# Disturbance propagation in chaotic extended systems with long-range coupling

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Propagation of initially localized perturbations is investigated in chaotic coupled map lattices with long-range couplings decaying as a power of the distance. The initial perturbation propagates exponentially fast along the lattice, with a rate given by the ratio of the maximal Lyapunov exponent and the power of the coupling. A complementary description in terms of a suitable comoving Lyapunov exponent is also given. [S1063-651X(97)50404-1]

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Propagation of fronts in spatially extended systems is a topic of wide interest, in several scientific contexts including fluid dynamics, dendritic growth, directional solidification, liquid crystals, chemical reactions, flame propagation, and epidemic spreading [1–4]. In one spatial dimension, partial differential equations [2], coupled map lattices (CML's) [5] and cellular automata [6] models have been studied. The fronts were either separating stable and unstable steady states [7–9], or regular and chaotic regions [10,11].

A similar phenomenon is the propagation of disturbances in fully chaotic states of CML's [12,13] and of the complex Ginzburg-Landau equation [14]. When the spatial coupling is local, e.g., for CML with nearest-neighbor interactions, two distinct regimes have been found to characterize the dynamics of fronts [12]. In the first regime, the velocity of the propagating front is calculated in the framework of a linear analysis, while in the second the value of the velocity is determinated by the full nonlinear evolution of the system. For a particular class of models, a transition between the two regimes has been observed when a parameter is varied [12]. The scenario is not expected to be different for couplings extending to more than two neighbors, since this affects only the limit velocity of propagation [15].

If more general spatial couplings are considered, even the very definition of a velocity can be nontrivial. For example, epidemic models in one dimension exhibit a finite propagation speed if the spatial coupling (i.e. the infection rate) decays exponentially or faster [3]. The same result for some CML models is reported in Ref. [16], where it was observed that the time for a localized disturbance to overcome a threshold value at a certain distance $l$ from the initially perturbed site grows linearly with $l$ if the coupling is local or exponentially decaying, while is almost independent of $l$ for couplings decaying with power laws (with exponent not too large), indicating that the velocity is ''infinite.''

This Rapid Communication focuses on the study of disturbance propagation in systems with long-range coupling, whose strength decays as a power law in space. The spatiotemporal evolution of an initially localized perturbation of a chaotic state is studied theoretically and numerically. The perturbation is found to spread exponentially fast along the lattice, and an expression for the corresponding rate is given. A comoving Lyapunov analysis confirms that the prediction is indeed correct.

In order to mimic a spatially extended system with long-range interaction, we introduce the CML model [16]

$$u(x,t+1)=f(\widetilde{u}(x,t)), \tag{1}$$

where the indices $x$ labels the sites of a chain of length $L$, and $t$ is the discrete time variable. The function $f(x)$ is a chaotic map of the interval, and

$$\widetilde{u}(x,t)=(1-\varepsilon)u(x,t)+\varepsilon\sum_{0<|x-y|\leq\Delta_L} w(x-y)u(y,t), \tag{2}$$

where $\Delta_L=(L-1)/2$ ($L$ is assumed to be odd). As usual, periodic boundary conditions are considered $u(x,t)=u(x\pm L,t)$. The coupling constant $\varepsilon$ is bounded between 0 and 1, and we consider the coupling strength to decay in space as

$$w(x)=\frac{c(\alpha)}{|x|^{\alpha}}, \tag{3}$$

where $\alpha>1$ to insure that the normalization constant $c(\alpha)=[\Sigma_y|y|^{-\alpha}]^{-1}$ is bounded. Obviously, this constant is independent of $L$ in the thermodynamic limit $L\to\infty$. For $\alpha\to+\infty$, model (2) reduces to the usual nearest-neighbors CML [5]. Notice that, at variance with the model of Ref. [16], the long-range coupling is not introduced as a perturbation of the nearest-neighbors CML.

To study the propagation of localized disturbances in system (2), let us consider two chaotic trajectores $\{u(x,t)\}$ and $\{v(x,t)\}$ generated by starting initial conditions, which differ only around a single site $x=0$. More precisely, we assume that $u(x,0)$ is a typical chaotic state, obtained after all transients have died out, and $v(x,0)=u(x,0)+u_0(x)$, where $u_0(x)$ is a function localized around the origin.

If only *linear* mechanisms of propagation are present [12,13], we can assume that the evolution of $\delta u(x,t)=u(x,t)-v(x,t)$ is ruled by the linearized dynamics

$$\delta u(x,t+1)=m(x,t)\Bigg[(1-\varepsilon)\,\delta u(x,t)+\varepsilon\sum_{0<|x-y|\leq\Delta_L} w(x-y)\,\delta u(y,t)\Bigg], \tag{4}$$

 

where $m(x,t)=f'(\tilde{u}(x,t))$ is the local multiplier along the assigned trajectory. This hypothesis is justified at least for large distances ($|x|\gg 1$) where the disturbance is small. Moreover, its validity has also been numerically checked by comparing the evolution given by Eq. (4) with that of $u(x,t)-v(x,t)$. As a matter of fact, nonlinear effects, i.e., saturation of the perturbation growth, are present only in the central part of the disturbance.

We now consider the spatial shape of the leading edge of the front and its temporal evolution in the limit $t\to\infty$ for the ideal case of an infinite lattice ($L=\infty$). As a starting point, we make the ansatz that for $|x|\gg 1$,

$$\delta u(x,t)=\frac{\phi(x,t)}{|x|^{\beta}}, \tag{5}$$

with $\phi(x,t)$ weakly dependent on $x$. Inserting Eq. (5) into Eq. (4), we obtain

$$\frac{\phi(x,t+1)}{|x|^{\beta}}=m(x,t)\left[(1-\varepsilon)\frac{\phi(x,t)}{|x|^{\beta}}+\varepsilon\sum_{y\neq x}w(x-y)\frac{\phi(y,t)}{|y|^{\beta}}\right]. \tag{6}$$

Due to the long-range coupling we can assume that, as a first approximation, a mean field description holds. This amounts to neglecting the spatial fluctuations of $\phi(x,t)$, and replacing $m(x,t)$ with its average $e^{\lambda}$, where $\lambda$ is the (maximal) Lyapunov exponent. Equation (6) is then split into two equations, one for the time evolution

$$\phi(t+1)=e^{\lambda}\phi(t), \tag{7}$$

and one for the spatial profile

$$\frac{1}{|x|^{\beta}}=\sum_{y\neq x}\frac{w(x-y)}{|y|^{\beta}}. \tag{8}$$

Moreover, at least for $x\gg 1$, the sum appearing in Eq. (8) can be approximated by neglecting the small terms coming from decaying tails, i.e., by extending the sum only between 1 and $x-1$ (for symmetry reasons, we can also consider $x>0$). The spatial shape of the front is thus conserved in time if the fixed-point condition

$$\frac{1}{x^{\beta}}\approx\sum_{0<y<x}\frac{c(\alpha)}{y^{\beta}(x-y)^{\alpha}} \tag{9}$$

is satisfied for $x$ large enough. Since the leading contributions to the sum came from the extrema, and can be estimated to be of order $|x|^{-\beta}$ and $|x|^{-\alpha}$, Eq. (9) is fulfilled for $\beta\leqslant\alpha$. By combining this result with that of Eq. (7), we obtain the expression for the asymptotic behavior of the front leading edge:

$$\delta u(x,t)\sim\frac{\exp(\lambda t)}{|x|^{\beta}}. \tag{10}$$

Let us define the front position $r(t)$ as the maximal distance from $x=0$ where $|\delta u(x,t)|\geqslant\theta$, with $\theta>0$ being a preassigned threshold. According to Eq. (10), $r(t)$ grows exponentially as

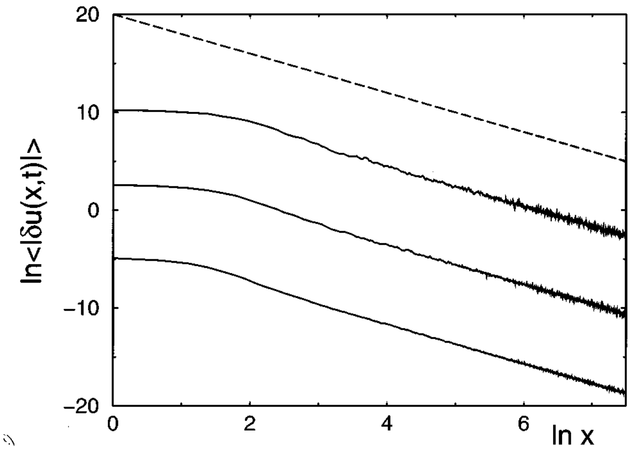


FIG. 1. Plot of the logarithm of the average disturbance amplitude $\langle|\delta u(x,t)|\rangle$ vs $\ln x$ at three different times for coupled tent maps with $\alpha=2$, $L=5001$, and $\varepsilon=\frac{1}{3}$. From bottom to top the three solid curves correspond to $t=20$, 40, and 60, respectively. The dashed line has a slope 2.

$$r(t)\sim\exp\left(\frac{\lambda}{\beta}t\right)\equiv\exp(S(\beta)t) \tag{11}$$

for $t\to\infty$, where the value of the $\beta$ parameter depends on the initial shape of the disturbance. From the above-reported arguments it is clear that, if the perturbation decays initially as $u_0(x)\sim|x|^{-\beta}$, with $\beta\leqslant\alpha$, then its profile will be preserved during the time evolution. For more general initial conditions the selected $\beta$ can be determined by means of the following argument. Let us consider $u_0(x)$ to be a superposition of several profiles, each one decaying as $|x|^{-\beta}$, but with different $\beta<\alpha$. In the linear approximation, each one of these components will propagate independently with a different rate $S(\beta)$. On the basis of general arguments [8], we expect that [for $u_0(x)$ sufficiently localized] the profile with slowest growth rate $S(\alpha)=\lambda/\alpha$ will be selected. These results are analogous to those found for CML with nearest-neighbors coupling [13] and for the complex Ginzburg-Landau equation [14], the main difference being that the tails of the front are exponential for the latter case.

We numerically tested the above predictions for a lattice of coupled tent maps [$f(z)=1-2|z|$], with several values of the coupling constant and chain lengths ranging from $L=1001$ to 20 001. In particular, we considered a single-site perturbation $u_0(x)=\delta_{x,0}$, where $\delta_{x,y}$ is the usual Kroneker delta. We computed the time evolution of $\delta u(x,t)$ according to Eq. (4), and averaged it over different reference trajectories. As shown in Fig. 1, the perturbation profile decays on average as $|x|^{-\alpha}$, while growing exponentially in time. We have also verified that an initial disturbance with a decaying profile $|x|^{-\beta}$ is conserved only for $\beta\leqslant\alpha$; otherwise a power-law decay exponent $\alpha$ is always found. Finally, we checked the validity of Eq. (7) by verifying that, for increasing $t$, the local growth rate on the tails of $\delta u$ approaches the Lyapunov exponent.

A direct numerical test of Eq. (11) is complicated by the presence of finite-size effects (see, however, Ref. [17] for results on a closely related a CML model). An indirect check is accomplished by considering a suitably defined comoving Lyapunov exponent $\Lambda$ [18]. For a CML with local coupling, $\Lambda(v)$ is defined as the asymptotic growth rate of a disturbance in a reference frame moving along the ''world line'' $x_v(t)=vt$, where $v$ is the frame velocity ($|v|\leqslant 1$). This amounts to assume $\delta u(x_v(t),t)\sim\exp(\Lambda(v)t)$. Within this scheme, the condition $\Lambda(v)=0$ defines the propagation speed of an initially localized disturbance [18,13].

In the present case, since the long-range coupling leads to exponentially fast propagation [see Eq. (11)], we define the comoving Lyapunov exponent $\Lambda(R)$ in a reference frame moving along the ''world line'' $x_R(t)=[\exp(Rt)-1]$. Therefore, on the leading edge and for sufficiently long time $t$, the relation

$$\delta u(x,t)\sim\exp(\Lambda(R)t)=\exp(\lambda t-\alpha\ln x) \qquad (12)$$

should hold. From this equation it is readily seen that the comoving exponent must be a linear function of the rate $R$, namely,

$$\Lambda(R)=\lambda-\alpha R. \qquad (13)$$

Notice that, in analogy with system with local coupling, the condition $\Lambda(R)=0$ gives exactly the growth rate $S(\alpha)=\lambda/\alpha$, in agreement with the above prediction.

As before, the finite size of the system prevents the numerical computation of $\Lambda$ for asymptotically large times, as requested by its very definition. Therefore, we computed its finite-time value

$$\Lambda(R,t)=\frac{1}{t}\left\langle \ln\left|\frac{\delta u(x_R(t),t)}{\delta u(0,0)}\right|\right\rangle, \qquad (14)$$

where $R=\ln|x+1|/t$, and $\langle\cdot\rangle$ is the average over different reference trajectories. As can be easily realized, the maximal accessible rate will decrease as $\ln L/t$. Therefore if the iteration time is doubled, the chain length should be increased by a factor $L$ in order to achieve the same maximal $R$. Due to the large amount of CPU time required by the iteration, it is thus not feasible to consider system sizes larger than $10^4$, and the accessible ranges of $R$ and $t$ values are limited by this constraint. Nevertheless, the results reported in Fig. 2 confirm that a linear behavior $\Lambda(R,t)=\lambda^*(t)-\alpha R$ is observed at each time and for $R$ not too small, but with an intercept $\lambda^*\neq\lambda$ due to the finite time of observation. Empirically, we found that this intercept converges according to the rule $\lambda^*(2nt)-\lambda^*(t)\simeq \text{const}/q^{n-1}$, for some $q>1$, so that we can extrapolate its asymptotic value $\lambda^*(\infty)$ on the basis of the available data. Indeed, for the situation reported in Fig. 2 ($\alpha=\frac{3}{2}$ and $\varepsilon=\frac{1}{3}$) we have estimated a value $\lambda^*(\infty)=0.335$, in excellent agreement with the corresponding Lyapunov exponent $\lambda=0.338$.

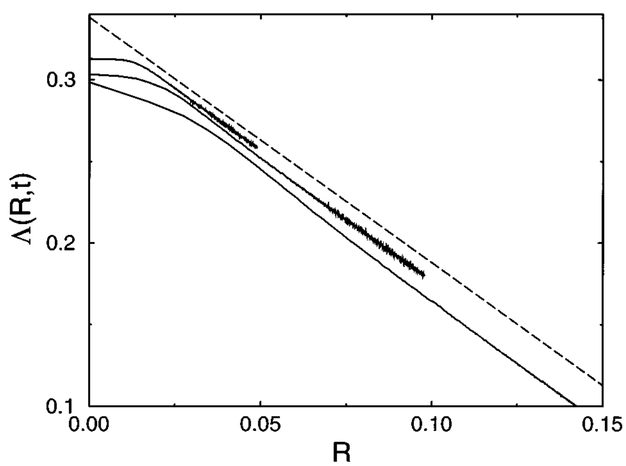


FIG. 2. The finite-time comoving Lyapunov exponents $\Lambda(R,t)$ as a function of rate $R$ at time $t=20$, 40, and 80 (from bottom to top), for coupled tent maps with $\alpha=\frac{3}{2}$, $L=5001$, and $\varepsilon=\frac{1}{3}$. The dashed line represents the asymptotic expression $\Lambda(R)=\lambda-\frac{3}{2}R$.

The deviation from Eq. (13) at small values of $R$ is due to transient effects. We have numerically observed that, for increasing $t$, the interval of $R$ values where deviations are observed reduces.

In conclusion, we have fully identified the mechanism that rules the disturbance propagation for systems with power law long-range couplings. This gives at any position in space an exponential increase in time, and a power law falloff with $x$. The power with which the perturbation decays, is, for generic initial conditions equal to the power describing the interaction falloff. Moreover, the time needed for the disturbance to propagate with finite amplitude at a given distance $l$ is inversely proportional to the Lyapunov exponents, and increases logarithmically with $l$.

As a final remark, we expect that this propagation mechanism can also be observed for fronts separating stable and unstable steady states, when a power law decay coupling is assumed for the spatial interaction. We hope that this work will stimulate further studies of front spreading into nonchaotic states of system with long-range coupling, since at the present moment a detailed analysis is still lacking [19].

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[1] M.C. Cross and P.H. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[2] P. Collet and J.P. Eckmann, *Instabilities and Fronts in Extended Systems* (Princeton University Press, Princeton, 1990).

[3] D. Mollison, J. R. Stat. Soc. B **39**, 283 (1977); and in *Mathematical Aspects of Rabies Epizootic*, edited by P.J. Bacon (Academic, London,1983).

[4] A.D. Kessler, J. Koplik, and H. Levine, Adv. Phys. **35**, 255 (1988).

[5] K. Kaneko, Prog. Theor. Phys. **72**, 980 (1984); I. Waller and

R. Kapral, Phys. Rev. A **30**, 2047 (1984).
[6] S. Wolfram *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).
[7] A. Kolmogorv, I. Petrovsky, and N. Piskunov, Bull. Univ. Moscow, Ser. Int. **A 1**, 1 (1937); D.G. Aronson and H.F. Weinberg, Adv. Math. **30**, 33 (1978).
[8] W. van Saarloos, Phys. Rev. A **37**, 211 (1988); **39**, 6367 (1989).
[9] R.D. Benguria and M.C. Depassier, Phys. Rev. Lett. **73**, 2272 (1994); A. Goriely, *ibid.* **75**, 2047 (1995); W. van Saarloos, M. van Hecke, and R. Holyst, Phys. Rev. E **52**, 1773 (1995).
[10] K. Kaneko, Prog. Theor. Phys. **74**, 1033 (1985).
[11] A.S. Pikovsky, Phys. Lett. A **156**, 223 (1991).
[12] A. Politi and A. Torcini, Europhys. Lett. **28**, 545 (1994).
[13] A. Torcini, P. Grassberger, and A. Politi, J. Phys. A **27**, 4533 (1995).
[14] A. Torcini, H. Frauenkron, and P. Grassberger, Physica D (to be published).
[15] S. Lepri, A. Politi, and A. Torcini, J. Stat. Phys. **82**, 1429 (1996).
[16] G. Paladin and A. Vulpiani, J. Phys. A **25**, 4911 (1994).
[17] A. Torcini, in *Chaos: The Interplay Betweeen Stochastic and Determistic Behaviour*, edited by P. Garbaczewski *et al.*, Lecture Notes in Physics, Vol. 487 (Springer-Verlag, Berlin, 1995), p. 537.
[18] R.J. Deissler and K. Kaneko, Phys. Lett. A **119**, 397 (1987).
[19] W. van Saarloos (private communication).