Synchronization

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Foreword

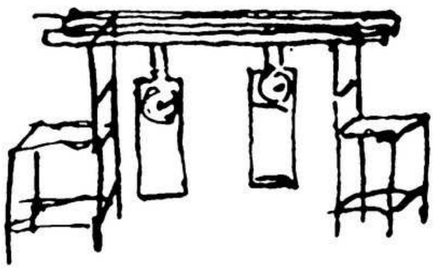
Everyone likes to synchronize

Tout le monde aime synchroniser

- Les Lucioles
- Danseurs et Danseuses
- et aussi . . . les oscillateurs

Synchronization of Two Clocks

Christiaan Huygens reported the first observation of synchronization:



"... It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."

Antiphase Synchronization

This problem is still nowdays studied:

Bennett, Schatz, Rockwood, Wiesenfeld,

"Huygens Clocks", Proc. R. Soc. Lond. A, vol. 458 (2002), pp. 563 - 579.

Summary

Periodic Oscillators

- Synchronization by external force
- Mutual Synchronization
- Synchronization by common noise
- Synchronization of ensembles of coupled oscillators

Chaotic Oscillators

- Complete Synchronization
- Phase Synchronization
- Generalized Synchronization

Phase Model I

Many physical, chemical, biological systems exhibit Rhytmic Oscillations

- A.T Winfree, The geometry of biological time (2001)
- G. Buzsaki, Rhythms of the Brain (2006)

A dissipative autonomous dynamical system exhibiting a stable periodic orbit γ is called an Oscillator, $\frac{dx}{dt} = f(x)$ we will study synchronization properties of these systems.

The motion along the orbit γ can be characterized by the time t from the last crossing t_n of a certain point x_0 on the orbit, and a phase can be introduced as

$$\theta = \frac{t - t_n}{t_{n+1} - t_n} 2\pi \qquad 0 \leq \theta \leq 2\pi$$

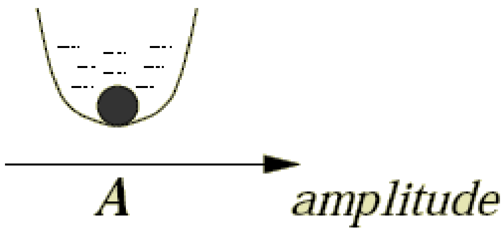
the dynamics on the orbit can now be rewritten simply as

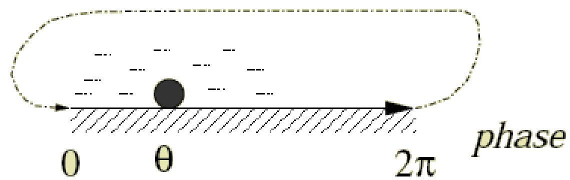
$$\dot{\theta} = \omega_0$$

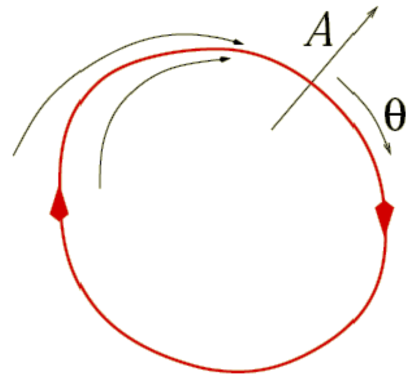
where ω_0 is the natural frequency of the oscillation.
Information on the amplitude oscillation are lost, but not on its phase.

Phase Model II

- $\dot{\theta} = \omega_0$ $\lambda = 0$ (Phase is Marginally Stable)
- $\dot{A} = -\eta(A - A_0)$ $\lambda = -\eta$ (Amplitude is Stable)







- Since the amplitude is stable it is difficult to modify it with small perturbations
- The phase is at the edge between stability and instability small perturbations (due to external forcing or coupling) can induce large modifications of the phase

Thus with a small forcing it is possible to adjust the phase and the frequency of the oscillations, without altering the amplitude:

this is the essence of the synchronization phenomenon

Synchronization by external forcing

Examples:

- Radio controlled clocks: low quality clocks can become precise due to adjustment by a periodic radio signal
- Cardiac pacemakers: heart beats are made regular by a sequence of pulses from an electronic generator

Kuramoto Approach (1984)

If an oscillator θ is forced by a second one ψ, for a small forcing term (e.g $\varepsilon \sin \omega t$) only the phase is affected:

$$\frac{d\theta}{dt} = \omega_0 + \varepsilon Q(\theta, \psi) \qquad \frac{d\psi}{dt} = \omega$$

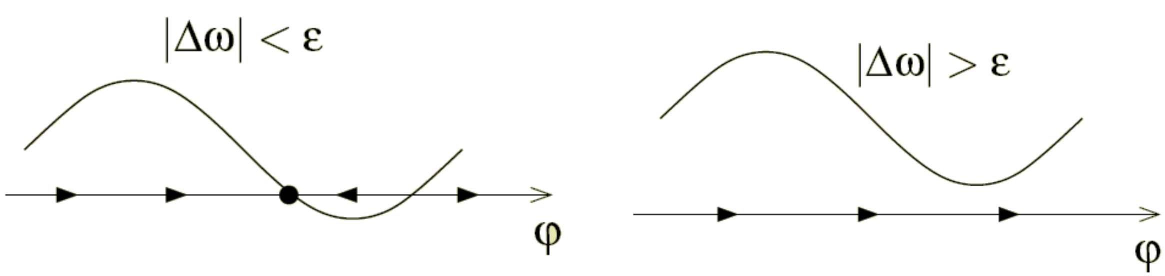
If $\omega \sim \omega_0$, then the phase difference $\varphi(t) = \theta(t) - \psi(t)$ evolves slowly in time, therefore by keeping only slow terms

$$\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin(\varphi) \qquad \Delta\omega = \omega_0 - \omega$$

one gets the Adler equation

Adler equation

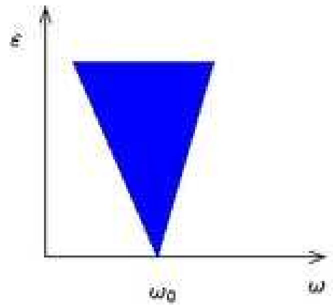
$$\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin(\varphi)$$

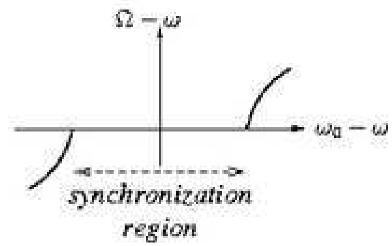


- $\Delta\omega < \varepsilon$ – stable fixed point – Synchronization Region or Arnold Tongue
  - Frequency entrainment $\Omega =< \dot{\theta} >= \omega$
  - Phase Locking $\varphi = \theta - \psi = const.$
- $\Delta\omega > \varepsilon$ – periodic orbit
  - Asynchronous quasi-periodic motion - 2 frequencies - Torus T^2

Arnold Tongues

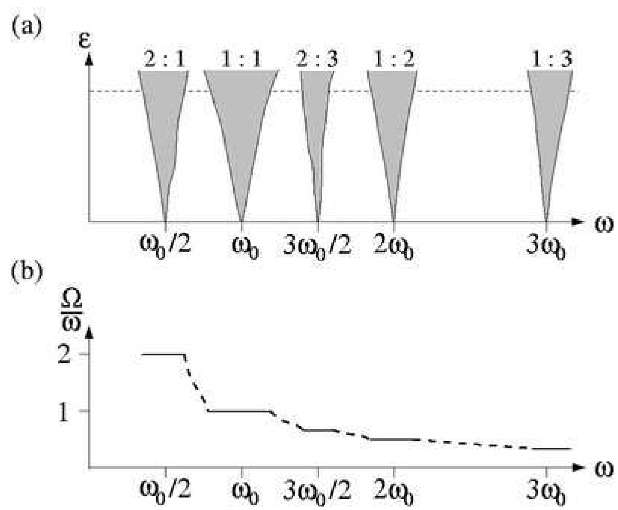
1:1 Synchronization





- Frequency entraînment
  $\Omega =< \dot{\theta} >= \omega$
- Phase Locking $\varphi = \theta - \psi = const.$

Higher order locking



- Frequency entraînment $\Omega =< \dot{\theta} >= \frac{n}{m}\omega$
- Phase Locking
  $m\theta = n\psi + const. = n\omega t + const$
- Devil staircase – At fixed ε: Horizontal plateaus at all possible rational frequency ratios, in between quasi-periodic motions.

The picture is preserved for moderate forcing, at strong forcing chaos can emerge.

Mutual Synchronization

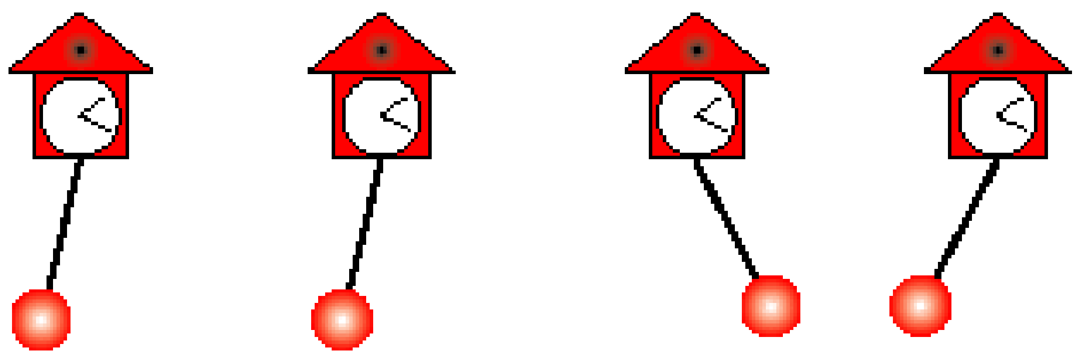
Let us consider 2 coupled oscillators, if the interaction is weak we can write

$$\frac{d\theta_1}{dt} = \omega_1 + \varepsilon Q_1(\theta_1, \theta_2) \quad ; \quad \frac{d\theta_2}{dt} = \omega_2 + \varepsilon Q_2(\theta_1, \theta_2)$$

For $\omega_1 \sim \omega_2$, the phase difference $\phi = \phi_1 - \phi_2$ is evolving slowly, by averaging and taking only slow terms one obtains once more The Adler equation

$$\frac{d\varphi}{dt} = \Delta\omega + \varepsilon \sin(\varphi)$$

The coupling among the oscillators can now be attractive or repulsive : In-phase or Anti-phase Synchronization



For strong ε one can observe oscillation death, this is due to the dissipative effect of coupling that tend to equalize the two states

Synchronization by Noise

$$\frac{d\theta_1}{dt} = \omega_1 + \varepsilon\eta(t) \quad ; \quad \frac{d\theta_2}{dt} = \omega_2 + \varepsilon\eta(t) \qquad <\eta(t)>= 0 \qquad <\eta(t)\eta(0)>= \delta(t)$$

For quite strong noise two oscillators subjected to the same noise can synchronize, since the noise induced phase dynamics is stable ($\lambda < 0$) and both phases follow the same pattern induced by noise.

Neuron reliability

Neurons can be considered as oscillators, their synchronization by common noise can be interpreted as a reliable response of the neuron to the presentation of the same noisy signal

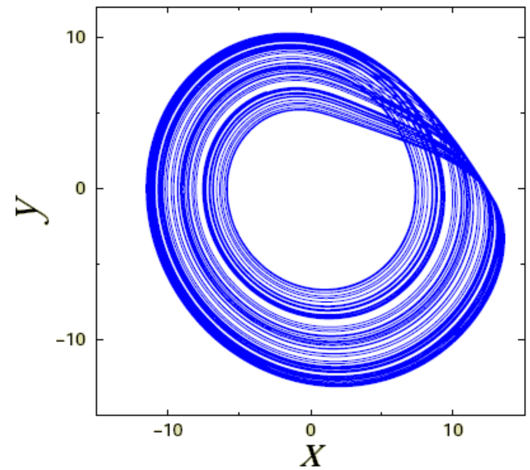
FIG. 4. Spike time reliability in *Aplysia* motoneuron with aperiodic inputs. Superposed voltage traces from 10 different trials recorded from a buccal motoneuron for 4 different input signals. *A*: broadband aperiodic input; *B*: band-stop input lacking frequencies around f_{DC}; *C*: band-stop control input lacking frequencies $\sim 0.55\ f_{DC}$ but containing frequen-

Hunter et al., J Neurophysiol 80 (3): 1427 (1998)

Phase of a chaotic oscillator

Rössler Oscillator

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + 0.15y \\ \dot{z} &= 0.4 + z(x - 0.85) \end{aligned}$$



This oscillator is chaotic:

$$\lambda_1 > 0 \quad \lambda_2 = 0 \quad \lambda_3 < 0$$

A phase can be introduced as

$$\phi = \arctan\left(\frac{y(t)}{x(t)}\right) \qquad \phi = \frac{t - t_n}{t_{n+1} - t_n} 2\pi \qquad 0 \leq \phi \leq 2\pi$$

Phase corresponds to the zero Lyapunov exponent (time, motion along the orbit)

Coupled chaotic oscillators

$$\dot{x}_{1,2} = -\omega_{1,2}y_{1,2} - z_{1,2} + C(x_{2,1} - x_{1,2})$$
$$\dot{y}_{1,2} = \omega_{1,2}x + 0.15y_{1,2}$$
$$\dot{z}_{1,2} = 0.2 + z_{1,2}(x - 10)$$

The two uncoupled oscillator are chaotic for $C = 0$:

$$\lambda_1 \ \& \ \lambda_2 > 0 \quad \lambda_3 = \lambda_4 = 0 \quad \lambda_5 \ \& \ \lambda_6 < 0$$

By increasing C we observe a transition form a regime where the phases rotate with different velocities

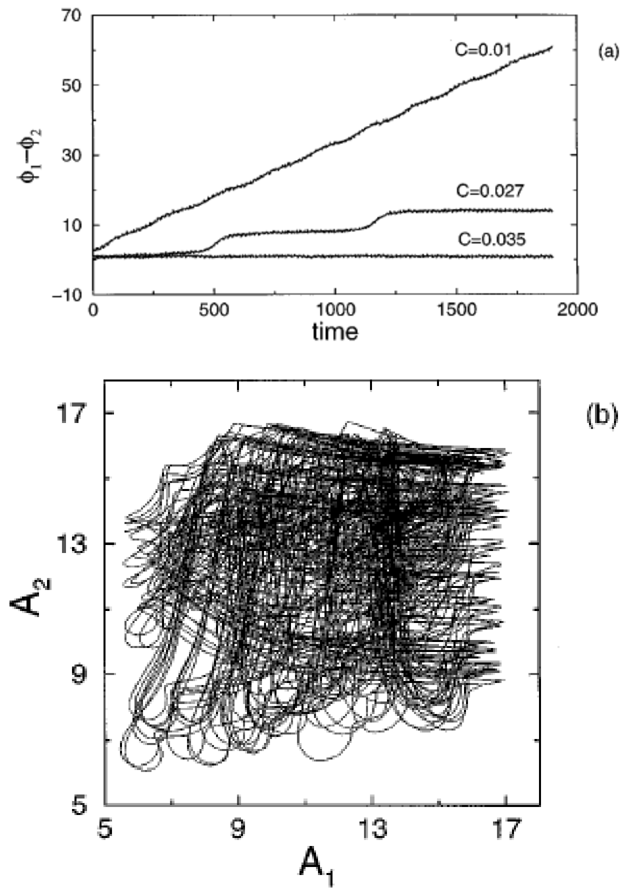
$$\varphi_1 - \varphi_2 \sim \Delta\Omega t$$

to a regime where they are phase locked

$$|\varphi_1 - \varphi_2| < const \qquad \Delta\Omega = 0$$

The amplitudes remain completely chaotic

Rosenblum et al. Phys. Rev. Lett. 76 (1996) 1804



Phase synchronization

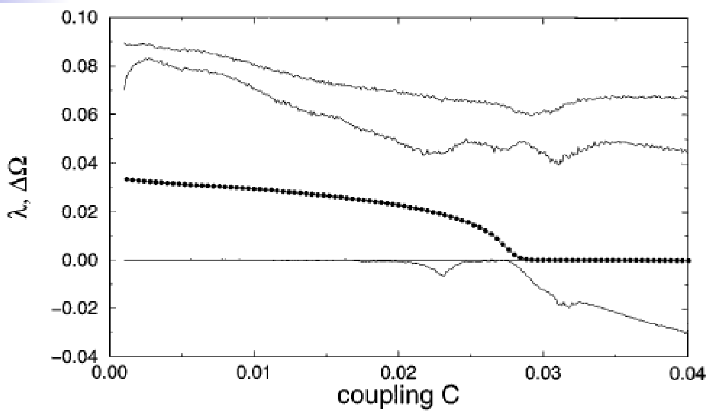


FIG. 3. The four largest Lyapunov exponents, one of which is always zero (lines) and $\Delta\Omega$ (circles) vs coupling C for system (3) with $\Delta\omega = 0.015$.

The transition to phase synchronization occurs when one of the two zero Lyapunov exponents, associated to the phases, becomes negative inducing an attraction of the two phases which then phase lock.

Qualitatively the phase dynamics of chaotic oscillators can be mimicked by an equation like this

$$\frac{d\phi}{dt} = \omega + F(A)$$

where the chaotic action of the amplitude can be seen as a small noise source, due to the weak coupling between phase and amplitude.

Therefore for 2 coupled oscillator we end up once more with the Adler equation plus a small noise term, due to the coupling with the amplitudes, and results similar to non chaotic oscillators are expected.

Phase synchronization

In order to observe phase synchronization one needs a zero Lyapunov exponent $\lambda = 0$, which leaves the phase free, therefore it is impossible in

- non autonomous system (i.e. periodically forced)
- systems with discrete time (i.e. maps)

Weak phase synchronization

One can have a weak form of phase synchronzations whenever

$$\varphi_1 - \varphi_2 \sim \Delta\Omega t \ \text{ with } \quad <\Delta\Omega>= 0$$

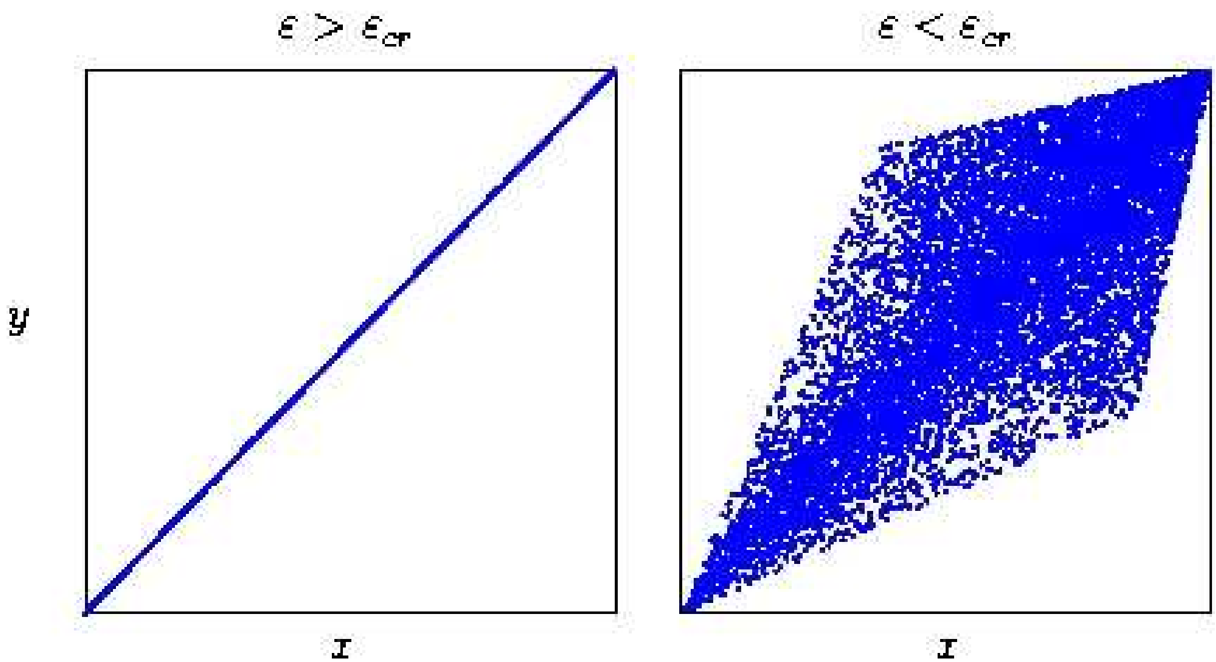
but $\varphi_1 - \varphi_2$ has a random walk behaviour, on average its value it is zero but $\varphi_1 - \varphi_2$ it is not confined and it can have a so-called diffusive behaviour
This can happen by coupling a chaotic (3d) and a hyper-chaotic (4d) Roessler oscillators

Complete chaotic synchronization

Let us consider two coupled identical chaotic systems, (chaotic maps)

$$x_{n+1} = f(x_n) + \varepsilon(f(y_n) - f(x_n)) \qquad y_{n+1} = f(y_n) + \varepsilon(f(x_n) - f(y_n))$$

for strong enough coupling $\varepsilon > \varepsilon_{cr}$ they can completely synchronize to a common chaotic state $x_n = y_n$



Transverse Lyapunov

To measure the distance between the two orbits let us introduce $v_n = \frac{x_n - y_n}{2}$ its evolution is given by

$$v_{n+1} = \frac{(1-2\varepsilon)[f(x_n) - f(y_n)]}{2}$$

For almost synchronized states $v_n << 1$ we can consider the linearization the Taylor expansion

$$f(y_n) = f(x_n) - 2f'(x_n)v_n \qquad \text{where} \qquad y_n = x_n - 2v_n$$

and therefore

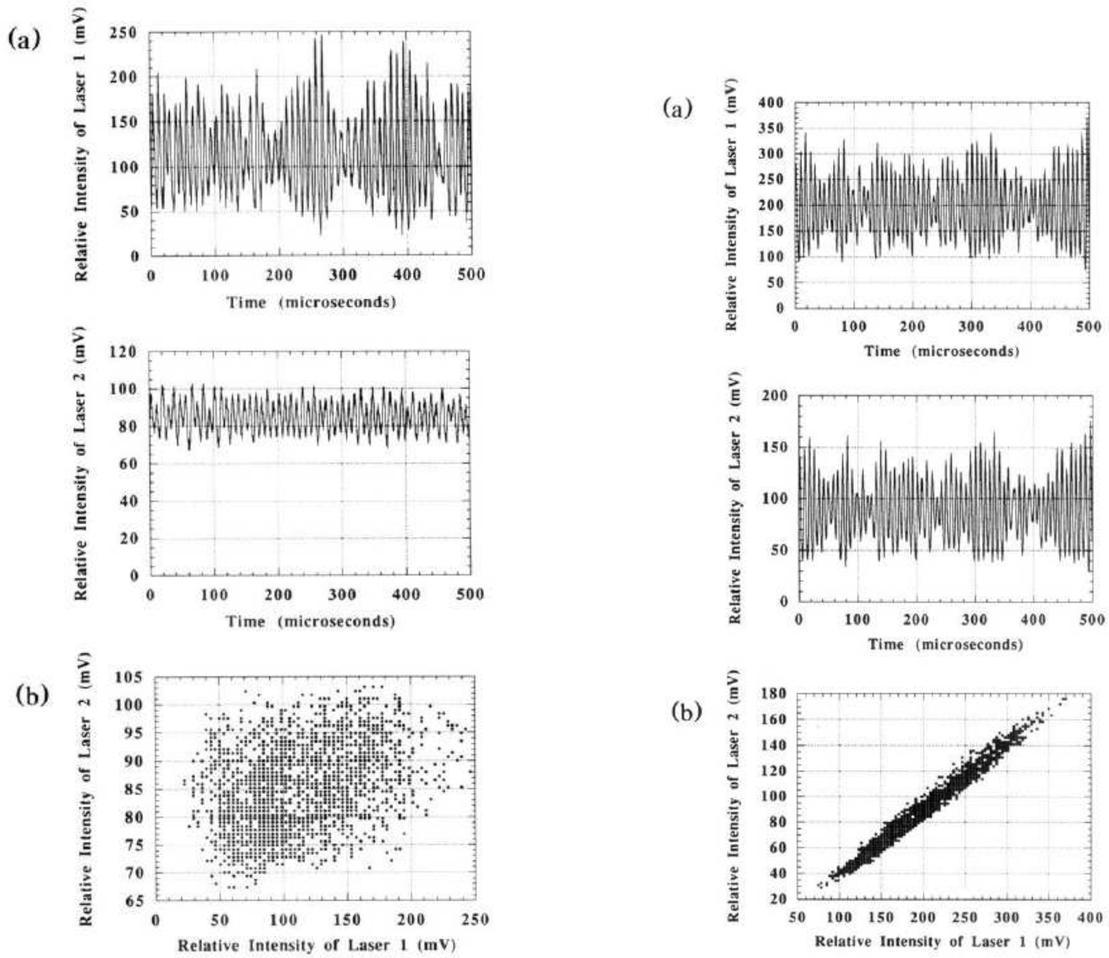
$$v_{n+1} = (1-2\varepsilon)f'(x_n)v_n = (1-2\varepsilon)\mathrm{e}^{\lambda}v_n = \mathrm{e}^{\lambda+\ln(1-2\varepsilon)}v_n$$

The vanishing of the transverse Lyapunov $\lambda_{\perp} = \lambda + \ln(1-2\varepsilon)$ tell us the synchronization threshold

$$\varepsilon_{cr} = \frac{1-\mathrm{e}^{-\lambda}}{2}$$

Chaotic lasers

Two coupled chaotic Nd:YAG lasers



Roy et al., Phys. Rev. Lett. 72 (1994) 2009

References

Texts employed for the lectures

- Synchronization: a universal concept in nonlinear sciences
  A. Pikovsky, M. Rosenblum, J. Kurths (Cambridge University Press, 2003)
- The synchronization of chaotic systems
  S Boccaletti, J Kurths, G Osipov, DL Valladares, CS Zhou, Physics Reports, 2002

Most popular texts

- Sync
  S. Strogatz (Hyperion Book, 2003)
- The geometry of biological time
  A.T. Winfree (Springer Verlag, 2001)
- Rhythms of the Brain
  G. Buzsaki (Oxford University Press, 2006)