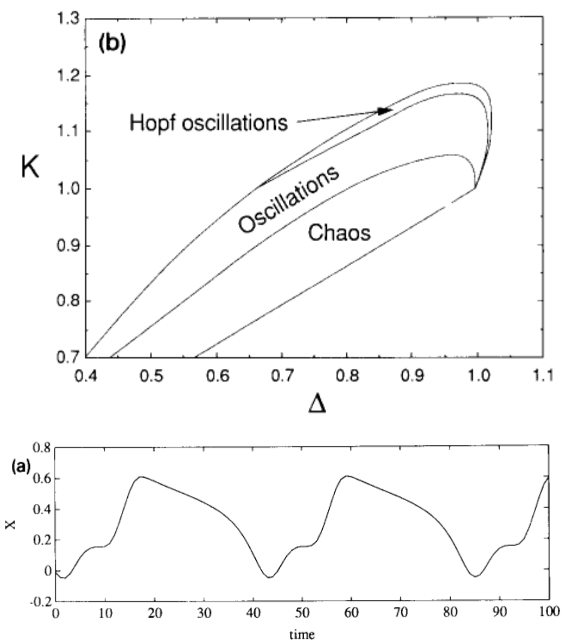




Steady Collective Solutions

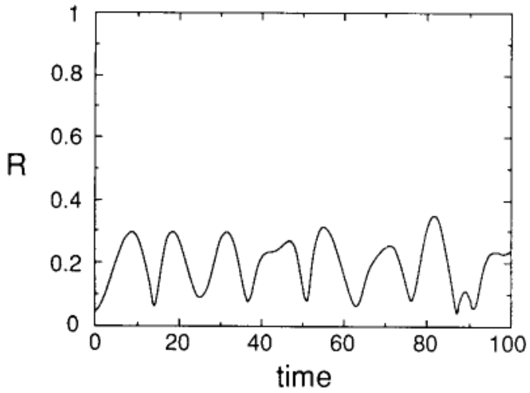
- Amplitude Death: for $K > 1$ and Δ large $z_j = 0$ is a stable fixed point
- Locking: each oscillator moves along a circle of radius $\sqrt{1-K}$ with the same frequency – $R \neq 0$
- Incoherence: each oscillator moves along a circle of radiun $\sqrt{1-K}$ with his own frequency – $R = 0$ in the thermodynamic limit

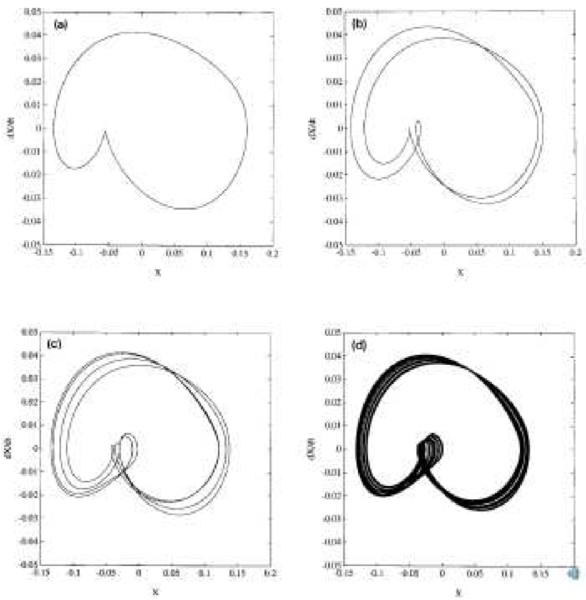
Unsteady Macroscopic Behaviours



- Hopf oscillations: for $K > 1$ the locked state loses stability via a hop bifurcation giving rise to small oscillations around the locked state – $R(t)$ is periodic
- Large oscillations: For $K < 1$ the locked state undergoes a saddle-node bifurcation to large oscillations – $R(t)$ is periodic
- Quasiperiodicity: the large oscillations lose stability via successive Hopf bifurcations adding a second and a third frequency – $R(t)$ moves on Torus T^2 and T^3
- Chaos: the single oscillators as well as $R(t)$ behaves in an erratic manner. Moreover there is a SIC for the collective dynamics and a broadband spectrum

Collective Chaos





Routes to Chaos

Two different Route to chaos are observable for the macroscopic dynamics

- Period Doubling: for fixed $K = 1.05$ by increasing Δ the period of the oscillations show a period-doubling cascade ending up in a low dimensional chaotic attractor
- Ruelle-Takens route to chaos: for fixed $K = 0.8$ by increasing Δ the oscillatory solution undergoes hopf bifurcations to a T^2 and then to a T^3 Torus and then to chaos

Collective Chaos has been observed also for identical non isochronous oscillators

Hakim & Rappel (1992) Nakagawa & Kuramoto (1993)

References

Texts employed for the lectures

- Chemical Oscillations, Waves, and Turbulence
  Y Kuramoto
- Synchronization: a universal concept in nonlinear sciences
  A. Pikovsky, M. Rosenblum, J. Kurths (Cambridge University Press, 2003)
- The Kuramoto model: a simple paradigm for synchronization phenomena
  JA Acebron, LL Bonilla, CJ Peréz Vicente, F Ritort, R Spigler Reviews of Modern physics 77 (2005) p. 137

Nice articles:

- From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators
  SH Strogatz, Physica D 143 (2000) 1-20