Dynamical systems in a nutshell

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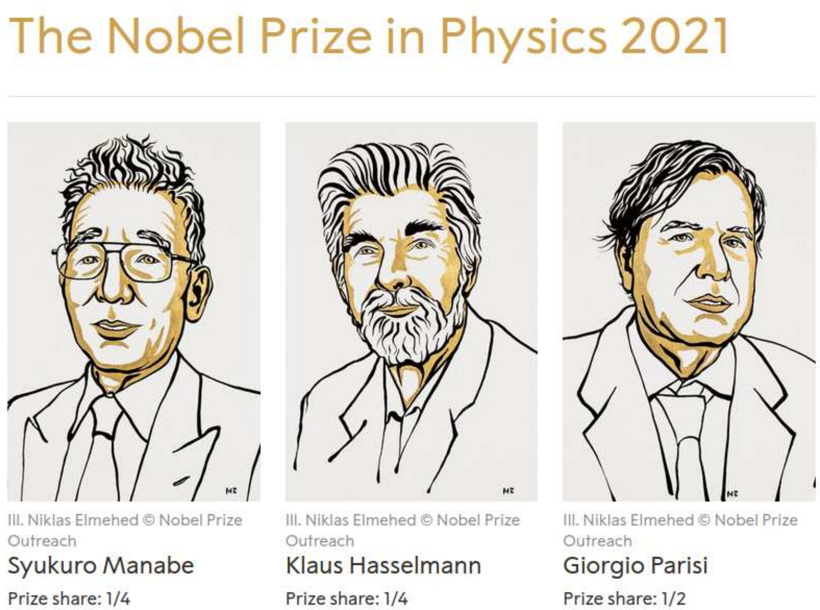
Nobel Prize in Physics 2021



The Nobel Prize in Physics 2021 was awarded "for groundbreaking contributions to our understanding of complex systems" with one half jointly to Syukuro Manabe and Klaus Hasselmann "for the physical modelling of Earth’s climate, quantifying variability and reliably predicting global warming" and the other half to Giorgio Parisi "for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales."

The ceremony took place on 10th December, 2021

Scientific Motivations

Scientific Background on the Nobel Prize in Physics 2021

"FOR GROUNDBREAKING CONTRIBUTIONS TO OUR UNDERSTANDING OF COMPLEX PHYSICAL SYSTEMS"

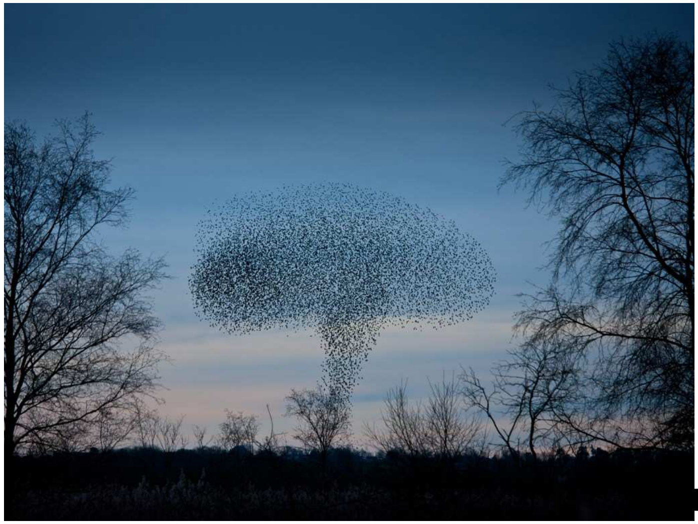
The Nobel Committee for Physics

This year's Nobel Prize in Physics focuses upon the complexity of physical systems, from the largest scales experienced by humans, such as Earth's climate, down to the microscopic structure and dynamics of mysterious and yet commonplace materials, such as glass . . .
A central emphasis is on the physical reality that the variability in the basic processes, from climate dynamics to frustrated materials, leads to the emergence of multiple length and time scales . . .

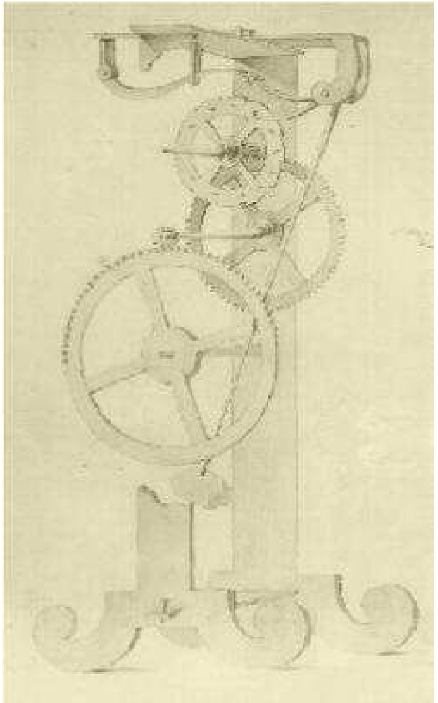
What is it a Complex System ?

Two different view of a complex system :

- A quite simple system that can give rise to a very complex behaviour : e.g. a pendulum/ a clock (Dynamical Point of View)
- A system made of a large number of interacting elements, so that the collective behaviour of those elements goes far beyond the simple sum of the individual behaviours. (Statistical Mechanics Point of View)
  - schools of fishes
  - swarm of birds FILM



A clock



Galileo Gailei was the first who had the idea to exploit the regularity of pendulum oscillations to realize a clock, however was the Dutch scientist Christian Huygens to realize it in 1656.

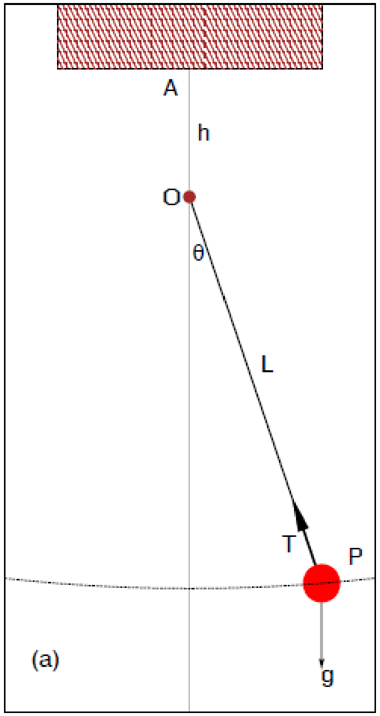
The first clock had an error less than 1 minute per day, an incredible good accuracy at the time.

Everyone will safely affirm that the oscillations of a pendulum are regular (**predictable**), but as we will see this is completely **FALSE**

From pendulum to chaos

Deterministic (Newton's law) $\neq$ Predictable

The nonlinear pendulum



Planar Pendulum

- Mass **M**
- attached to a Pivot O
- via a massless and inextensibile wire of length **L**
- 2 forces acting on the mass :
  - gravity $\mathbf{F_g} = M\mathbf{g}$
  - wire tension $\mathbf{T} = M g \cos\theta$

By applying the Newton's second law (**F=m a**) it emerges that the system can be described simply by the angle θ between the wire and the vertical axis and by the angular velocity $\omega = d\theta/dt$.

More at the Blackboard

The nonlinear pendulum

- **Along the inextensible wire – Equilibrium** $\boxed{\mathbf{T} = M g \cos\theta = M a_p}$
- **Tangent to the circle – Dynamics** $\boxed{-M g \sin\theta = M L \frac{d\theta^2}{dt^2} = M a_t}$

Therefore the equation of motion turns out to be independent of the mass

$$\frac{d\theta^2}{dt^2} = -\frac{g}{L}\sin\theta$$

This is a NONLINEAR second order ordinary differential equation (ODE), difficult to solve.

This can be rewritten as a system of two first order ODEs :

$$\dot{\theta} = \omega \quad ; \quad \dot{\omega} = -\frac{g}{L}\sin\theta$$

- **Two first order ODEs $\longrightarrow$ 2 Degrees of Freedom – 2 dimensional Phase Space**
- **Trajectories' visualization**
- **Approximate simplified equation for small angles**

The linear pendulum

For **small oscillations** $\sin\theta \simeq \theta$ ($x \simeq L\theta$) therefore

$$\frac{d\theta^2}{dt^2} = \ddot{\theta} = -\frac{g}{L}\theta = -\omega_0^2\theta$$

this a **LINEAR ODE** and it can be solved analytically, the angle **oscillates periodically** in time

$$\theta(t) = A\cos(\omega_0 t + \phi) \qquad \dot{\theta}(t) = -A\omega_0\sin(\omega_0 t + \phi) \qquad \omega_0 = \sqrt{\frac{g}{L}}$$

The period T of the motion is

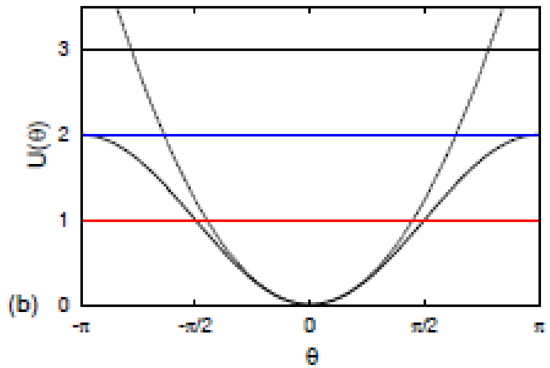
- $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}$
- T is independent of the **amplitude** A and of the **phase** ϕ

However how can we determine amplitude A and phase ϕ ?

The linear pendulum

The total energy is conserved

- **Kinetic Energy** – $K = \frac{1}{2} M v_t^2 \simeq \frac{1}{2} M L^2 \left(\frac{d\theta}{dt} \right)^2$
- **Potential Energy**
- Force = minus derivative of the potential
- $F = M a_t \simeq -M g \theta = -\frac{dU}{dx}$
- $U = M g L \frac{\theta^2}{2}$
- **Total Energy** – $E = K + U = \frac{1}{2} M L^2 \left(\frac{d\theta}{dt} \right)^2 + M g L \frac{\theta^2}{2}$



At time $t = 0$ the pendulum is at rest $\dot{\theta} = 0$:

- $\dot{\theta}(t = 0) = -A \omega_0 \sin(\phi)$ therefore $\phi = 0$
- $\theta(t = 0) = A \cos(\phi) = A$

The total energy at $t = 0$ is

$E = U = M g L \frac{\theta^2(t=0)}{2} = M g L \frac{A^2}{2}$ and $A = \sqrt{\frac{2E}{mgL}}$ is determined by the initail energy E

The linear approximation works only for small θ angles, because for large θ the potential energy diverges to infinite

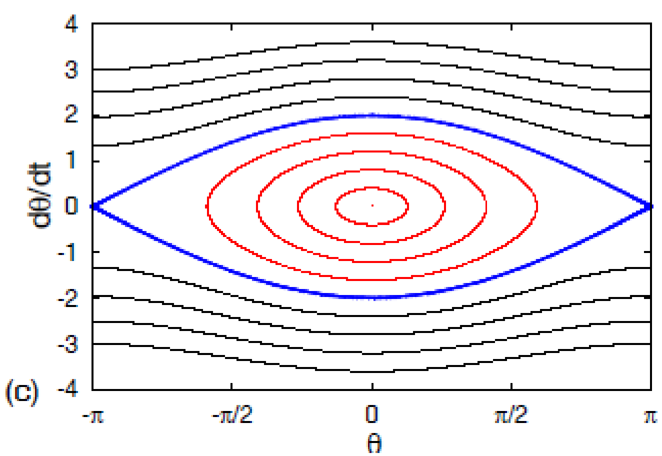
$$U = M g L \frac{\theta^2}{2} \to \infty \quad \text{for} \quad \theta \to \pm\infty$$

Trajectories' visualization

Since the **nonlinear ODE** for large oscillations cannot be solved analytically

$$\frac{d\theta^2}{dt^2} = -\frac{g}{L}\sin\theta \qquad ,$$

but only numerically we plot the dynamics graphically in the Phase Space $\left(\theta; \frac{d\theta}{dt}\right)$



Each curve in the Phase Space is called **a trajectory**

- **Oscillations** (closed orbits)
- **Rotations** (open orbits)
- The **separatrix** corresponds to the pendulum starting with **zero velocity** from **the unstable equilibrium position** $\theta = \pi$ and returning to it with zero velocity in an **infinite time**.

- The motion continues for ever, the system is conservative,
- The motion repeats **periodically**
- Liouville's Theorem : Volumes in Phase Space are preserved

The separatrix

Potential Energy

- $F = Ma_t = -Mg\sin\theta = -\frac{dU}{dx}$
- $U(\theta) = MgL\int_{-\pi}^{\theta} d\theta \sin\theta = MgL(1-\cos\theta)$

The system is conservative, its total energy is **constant**

$$E = K + U = \frac{1}{2}ML^2\left(\frac{d\theta}{dt}\right)^2 + MgL(1-\cos\theta)$$

Separatrix

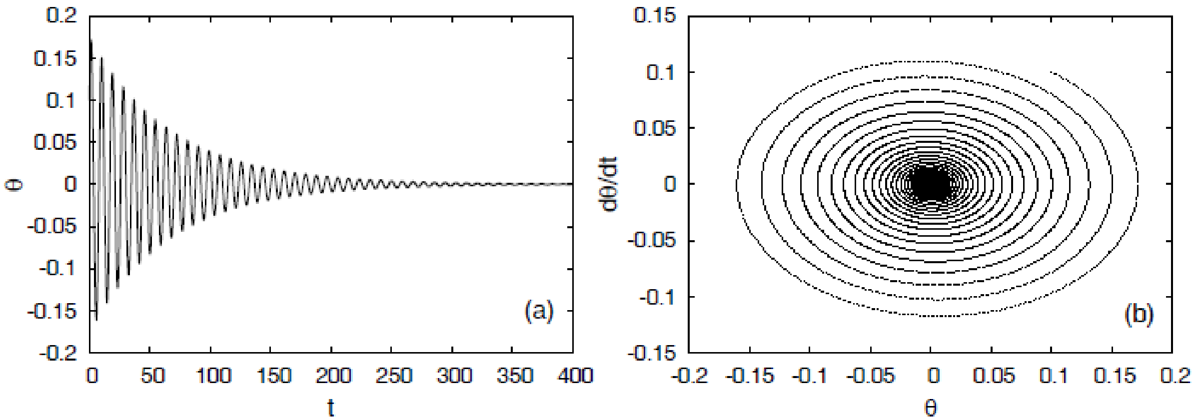
The separatrix divides oscillations from rotations, its energy can be estimate by setting the pedulum at rest at the unstable fixed point, i.e. $\theta = \pi$ and $\dot{\theta} = 0$, therefore it will need an **infinite** time to make one complete oscillation.

$$E_{sep} = 2MgL$$

The damped nonlinear pendulum

The previous picture is not realistic, the **friction** due to the air drag on the pendulum is always present. This force is proportional to the velocity $d\theta/dt$ and it acts against the motion (**Stokes' law**):

$$\frac{d\theta^2}{dt^2} = -\gamma \frac{d\theta}{dt} - \frac{g}{L}\sin\theta \qquad \gamma \text{ is the damping constant}$$



The energy is no more conserved, the friction dissipates energy, and the pendulum ends up always in the resting state $\theta = 0$ In mathematical language:

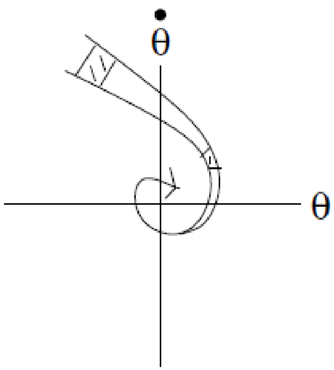
- the system is **dissipative**
- the rest state $(\theta, d\theta/dt) = (0, 0)$ is an **attractor** for the dynamics : **a stable fixed point**

Dissipative vs Conservative

The Liouville Theorem

GO TO THE BLACKBOARD !!!

Dissipative Systems

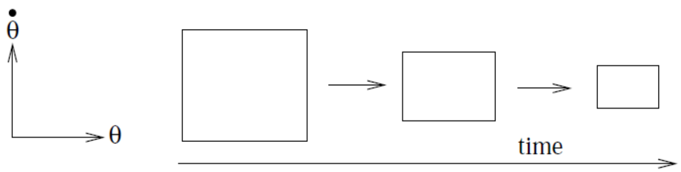


Dissipative systems :

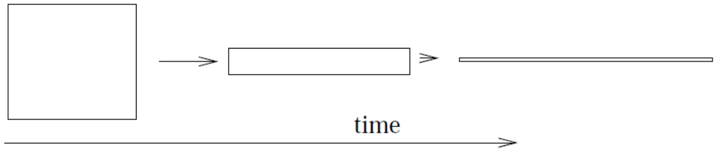
- are not **time reversible**
- areas in phase space are **contracted**

The area occupied by a set of initial conditions diminishes in time

There are different ways to contract such area



All directions are contracted in the same way



One direction is expanded,

but the other contracts even more

Area contraction does not necessarily imply that all the directions in phase space will be contracted, someone can even **expand**

The driven damped pendulum

O Botafumeiro - Santiago de Compostela

A giant **censer** of 53 kg (1.60 mt height) hanging from the vault of the Cathedral by a rope of 20 meter.

Due to dissipation the pendulum tends to stop, by varying periodically the length of the rope it is possible to maintain it in motion: **parametric energy pumping !**

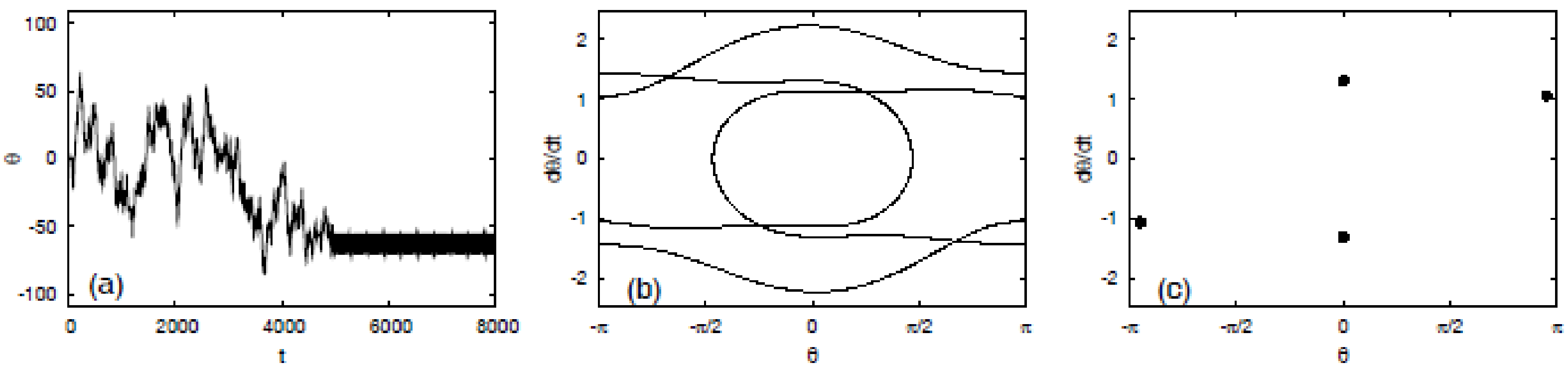
FILM and Derivation of the Equation at the BlackBoard

The driven damped pendulum

The equation of motion of the driven damped nonlinear pendulum where the pivot moves periodically in time as $h(t) = h_0 \cos(\omega t)$ is

$$\frac{d\theta^2}{dt^2} = -\gamma \frac{d\theta}{dt} - \left(\frac{g}{L} - \frac{h_0 \omega^2}{L} \cos(\omega t)\right) \sin\theta$$

the period of the forcing is $T_0 = 2\pi/\omega$

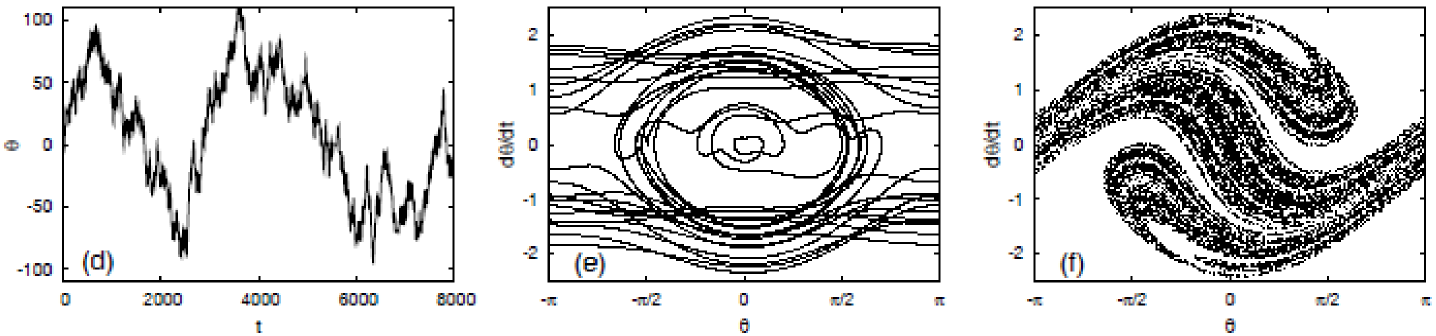


For a certain choice of parameters L, h_0, ω one observes

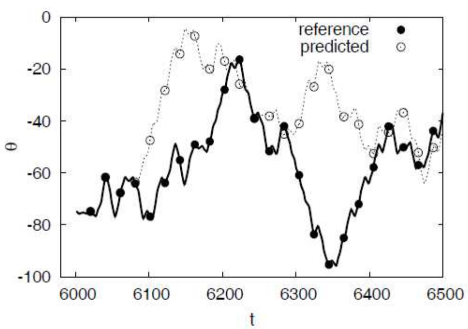
- (a) After a **transient** the dynamics ends up on a periodic orbit
- (c) By observing the trajectory at stroboscopic times $t_n = nT_0$ only 4 points remain : Periodic Motion of Period $4T_0$ (**Stroboscopic Map**)

The driven damped pendulum

For a different choice of parameters L, h_0, ω



- (d) The dynamics is always irregular
- (e) The Phase Plane is almost filled
- (f) The stroboscopic observation reveals a **Chaotic Attractor**



Two initial conditions differing **less than 1 part/100,000** give rise to different trajectories :

Sensitivity to Initial Conditions (SIC)

Deterministic but NOT predictable : chaotic

Dynamical Systems

The state of a generic system is characterized by the set of the variables describing the system

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

where d is the dimension of the Phase Space.

For the pendulum $d = 2$ and $x_1(t) = \theta(t)$, $x_2(t) = d\theta(t)/dt$

The evolution in time of the variables is ruled by a set of ODE's - **The Flow**

$$\frac{dx_1}{dt} = f_1(x_1(t), x_2(t), \ldots, x_d(t))$$

$$\vdots$$

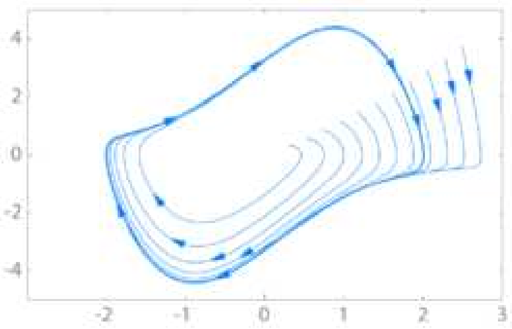
$$\frac{dx_d}{dt} = f_d(x_1(t), x_2(t), \ldots, x_d(t))$$

Dissipative Systems

- Phase Space volumes are **contracted**
- The set of points asymptotically reached by the trajectories lives in a space of dimension $D < d$ - D can be non integer: **Fractal dimension**
- Such set is called **Attractor**

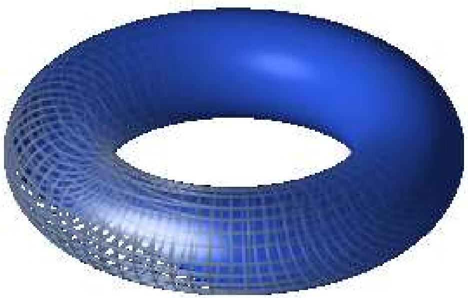
Regular Attractors

Fixed Point



Limit Cycle

Periodic Motion characterized by 1 single period

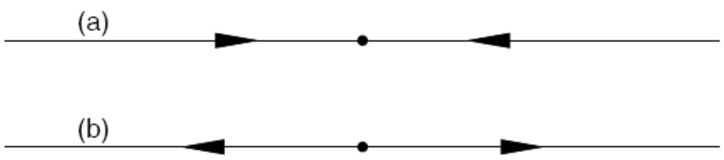


Torus T^2

Quasi-periodic Motion characterized by 2 incommensurable periods

Torus T^N

Stability of a Fixed Point



Let us consider a 1d system with a fixed point $x = x^*$:

$$\frac{dx}{dt} = f(x) \qquad f(x^*) = 0$$

Is the dynamics **attracted** or **repelled** by x^* ?

To answer we perform a **linear stability analysis**, we perturb the fixed point by δx_0 and look for its evolution:

$$x(0) = x^* + \delta x_0 \rightarrow x(t) = x^* + \delta x$$

therefore

$$\frac{dx}{dt} = f(x^* + \delta x) \rightarrow \frac{dx^*}{dt} + \frac{d\delta x}{dt} \sim f(x^*) + f'(x^*)\delta x \rightarrow \frac{d\delta x}{dt} = f'(x^*)\delta x$$

by setting $\gamma = f'(x^*)$ we get $\delta x(t) = \delta x_0 \mathrm{e}^{\gamma t}$

- If the Floquet exponent $\gamma < 0 \rightarrow$ **Stable Fixed Point**
- If the Floquet Exponent $\gamma > 0 \rightarrow$ **Unstable Fixed Point**
- If the Floquet Exponent $\gamma = 0 \rightarrow$ **Marginally Stable**

Fixed Points in 2D

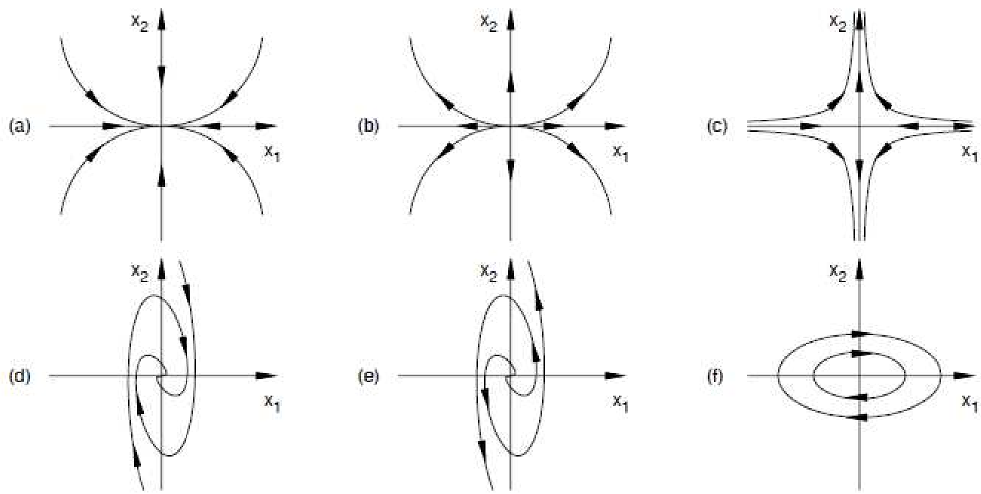


Table 2.1 Classification of fixed points (second column) in $d = 2$ for non-degenerate eigenvalues. For the case of ODEs see the second column and Fig. 2.5 for the corresponding illustration. The case of maps correspond to the third column.

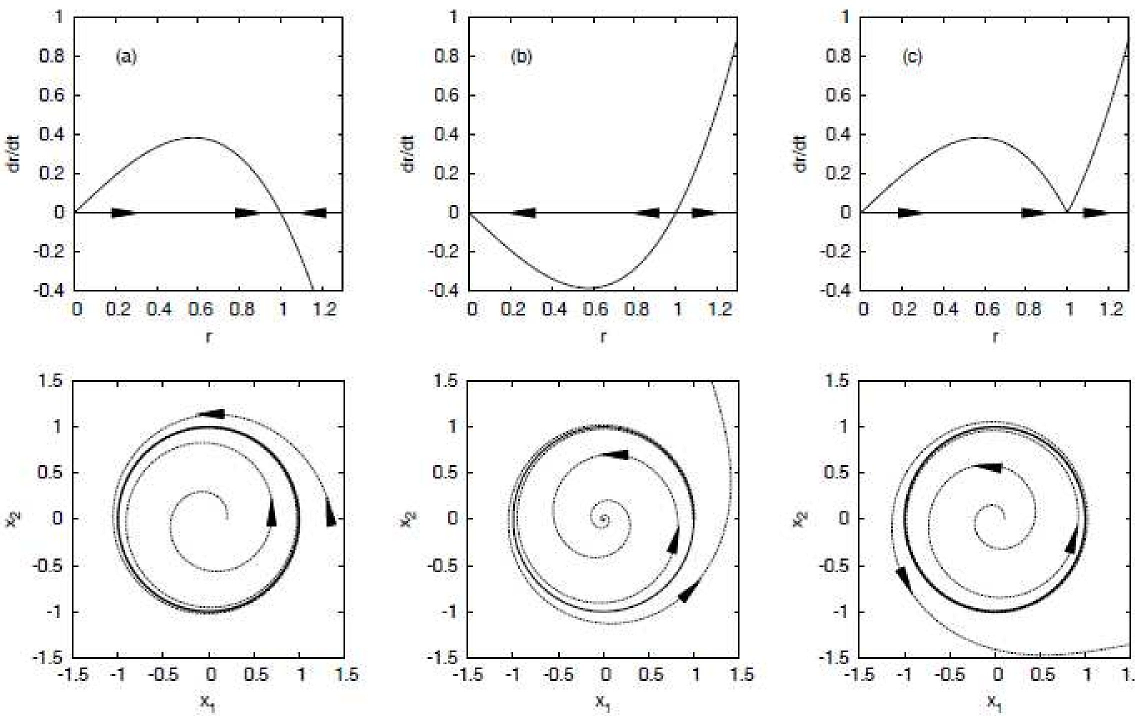
Case	Eigenvalues (ODE)	Type of fixed point	Eigenvalues (maps)
(a)	$\lambda_1 < \lambda_2 < 0$	stable node	$\rho_1 < \rho_2 < 1$ & $\theta_1 = \theta_2 = k\pi$
(b)	$\lambda_1 > \lambda_2 > 0$	unstable node	$1 < \rho_1 < \rho_2$ & $\theta_1 = \theta_2 = k\pi$
(c)	$\lambda_1 < 0 < \lambda_2$	hyperbolic fixed point	$\rho_1 < 1 < \rho_2$ & $\theta_1 = \theta_2 = k\pi$
(d)	$\lambda_{1,2} = \mu \pm i\omega$ & $\mu < 0$	stable spiral point	$\theta_1 = -\theta_2 \neq \pm k\pi/2$ & $\rho_1 = \rho_2 < 1$
(e)	$\lambda_{1,2} = \mu \pm i\omega$ & $\mu > 0$	unstable spiral point	$\theta_1 = -\theta_2 \neq \pm k\pi/2$ & $\rho_1 = \rho_2 > 1$
(f)	$\lambda_{1,2} = \pm i\omega$	elliptic fixed point	$\theta_1 = -\theta_2 = \pm(2k+1)\pi/2$ & $\rho_{1,2} = 1$

More at the blackboard !!!

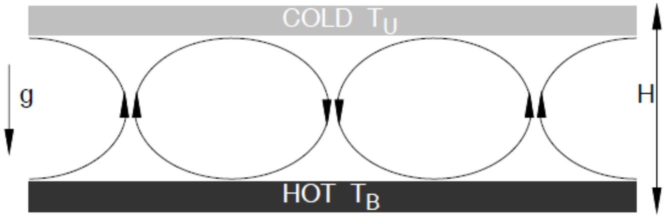
Limit Cycles

A Limit Cycle is a Closed and Isolated Trajectory

- LCs are examples of naturally oscillating systems without being forced periodically (Autonomous ODE)
- LCs are characterized by a typical period, amplitude and wave form determined by the equation structure
- The LCs are nonlinear phenomena emerging in dissipative systems
- LCs exist for autonomous ODEs in dimension $d \geq 2$ (amplitude and phase)



Fluid Convection



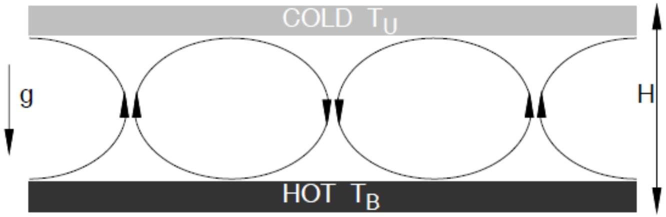
- The fluid density depends on the temperature
  $\rho = \rho(T)$
- Forces : **Buoyancy** (Spinta di Archimede) vs **Viscosity**

- If $T_U > T_B$
  - Usual **Heat Conduction**
  - Linear Profile of Temperature $T(z) = T_B + z\frac{T_U - T_B}{H}$
  - the state is **stable**
- If $T_U < T_B$
  - **The situation is unstable** : fluid below is less dense than fluid above **Expandible Fluid**
  - **Buoyancy** Towards Upwards
  - **Dissipative Effects** act against the **Buoyancy**

The heat can be transfered from on plate to the other via 2 mechanisms :

- **Fluid Motion** with a time $\tau_m = \frac{\eta}{\rho_0 g \alpha H \Delta T}$
- **Heat Diffusion** with a time $\tau_c \propto H^2/k$

Fluid Convection



- The fluid density depends on the temperature
- Forces : **Buoyancy** (Spinta di Archimede) vs **Viscosity**

- The dynamics is controlled by the Rayleigh number

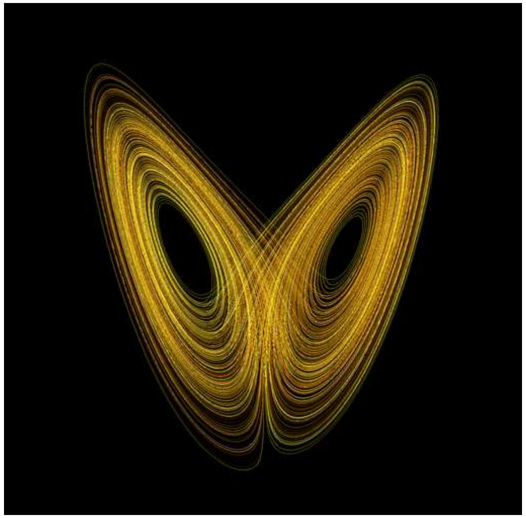
$$R_a = \frac{\rho_0 g \alpha H^3 \Delta T}{k\nu} = \frac{\tau_c}{\tau_m} \qquad \Delta T = T_B - T_U$$

  - g is the gravity constant
  - $\rho(\Delta T) = \rho_0(1 - \alpha \Delta T)$, α thermal dilatation coefficient
  - k thermal diffusivity - **Heat equation** $\dot{T} = K\nabla^2 T$
  - ν fluid viscosity
- If $Ra > Ra_c$ the **heat conduction** is replaced by the **convective motions**
- The fluid motion is **faster** then the heat diffusion $\tau_c > \tau_m$
- If $Ra >> Ra_c$ the steady convection state is replaced by **erratic dynamics**

Rayleigh-Bénard convection is fundamental for atmosphere, stars, earth magmatic mantile etc.

Lorenz Attractor 1960

A three dimensional simplified model for the description of convective motions in a fluid



$$\frac{dX}{dt} = \sigma(Y - X)$$
$$\frac{dY}{dt} = -XZ + rX - Y$$
$$\frac{dZ}{dt} = XY - bZ$$

- $X(t)$ **is the amplitude of the convective motion**
- $Y(t)$ **is the temperature difference between ascending and descending fluid**
- $Z(t)$ **is the deviation from the linear temperature profile**

The parameters have physical meaning

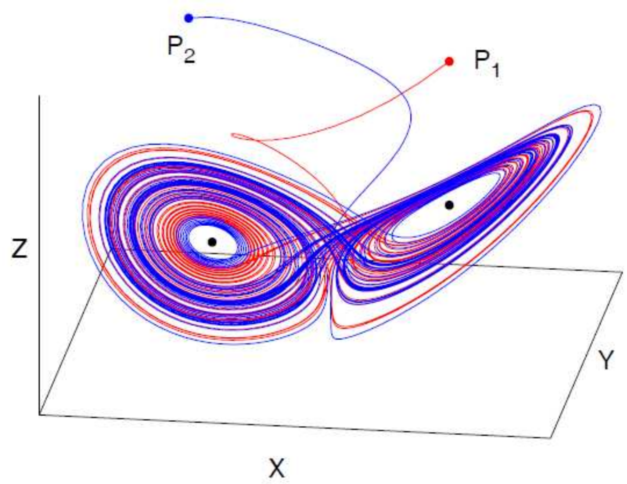
$$r = \frac{Ra}{Ra_c} \qquad \sigma = \frac{\nu}{k}$$

b **is a geometrical factors linked to the rolls wave lenght**

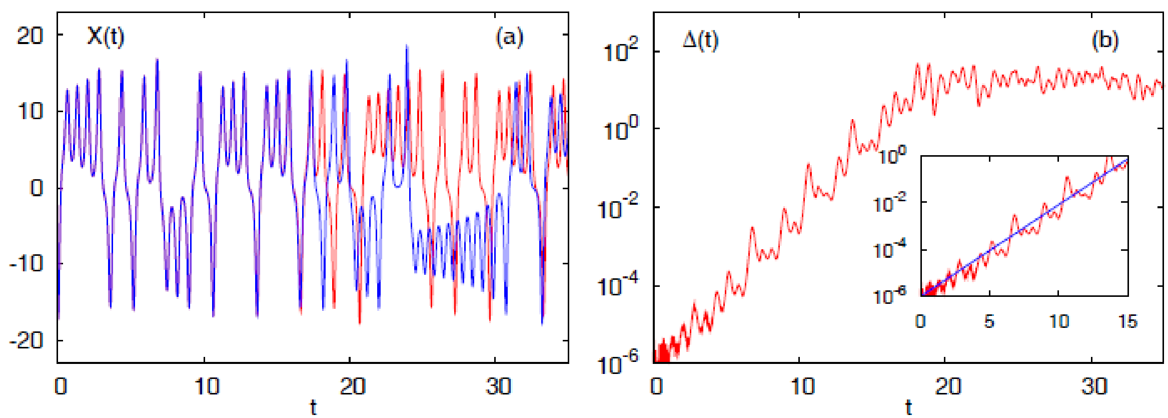
Blackboard !!!

Lorenz Attractor 1960

Distant initial points



Nearby initial conditions



Stroboscopic Maps

Flux in 3 Dimensions

$$\dot{\vec{x}} = \vec{\mathbf{F}}_c(\vec{x}) \qquad \vec{x} \in \mathcal{R}^3 \qquad c \in \mathcal{R}$$

c is the control parameter of the dynamics

Stroboscobic Map

We measure the values of $\vec{x}(t)$ at regular time intervals $t = nT_0$

- $\dot{\vec{x}} = \frac{\vec{x}(nT_0+T_0)-\vec{x}(nT_0)}{T_0} = \vec{\mathbf{F}}_c(\vec{x}(nT_0))$
- by rescaling time as $t' = t/T_0$, one can always assume $T_0 = 1$
- $\vec{x}(nT_0 + T_0) = \vec{x}_{n+1}$ and $\vec{x}(nT_0) = \vec{x}_n$
- $\vec{x}_{n+1} - \vec{x}_n = \vec{\mathbf{F}}_c(\vec{x}_n)$

Finally we get the stroboscobic map

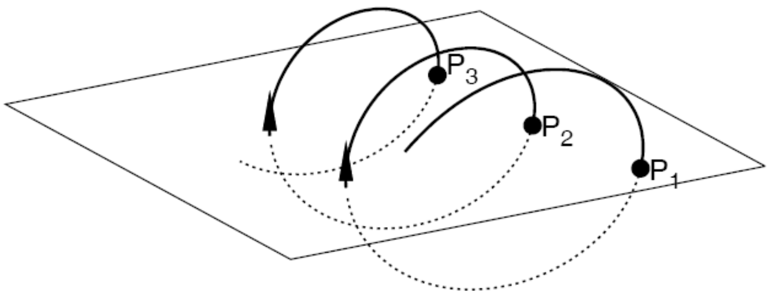
$$\vec{x}_{n+1} = \vec{\mathbf{F}}_c(\vec{x}_n) + \vec{x}_n = \vec{\mathbf{G}}_c(\vec{x}_n)$$

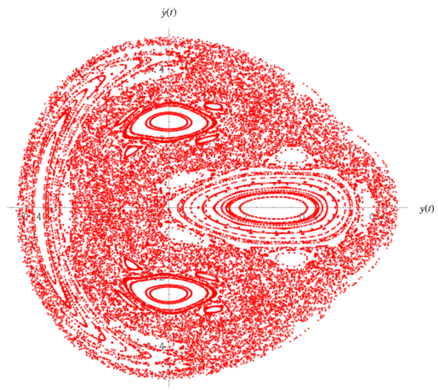
The dynamical evolution is obtained by applying n times the map G_c to the initial condtion $\vec{x}_0$:

$$\vec{x}_{n+1} = \vec{\mathbf{G}}_c(\vec{x}_n) = \vec{\mathbf{G}}_c^2(\vec{x}_{n-1}) = \ldots = \vec{\mathbf{G}}_c^n(\vec{x}_0)$$

Poincaré Section

For $d > 3$ the visualization of the trajectories is impossible !

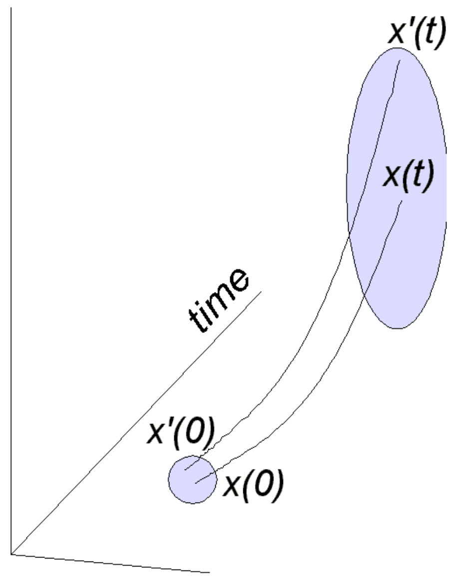




Poincaré Section for a 3 dimensional flow $d\mathbf{x}/dt$

- define a plane
- consider the successive points P_n in which the trajectory crosses the plane (from the same side)
- $\mathbf{P_2} = \mathbf{F}(\mathbf{P_1}), \mathbf{P_3} = \mathbf{F}(\mathbf{P_2}) = \mathbf{F}^2(\mathbf{P_1})$
  $\mathbf{P_N} = \mathbf{F}(\mathbf{P_{N-1}}) = \mathbf{F}^2(\mathbf{P_{N-2}}) = \ldots = F^N(\mathbf{P_1})$ **t**
- $\mathbf{F}$ is the **2d Poincaré Map** (discrete time) - time intervals are **NOT** constant
  - Periodic Orbit $\mathbf{F} \rightarrow$ 1 or more isolated points
  - Torus T^2 $\mathbf{F} \rightarrow$ Closed Curve
  - Strange Attractor $\mathbf{F} \rightarrow$ A very reach and foliated structrure

Deterministic Chaos



At time $t = 0$ consider two slightly different initial conditions for 2 orbits:

$$x(0) \qquad x'(0) = x(0) + \delta x_0$$

and follow their time evolution ruled by some ODE or Map

$$x(t) \qquad x'(t) = x(t) + \delta x(t)$$

If their distance increases **exponentially** in time for any generic $x(0)$

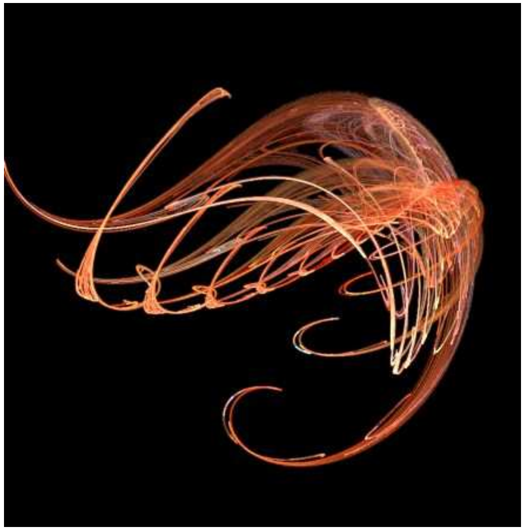
$$\delta x(t) \sim e^{\lambda t} \delta x_0 \qquad \lambda > 0$$

the system exibits **SIC**

If the **Maximal Lyapunov exponent** λ is **positive** the system is **Chaotic**

The chaotic behaviour is deterministic, ruled by a set of ODEs, no noise, no stochastic term in the system, despite this the information concerning the initial condition is lost within a short time

Strange Attractor



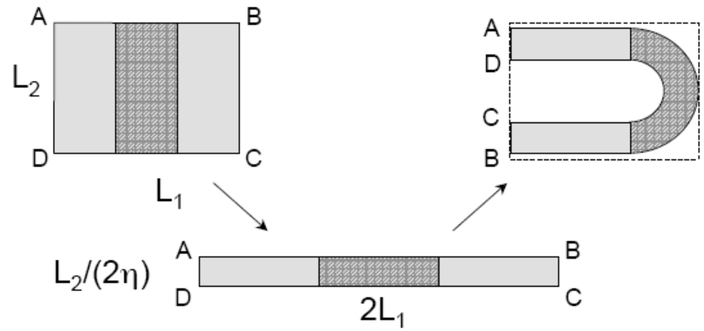
Attractors exhibiting **SIC** are **Strange**

- the volume should contract because the system is dissipative
- but at the same time the orbits should exponentially diverge
- the only possibility is that the attractor is folded , he has an extremely foliated structure

To have volume reduction it is not necessary that all the directions will contract, someone can expand, but the average expansion rate should be smaller than the average contraction rate

- the direction along which there is SIC are termed **Unstable Manifolds** and are characterized by **positive Lyapunov exponents** $\lambda > 0$
- the direction along which the system is contracted are termed **Stable Manifolds** and are characterized by **negative Lyapunov exponents** $\lambda < 0$

Smale's Horseshoe Map

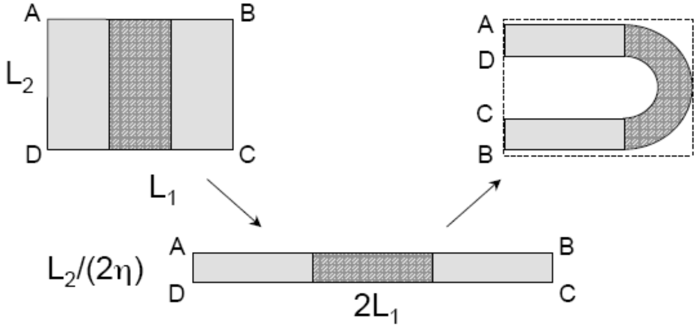


- The rectangle ABCD is **stretched** by a factor **2** along x direction
- The rectangle ABCD is **contracted** by a factor 2η ($\eta > 1$) along y direction
- The stripe is then **folded** and reinserted in the original rectangle (without changing its area)
- The operation is repeated many times thus obtaining a very foliated structure typical of **strange attractors**

The operations correspond to properties of **strange attractors**

- **Contraction** - The area of ABCd is reduced by a factor $\eta \longrightarrow$ **Dissipative System**
- **Folding** - The orbit remains in a finite portion of the space $\longrightarrow$ **Attractor**
- **Stretching** - The orbit is stretched by a factor 2 along x $\longrightarrow$ **Sensitivity to the Initial Conditions**

Stable and Unstable Manifold



Let us consider once more the Smale's Attractor:

- along x the distance among 2 points is doubled at each iteration

$$\Delta X_{n+1} = 2\Delta X_n = \mathrm{e}^{\ln 2}\Delta X_n$$
$$\Delta X_{n+1} = 2^n \Delta X_0 = \mathrm{e}^{n\ln 2}\Delta X_0$$

$$\lambda_1 = \ln 2 > 0$$

- along y the distance among 2 points is contracted at each iteration

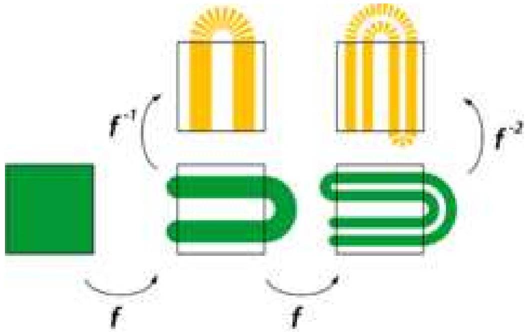
$$\Delta Y_{n+1} = \frac{1}{2\eta}\Delta Y_n = \mathrm{e}^{-\ln 2\eta}\Delta Y_n \qquad \lambda_2 = -\ln 2\eta < 0$$

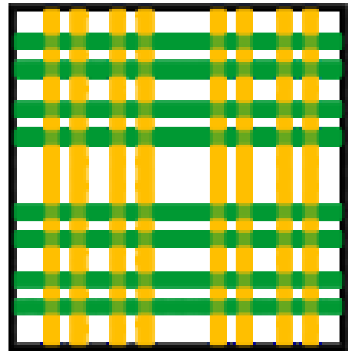
- the volumes are contracted

$$\Delta X_{n+1}\times\Delta Y_{n+1} = \frac{1}{\eta}\Delta X_n \Delta Y_n \qquad \frac{1}{\eta} = \mathrm{e}^{\lambda_1}\times\mathrm{e}^{\lambda_2} = \mathrm{e}^{\lambda_1+\lambda_2} < 1 \qquad \lambda_1+\lambda_2 < 0$$

x is the **Unstable manifold** and y is the **Stable manifold**

Attractor Dimension





The attractor dimension D, is the phase space dimension for which elements of initial conditions of dimension D on the attractor are neither expanded nor contracted

- initial conditions over a surface are contracted $\Delta X_n \times \Delta Y_n \sim e^{(\lambda_1+\lambda_2)n} \Delta X_0 \times \Delta Y_0$ therefore $D < 2$
- initial conditions along a line are expanded $\Delta X_n \sim e^{\lambda_1 n} \Delta X_0$ therefore $D > 1$

The dimension can be implicitely expressed as

$$\sum_{i=1}^{D} \lambda_i = 0 \qquad \text{usually D is not an integer Fractal Dimension}$$

By performing a linear interpolation one gets

$$D = 1 + \frac{\lambda_1}{|\lambda_2|} = 1 + \frac{\ln 2}{|\ln 2\eta|} \qquad \text{for } \eta = 4 \rightarrow D = 1.5$$

Lyapunov Spectrum

For N dimensional system one has N manifolds characterized by a Lyapunov spectrum

$$\lambda_i \qquad i = 1, \ldots, N$$

Dissipative Systems $\sum_{i=1}^{N} \lambda_i < 0$

- **Stable Fixed Point** $0 > \lambda_1 > \lambda_2 > \lambda_3 > \ldots$
- **Stable Limit Cycle** $\lambda_1 = 0 \quad ; 0 > \lambda_2 > \lambda_3 > \ldots$
- **Torus** T^k $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0 \quad ; 0 > \lambda_{k+1} > \lambda_{k+2} > \ldots$
- **Chaos** at least $\lambda_1 > 0$

Autonomous Continuous Systems without Fixed points

At least one Lyapunov exponent is zero, perturbation along the trajectory are neither contracted nor expanded

Fractal Dimension

$$D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|} \qquad \lambda_k > 0 \quad \lambda_{k+1} < 0$$

upper limit for the attractor dimension

References

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- Chaos in Dynamical Systems
  E. Ott (Cambridge University Press, 1993)
- Nonlinear dynamics and Chaos
  S. H. Strogatz (Westview Press, 2000)