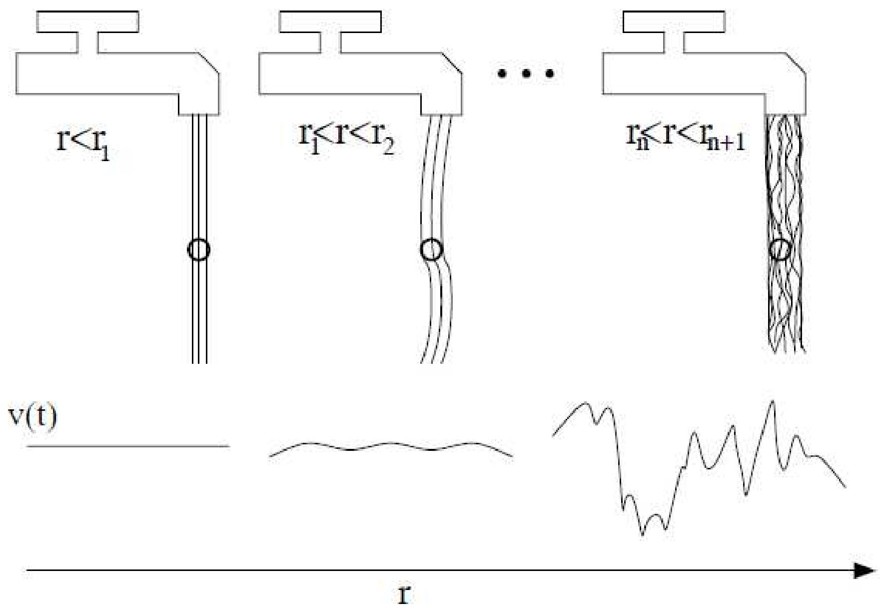
Scenarios for the transition to chaos

Alessandro Torcini

alessandro.torcini@u-cergy.fr

UNIVERSITÉ de Cergy-Pontoise

Chaos in a Faucet



The dynamics is controlled by the **Reynolds number**

$$r = \frac{LU}{\nu}$$

- L size of the open hole of the faucet
- U average velocity of the water
- ν viscosity of the fluid

- $r < r_1$ Laminar Motion ($V = const.$) – Fixed point
- $r_1 < r < r_2$ Periodic Oscillations in the velocity
- $r_n < r < r_{n+1}$ Irregular motion, **Turbulent regime**

Which is the mechanism leading from laminar to turbulent regime ?

The Landau-Hopf scenario

Landau says (1944):

The turbulent behaviour in fluids with high Reynolds numbers is due to the superposition of a growing number of regular oscillations with different frequencies.

- $r < r_1 \; v(t) = U$ — **Fixed Point**
- $r_1 < r < r_2 \; v(t) = U + A_1 \sin(\omega_1 t + \phi_1)$ — **Limit Cycle**
- $r_2 < r < r_3 \; v(t) = U + A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$ — **Torus T^2**
- $\ldots$
- $r_n < r < r_{n+1} \; v(t) = U + \sum_{k=1}^{n} A_k \sin(\omega_k t + \phi_k)$ — **Torus T^n**

$\omega_1, \omega_2, \ldots, \omega_n$ are incommensurable frequencies, i.e. they cannot be summed linearly with integer coefficients to give a zero result.

This scenario was considered valid until seventies, without experimental confirmations, and indeed

it was wrong !

The Ruelle-Takens scenario

Ruelle and Takens (1971) however proved that a Torus T^3 is structurally unstable and therefore the Landau-Hopf scenario cannot go beyond a quasiperiodic motion with 2 frequencies.

Def Structurally Stable System

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$$

a property of this system is structurally stable is it is valid also for the perturbed system

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \delta\mathbf{F}_r(\mathbf{x})$$

where $\delta\mathbf{F}_r$ is a very small perturbation of the original system, but it is a generic (non ad hoc) perturbation.

In theory, it can exist a system $\mathbf{F}_r$ exhibiting the Landau-Hopf scenario, but a small modification will destroy it, apart very special perturbation. **This implies that experimentally it will be never observed.**

Ruelle-Takens predicted that Chaos can appear already in ODEs with three degrees of freedom

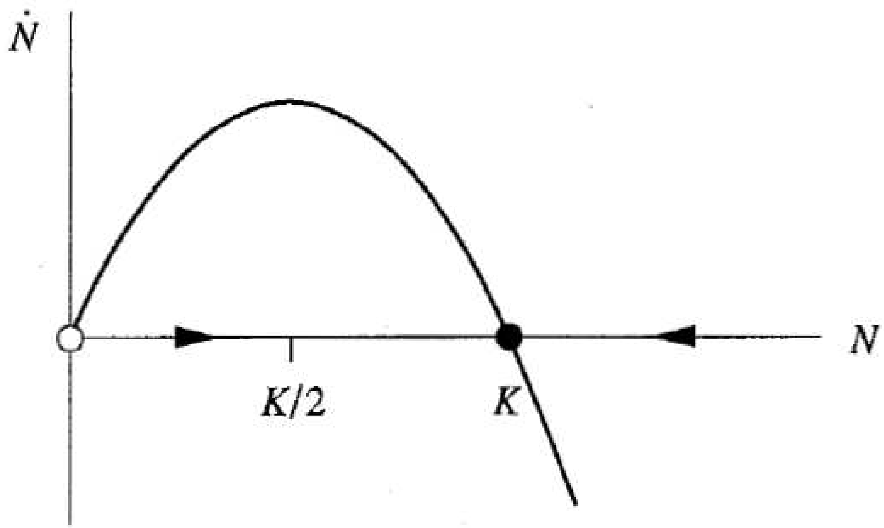
Population Growth Model

The simplest model for the growth of population of organisms was suggested by Malthus in 1798 and reads as

$$\dot{N}(t) = rN(t) \qquad N(t) = N_0 \mathrm{e}^{rt}$$

- $N(t)$ is the population at time t
- r is the reproductive power of each individual

The model is too simple leading to exponential growth, but due to resources' limitation , above a critical value K **carrying capacity** the death rate is higher than birth rate ($\dot{N} < 0$).



Logistic equation by Verhulst

$$\dot{N}(t) = rN(t)\left(1 - \frac{K}{N}\right)$$

Two fixed points

- $N^* = 0$ Unstable
- $N^* = K$ Stable

The populations always approaches the carrying capacity **NO CHAOS**

Logistic Map

A continuous time model for populations is not the best choice, since populations grow or decrease from one generation n to the next $n+1$

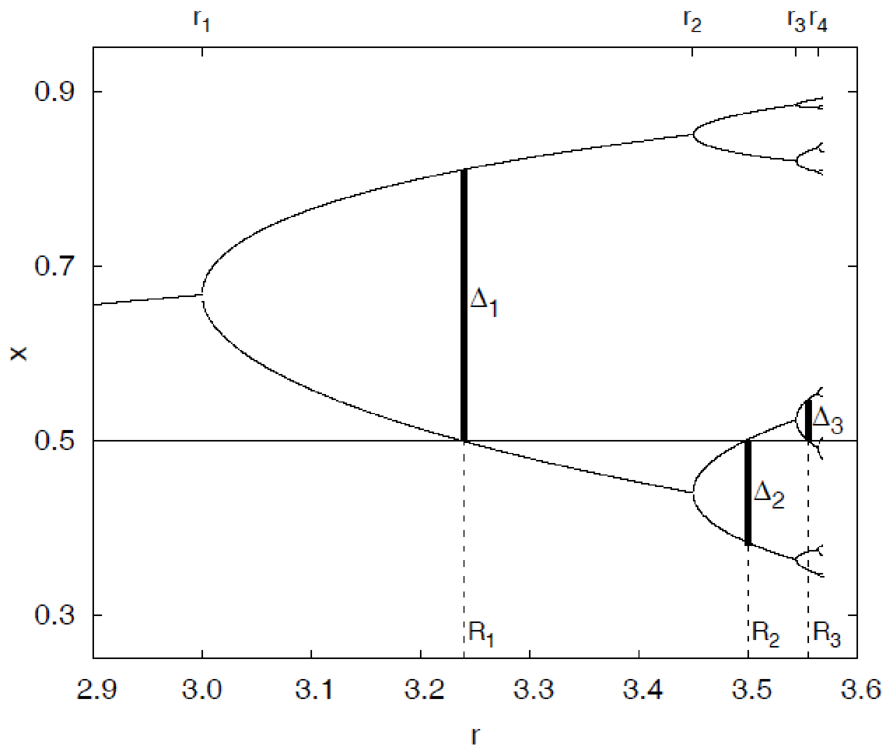
$$x_{n+1} = rx_n(1 - x_n) = f(x_n)$$

- one dimensional **non-invertible** map $\rightarrow$ Chaos
- the map is well defined for $x \in [0 : 1]$ and $r \in [0 : 4]$

Linear Stability Analysis

- $r < 1$ An unique stable fixed point $x^* = 0$ (**Population Extinction**)
- $1 < r < r_1 = 3$ Two fixed points $x^* = 0$ — **Unstable** and $x^* = 1 - \frac{1}{r}$ — **Stable**
- $r_1 < r < r_2 = 3.448$ The two fixed points are **unstable**, but the sytem exhibits **a stable period-2 orbit**
- $r_k < r < r_{k+1}$ The two fixed points are **unstable**, but the sytem exhibits **a stable period-2^k orbit**

Period Doubling Transition I



$r_1 = 3, r_2 \simeq 3.449\ldots, r_3 \simeq 3.544\ldots$

$$r_\infty = \lim_{n\to\infty} r_n = 3.569945\ldots$$

The sequence of parameter values R_n for which one has **super-stable** periodic orbit of period $2n$ is also a series with the same limiting value $R_\infty = r_\infty$

For $r > r_\infty \to$ **CHAOS**

Universality properties of the logistic map (Feigenbaum 1975)

- $\frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \frac{R_n - R_{n-1}}{R_{n+1} - R_n} \simeq \delta = 4.6692\ldots$
- $\frac{\Delta_n}{\Delta_{n+1}} \simeq -\alpha = -2.5029\ldots$
  Δ_n is the distance among the two points of the super-stable orbit which are closer to 1/2, the sign indicates that they lie on opposite sides with respect to 1/2

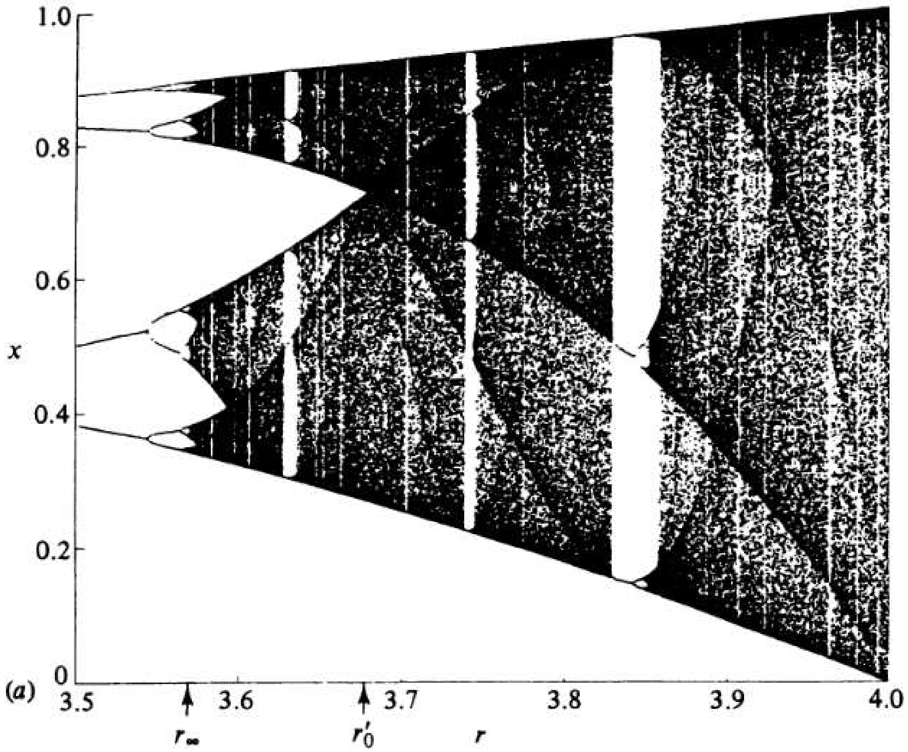
Period Doubling Transition II

- The Feigenbaum's costants are the same for any **quadratic unimodal map**
- For generic unimodal maps with non quadratic maximum (behaving like $|x - x_c|^z$ in proximity of the maximum) α and δ are again universal constants whose value depend on z
- Also ODEs can exhibit the period doubling scenario, with the same constants, this means that hidden in in the system there is a unimodal quadratic map
- This universality has been verified also **experimentally** in many many contexts, the first verification was done by Libchaber, Fauve, La-Roche in 1983 in Rayleigh-Benard convention, and they found $\delta \simeq 4.4$
- The **Renormalization Group** approach can be used to derive the Feigenbaum parameters analytically

What happens now for $r_\infty < r < 4$?

We have chaotic behaviours but not only that . . .

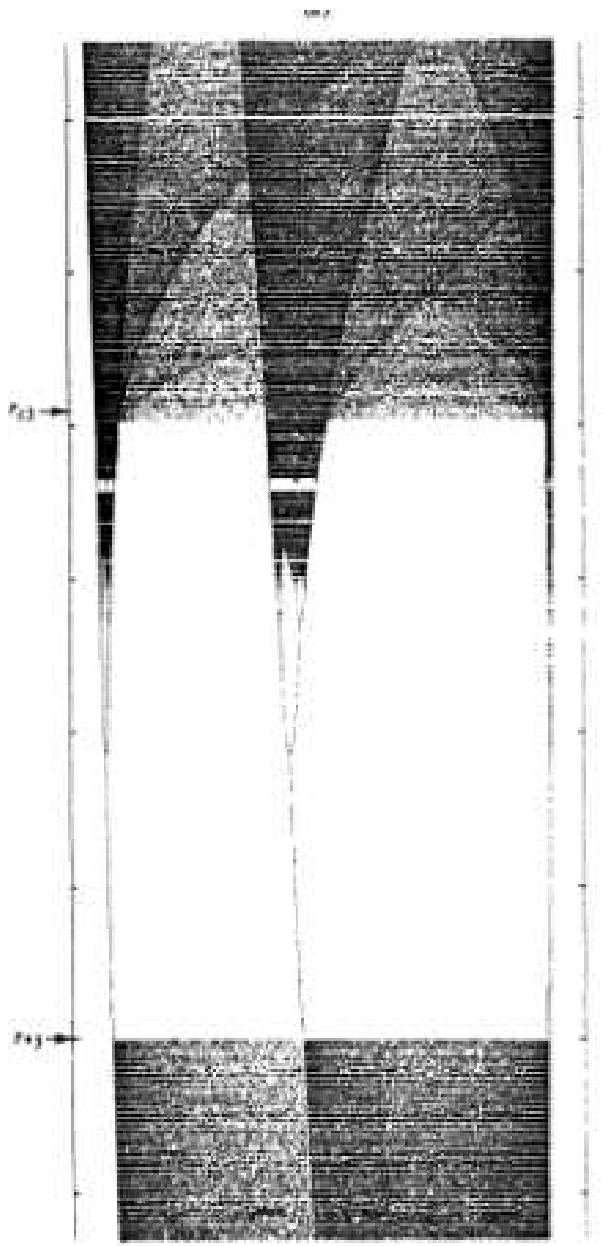
Inverse Chaotic Cascade



- $r'_0 < r < 4$ An unique chaotic attractor
- $r'_1 < r < r'_0$ A **two band** chaotic attractor — A single band chaotic attractor is recovered for f^2
- $r'_2 < r < r'_1$ A **four band** chaotic attractor — A single band chaotic attractor is recovered for f^4
- $\lim_{n\to\infty} r'_n = r_\infty$
- $\frac{r'_{n-1} - r'_n}{r'_n - r'_{n+1}} \to \delta$

The situation is more intricated a period three window is present

Period Three Window



- At $r = r_{*3}$ a Period Three Orbit is born
- A period doubling cascade is observabel for orbits of period 3×2^m
- The system becomes chaotic and a chaotic band merging is now observable (3×2^m bands $\rightarrow 3 \times 2^{m-1}$ bands)
- The chaos in three bands is observable
- Finally at $r = r_{c3}$ an unique chaotic band is observable of size similar to that of the attractor just before r_{*3}
- **There are an infinite number of windows of arbitrarly high period within the chaotic range $r_\infty \leq r \leq 4$**
  - These windows are **dense** in the chaotic range
  - The probability to choose at random a value of r in $[r_\infty, 4]$ and to observe chaos is not zero

Sarkovskii Theorem (1964)

Let us consider the following ordering of positive integers

$$3, 5, 7, \ldots, 2 \times 3, 2 \times 5, 2 \times 7, \ldots, 2^2 \times 3, 2^2 \times 5, 2^2 \times 7, \ldots,$$

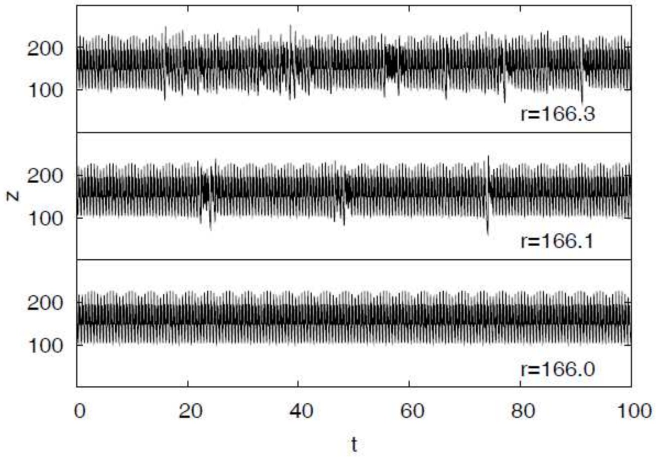
$$2^n \times 3, 2^n \times 5, 2^n \times 7, \ldots, 2^n, \ldots, 2^3, 2^2, 2, 1$$

The theorem says that

Given a 1d continous map $f(x)$ of the real line, then if the map admits a periodic orbit of period p , then the map admits also all the periodic orbits with period after p in the ordering above.

- If the map has an orbit of period p, which is **not a power of 2**, then it has **infinite number of periodic orbits**
- If an orbit of **period three** exists then the system admits periodic orbit of **any possible period**
  - For the logistic map, when it admits the **stable period three orbit** all the other periodic orbits should exist , but they are all **unstable**
  - Li and Yorke (1975) have also shown that the existence of a period three orbit implies the existence of an uncountable set of orbits which remain **aperiodic** for ever (they term this **chaos**). But this set has **zero Lebesgue measure** and these orbits are **unstable**.

Pomeau-Manneville Scenario 80



Lorenz Model

$$\frac{dX}{dt} = \sigma(Y - X)$$
$$\frac{dY}{dt} = -XZ + rX - Y$$
$$\frac{dZ}{dt} = XY - bZ$$

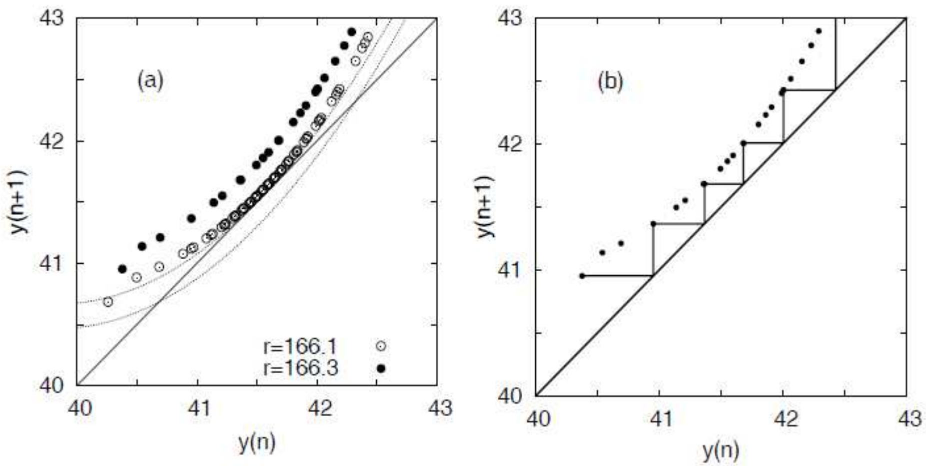
$$\sigma = 10, b = 8/3$$

Intermittency phenomena are observables in chemical and fluid systems: long **laminar** (regular) behaviours are interrupted by chaotic **bursting**

In the Lorenz model

- For $r < r_c = 1.66.05\ldots$ one observe periodic motions
- for $r > r_c$ bursting events are observable
- for $r >>> r_c$ irregular oscillations dominate the dynamics

Pomeau-Manneville Scenario 80



Poincaré Map

$$y(n+1) = f(y(n)) \quad \text{for} \quad x = 0, \quad y > 0$$

- For $r < r_c$ two intersections with the bisectrix are present – one stable and one unstable
- $r = r_c$ the map is tangent to the bisectrix – **Tangent Bifurcation**
- For $r > r_c$ a **channel** is formed where the the orbits stay long periods nearby the bisectrix, then escape making irregular motion, the it is trapped again
- The duration of the **laminar periods** grows proportionally to $1/\sqrt{(r - r_c)}$
- Experimental evidences of intermittency have been reported by Bergé in 1980 for Rayleigh-Benard convection

References

Texts employed for the lectures

- Ordre dans le chaos

  P. Bergé, Y. Pomeau, C. Vidal (John Wiley & Sons, New York, 1984)
- Chaos: from simple model to complex systems

  M. Cencini, F. Cecconi, A. Vulpiani (World Scientific, Singapore, 2010)

Most popular texts

- Chaos in Dynamical Systems

  E. Ott (Cambridge University Press, 1993)
- Nonlinear dynamics and Chaos

  S. H. Strogatz (Westview Press, 2000)