## Comment on "Universal Scaling Law for the Largest Lyapunov Exponent in Coupled Map Lattices"

In a recent Letter, Yang, Ding, and Ding [1] studied the maximal Lyapunov exponent $\Lambda$ for chains of coupled map lattices (CML) in the limit of small diffusive coupling $\varepsilon$. They conclude that $\Lambda$ exhibits a universal scaling behavior of the type $\varepsilon^{1/p}$ in unimodal maps with a single $p$th order maximum. We claim that they have neglected a second correction term which, although typically small, provides the leading behavior in the small coupling limit [2–4]. The results in [1] are exact only if the second contribution vanishes, which occurs when the single map exhibits no temporal multifractality.

Following [4], the variation of the Lyapunov exponent can be formally written as $\Lambda(\varepsilon) - \Lambda(0) \equiv \lambda_1 + \lambda_2$, where $\lambda_1$ is the correction to the single map exponent $\Lambda(0)$ due to changes in the probability distribution of the local multipliers [$f'(x_i(n))$ using the same notations as in [1]] induced by the coupling. This is the only correction term considered in [1], where it is argued that it scales as

$$\lambda_1 \simeq -\varepsilon^{1/p}, \tag{1}$$

the minus sign implying that the coupling decreases the Lyapunov exponent.

The further term $\lambda_2$ takes into account the corrections arising from the increase in the phase space dimension (it is already present in the case of two maps [2]). Let us briefly summarize some of the arguments used in [3] to estimate $\lambda_2$. By tracing back all terms contributing to the amplitude of the perturbation in a given site, one identifies a tree in the space-time plane made of several directed walks. Along each walk we have a multiplicative process such as in the single map. The Lyapunov exponent in the CML results from the linear superposition of all such contributions. Roughly speaking, the sum of almost independent terms tends to shift the actual value of the growth rate from the geometric average (invoked for the Lyapunov exponent of the single map) to the linear average of the local multipliers. As a result, it turns out that [5]

$$\lambda_2 \simeq |\ln(\varepsilon)|^{-1}. \tag{2}$$

By comparing Eqs. (1) and (2), it is clear that, in the limit $\varepsilon \to 0$, $\lambda_2$ eventually overcomes $|\lambda_1|$. Even more striking is that $\lambda_2$ is positive, i.e., the diffusive coupling increases the instability.

The contribution $\lambda_2$ has been derived analytically in [3] by considering products of random matrices. Here, we show numerically that the same conclusion holds for the class of CML studied in [1]. To be more specific, let us consider the map identified by the parameters $p = 1.5$ and $\mu = 4$ in [1]. The corresponding dependence of $\Sigma = [\Lambda(\varepsilon) - \Lambda(0)]|\log_{10}(\varepsilon)|$ vs $\log_{10}(\varepsilon)$ is reported in Fig. 1. From the figure, it is evident that $\Sigma$ tends to a constant value for $\varepsilon \to 0$, a behavior that implies a logarithmic increase of $\Lambda$. Notice also the change in the sign of $\Sigma$: At moderately small values of $\varepsilon$, the main contribution comes from $\lambda_1$ so that $\Lambda$ decreases with $\varepsilon$. The converse is true at sufficiently small values of $\varepsilon$. Therefore the assumption of the authors in [1] that $\lambda_2$ is of order $\mathcal{O}(\varepsilon)$ is not verified in the general case [6].

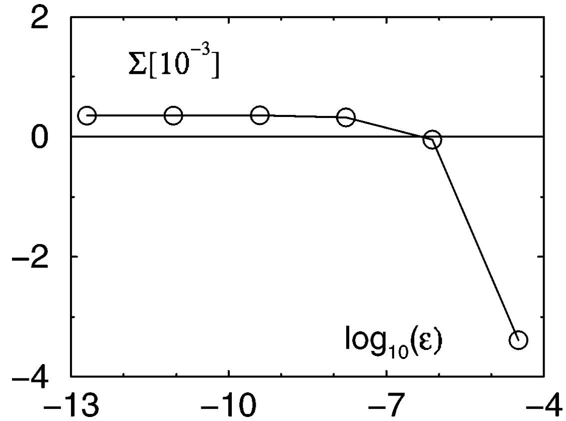


FIG. 1. The quantity $\Sigma$ vs $\log_{10}(\varepsilon)$ is reported for a chain $L = 100$ of CML $f(x) = 1 - |1 - 2x|^p$ with $p = 1.5$. The average is performed over $4 \times 10^8$ iterations for each $\varepsilon$ value.

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A. Torcini,[1] R. Livi,[2] A. Politi,[3] and S. Ruffo[4]
[1]Wuppertal University, D-42097 Wuppertal, Germany
[2]Bologna University, I-40127 Bologna, Italy
[3]Istituto Nazionale di Ottica, I-50125 Firenze, Italy
[4]Firenze University, I-50125 Firenze, Italy

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[4] R. Livi *et al.*, in *Cellular Automata and Cooperative Systems*, edited by E. Goles *et al.* (Kluwer, Dordrecht, 1993), p. 411.
[5] Strictly speaking, this is demonstrated in [3] only for positive definite multipliers, but numerics and nonrigorous arguments [4] show that it holds more generally.
[6] $\lambda_2$ is exactly zero when the growth rate along the directed walks is asymptotically the same so that linear and geometric averages coincide. This is true for the logistic map at the Ulam point studied in [1] for $p = 2$ and $\mu = 4$.

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