Supplemental Material on
"A reduction methodology for fluctuation driven population dynamics"

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CHARACTERISTIC FUNCTION AND PSEUDO-CUMULANTS

Here we report in full details the derivation of the model (10) , already outlined in the Letter, in terms of the characteristic function and of the associated pseudo-cumulants. In particular, the characteristic function for $V_{\boldsymbol{x}}$ is defined as

$$\mathcal{F}_{\boldsymbol{x}}(k) = \langle e^{ikV_{\boldsymbol{x}}} \rangle = \text{P.V.} \int_{-\infty}^{+\infty} e^{ikV} w(V,t|\boldsymbol{x})\, \mathrm{d}V \;,$$

which for a Lorentzian distribution becomes :

$$\text{P.V.} \int_{-\infty}^{+\infty} e^{ikV} \frac{a_{\boldsymbol{x}}}{\pi[a_{\boldsymbol{x}}^2 + (V - v_{\boldsymbol{x}})^2]} \mathrm{d}V = e^{ikv_{\boldsymbol{x}} - a_{\boldsymbol{x}}|k|} \;.$$

In order to derive the FPE in the Fourier space, let us proceed with a more rigourous definition of the characteristic function, namely

$$\mathcal{F}_{\boldsymbol{x}} \equiv \lim_{\varepsilon\to+0} \langle e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} \rangle \;.$$

Therefore by virtue of the FPE (Eq. (3) in the Letter) the time derivative of the characteristic function takes the form

$$\partial_t \mathcal{F}_{\boldsymbol{x}} = \lim_{\varepsilon\to+0} \text{P.V.} \int\limits_{-\infty}^{+\infty} e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} \frac{\partial w_{\boldsymbol{x}}}{\partial t} \mathrm{d}V_{\boldsymbol{x}} = - \lim_{\varepsilon\to+0} \text{P.V.} \int\limits_{-\infty}^{+\infty} e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} \frac{\partial}{\partial V_{\boldsymbol{x}}} \left((I_{\boldsymbol{x}} + V_{\boldsymbol{x}}^2) w_{\boldsymbol{x}} - \sigma_{\boldsymbol{x}}^2 \frac{\partial}{\partial V_{\boldsymbol{x}}} w_{\boldsymbol{x}} \right) \mathrm{d}V_{\boldsymbol{x}}$$

$$= - \lim_{\varepsilon\to+0} \lim_{B\to+\infty} \int\limits_{-B}^{B} e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} \frac{\partial}{\partial V_{\boldsymbol{x}}} \left((I_{\boldsymbol{x}} + V_{\boldsymbol{x}}^2) w_{\boldsymbol{x}} - \sigma_{\boldsymbol{x}}^2 \frac{\partial}{\partial V_{\boldsymbol{x}}} w_{\boldsymbol{x}} \right) \mathrm{d}V_{\boldsymbol{x}} \;.$$

Performing a partial integration, we obtain

$$\partial_t \mathcal{F}_{\boldsymbol{x}} = - \lim_{\varepsilon\to+0} \lim_{B\to+\infty} \left(e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} q_{\boldsymbol{x}}(V_{\boldsymbol{x}}) \Big|_{-B}^{B} - \int\limits_{-B}^{B} \frac{\partial e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|}}{\partial V_{\boldsymbol{x}}} q_{\boldsymbol{x}}(V_{\boldsymbol{x}}) \mathrm{d}V_{\boldsymbol{x}} \right) \;, \tag{1}$$

where the probability flux for the $\boldsymbol{x}$-subpopulation is defined as

$$q_{\boldsymbol{x}} = (I_{\boldsymbol{x}} + V_{\boldsymbol{x}}^2) w_{\boldsymbol{x}} - \sigma_{\boldsymbol{x}}^2 \frac{\partial w_{\boldsymbol{x}}}{\partial V_{\boldsymbol{x}}} \;.$$

As the membrane potential, once it reaches the threshold $+B$, is reset to $-B$ this sets a boundary condition on the flux, namely $q_{\boldsymbol{x}}(B) = q_{\boldsymbol{x}}(-B)$ for $B \to +\infty$; therefore,

$$e^{ikB - \varepsilon B} q_{\boldsymbol{x}}(B) - e^{-ikB - \varepsilon B} q_{\boldsymbol{x}}(-B) = 2ie^{-\varepsilon B} \sin kB\, q_{\boldsymbol{x}}(B) \overset{B\to+\infty}{\longrightarrow} 0$$

and the first term in Eq. (1) will vanish, thus the time derivative of the characteristic function is simply given by

$$\partial_t \mathcal{F}_{\boldsymbol{x}} = \lim_{\varepsilon\to+0} \lim_{B\to+\infty} \int\limits_{-B}^{B} ike^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} \left((I_{\boldsymbol{x}} + V_{\boldsymbol{x}}^2) w_{\boldsymbol{x}} - \sigma_{\boldsymbol{x}}^2 \frac{\partial w_{\boldsymbol{x}}}{\partial V_{\boldsymbol{x}}} \right) \mathrm{d}V_{\boldsymbol{x}} \;.$$

Hence, after performing one more partial integration for the remaining $V_{\boldsymbol{x}}$-derivative term, we obtain

$$\partial_t \mathcal{F}_{\boldsymbol{x}} = \lim_{\varepsilon\to+0} \text{P.V.} \int\limits_{-\infty}^{+\infty} e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} \left[ik \left(I_{\boldsymbol{x}} + V_{\boldsymbol{x}}^2 \right) w_{\boldsymbol{x}} - \sigma_{\boldsymbol{x}}^2 k^2 w_{\boldsymbol{x}} \right] \mathrm{d}V_{\boldsymbol{x}}$$

$$= ik \left(I_{\boldsymbol{x}} \mathcal{F}_{\boldsymbol{x}} + \lim_{\varepsilon\to+0} \text{P.V.} \int\limits_{-\infty}^{+\infty} e^{ikV_{\boldsymbol{x}} - \varepsilon|V_{\boldsymbol{x}}|} V_{\boldsymbol{x}}^2 w_{\boldsymbol{x}} \mathrm{d}V_{\boldsymbol{x}} \right) - \sigma_{\boldsymbol{x}}^2 k^2 \mathcal{F}_{\boldsymbol{x}} \tag{2}$$

and finally

$$\partial_t \mathcal{F}_{\boldsymbol{x}} = ik[I_{\boldsymbol{x}} \mathcal{F}_{\boldsymbol{x}} - \partial_k^2 \mathcal{F}_{\boldsymbol{x}}] - \sigma_{\boldsymbol{x}}^2 k^2 \mathcal{F}_{\boldsymbol{x}} \ , \tag{3}$$

which is Eq. (4) in the Letter.

Under the assumption that $\mathcal{F}_{\boldsymbol{x}}(k,t)$ is an analytic function of the parameters $\boldsymbol{x}$ one can calculate the average characteristic function for the population $F(k,t) = \int \mathrm{d}\eta \int \mathrm{d}J \mathcal{F}_{\boldsymbol{x}}(k,t) g(\eta) h(J)$ and the corresponding FPE via the residue theorem, with the caution that different contours have to be chosen for positive (upper half-planes of complex η and J) and negative k (lower half-planes). Hence, the FPE is given by

$$\partial_t F = ik \left[H_0 F - \partial_k^2 F \right] - |k| D_0 F - S_0^2 k^2 F \ , \tag{4}$$

where $H_0 = I_0 + \eta_0 + J_0 r$, $D_0 = \Delta_\eta + \Delta_J r$ and $S_0^2 = \sigma^2(\eta_0 + i\Delta_\eta k/|k|, J_0 + i\Delta_J k/|k|) = \mathcal{N}_R + i\mathcal{N}_I$.

For the logarithm of the characteristic function, $F(k) = e^{\Phi(k)}$, one obtains the following evolution equation

$$\partial_t \Phi = ik[H_0 - \partial_k^2 \Phi - (\partial_k \Phi)^2] - |k| D_0 - S_0^2 k^2 \ . \tag{5}$$

In this context the Lorentzian Ansatz amounts to set $\Phi_L = ikv - a|k|$ [1], by substituting Φ_L in (5) for $S_0 = 0$ one gets

$$\begin{aligned} \dot{v} &= H_0 + v^2 - a^2 \ , \\ \dot{a} &= 2av + D_0 \ , \end{aligned} \tag{6}$$

which coincides with the two dimensional mean-field model found in [2] with $r = a/\pi$.

In order to consider deviations from the Lorentzian distribution, we analyse the following general polynomial form for Φ:

$$\Phi = -a|k| + ikv - \sum_{n=2}^{\infty} \frac{q_n |k|^n + i p_n |k|^{n-1} k}{n} \quad . \tag{7}$$

The terms entering in the above expression are dictated by the symmetry of the characteristic function $\mathcal{F}_{\boldsymbol{x}}(k)$ for real-valued $V_{\boldsymbol{x}}$, which is invariant for a change of sign of k joined to the complex conjugation. For this characteristic function neither moments, nor cumulats can be determined [3].

Hence, we can choose the notation in the form which would be most optimal for our consideration. Specifically, we introduce $\Psi = k\partial_k \Phi$,

$$\Psi = -(a\mathrm{sign}(k) - iv)k - (q_2 + ip_2 \mathrm{sign}(k))k^2 - (q_3 \mathrm{sign}(k) + ip_3)k^3 - \dots \ . \tag{8}$$

Please notice that

$$\Psi(-k) = \Psi^*(k) \quad [\text{as well as } \Phi(-k) = \Phi^*(k)] \,. \tag{9}$$

In this context Eq. (5) becomes

$$\partial_t \Psi = ikH_0 - |k| D_0 - ik\partial_k \left(k\partial_k \frac{\Psi}{k} + \frac{\Psi^2}{k} \right) - 2S_0^2 k^2 \ . \tag{10}$$

It is now convenient to introduce the *pseudo-cumulants*, defined as follows:

$$W_1 \equiv a - iv \ , \qquad W_n \equiv q_n + ip_n \ . \tag{11}$$

From Eq. (10) we can thus obtain the evolution equation for the pseudo-cumulants W_m, namely

$$\dot{W}_m = (D_0 - iH_0)\delta_{1m} + 2(\mathcal{N}_R + i\mathcal{N}_I)\delta_{2m} + im\Big(-mW_{m+1} + \sum\nolimits_{n=1}^{m} W_n W_{m+1-n} \Big) \ , \tag{12}$$

where for simplicity we have assumed $k > 0$ and employed the property (9). Moreover, we have omitted the $k\delta(k)$ contribution, since it vanishes.

The evolution of the first two pseudo-cumulant reads as:

$$\dot{W}_1 = D_0 - iH_0 - iW_2 + iW_1^2 \ , \tag{13}$$

$$\dot{W}_2 = 2(\mathcal{N}_R + i\mathcal{N}_I) + 4i(-W_3 + W_2 W_1) \ . \tag{14}$$

Or equivalently

$$\dot{r} = (\Delta_\eta + \Delta_J r + p_2)/\pi + 2rv \ , \tag{15a}$$
$$\dot{v} = I_0 + \eta_0 + J_0 r - \pi^2 r^2 + v^2 + q_2 \ , \tag{15b}$$
$$\dot{q}_2 = 2\mathcal{N}_R + 4(p_3 + q_2 v - \pi p_2 r) \ , \tag{15c}$$
$$\dot{p}_2 = 2\mathcal{N}_I + 4(-q_3 + \pi q_2 r + p_2 v) \ , \tag{15d}$$

which is Eq. (10) in the Letter.

FIRING RATE AND MEAN MEMBRANE POTENTIAL FOR PERTURBED LORENTZIAN DISTRIBUTIONS

In the following we will demonstrate that the definitions of the firing rate r and of the mean membrane potential v in terms of the PDF $w(V,t)$, namely:

$$r = \lim_{V\to\infty} V^2 w(V,t) \qquad \text{and} \qquad v = \text{P.V.} \int_{-\infty}^{+\infty} V w(V,t)\,\mathrm{d}V \ ,$$

obtained in [2] for a Lorentzian distribution, are not modified even by including in the PDF the correction terms $\{q_n, p_n\}$.

The probability density for the membrane potentials $w(V,t)$ is related to the characteristic function $F(k)$ via the follwoing anti-Fourier transform

$$w(V,t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} F(k)\, e^{-ikV} \mathrm{d}k$$

with $F(k) = e^{\Phi(k)}$. By considering the deviations of $\Phi(k)$ from the Lorentzian distribution up to the second order in k, we have

$$\begin{aligned} 2\pi \ w(V,t) &= \int_{-\infty}^{+\infty} e^{ikv - a|k| - q_2\frac{k^2}{2} - ip_2\frac{k|k|}{2}} e^{-ikV} \mathrm{d}k \\ &\approx \int_{-\infty}^{+\infty} e^{-iky - a|k|} \left(1 - q_2 \frac{k^2}{2} - ip_2 \frac{k|k|}{2}\right) \mathrm{d}k \\ &= \int_{-\infty}^{+\infty} \left(1 + \frac{q_2}{2}\left[(1-\theta)\frac{\partial^2}{\partial y^2} - \theta \frac{\partial^2}{\partial a^2}\right] - \frac{p_2}{2}\frac{\partial^2}{\partial y \partial a}\right) e^{-iky - a|k|} \mathrm{d}k \ , \end{aligned}$$

where $y = V - v$ and θ is an arbitrary parameter. Thus one can rewrite

$$w(y,t) \approx \left(1 + \frac{q_2}{2}\left[(1-\theta)\frac{\partial^2}{\partial y^2} - \theta\frac{\partial^2}{\partial a^2}\right] - \frac{p_2}{2}\frac{\partial^2}{\partial y \partial a}\right) \frac{a}{\pi(a^2 + y^2)} \ . \tag{16}$$

From the expression above, it is evident that q_2 and p_2, as well as the higher-order corrections, do not modify the firing rate definition reported in [2] for the Lorentzian distribution, indeed

$$r = \lim_{V\to\infty} V^2 w(V,t) = \frac{a}{\pi} \ .$$

Let us now estimate the mean membrane potential by employing the PDF (16), where we set the arbitrary parameter θ to zero without loss of generality, namely

$$w(V,t) = \left(1 + \frac{q_2}{2}\frac{\partial^2}{\partial V^2} - \frac{p_2}{2}\frac{\partial^2}{\partial V \partial a} + \dots\right) w_0(V,t) \ , \tag{17}$$

where $w_0(V,t) = \pi^{-1} a/[a^2 + (V-v)^2]$. The mean membrane potential is given by

$$\begin{aligned} \langle V \rangle = \text{P.V.} \int_{-\infty}^{+\infty} V w(V,t)\,\mathrm{d}V &= \text{P.V.} \int_{-\infty}^{+\infty} \left(V w_0 - \frac{q_2}{2}\frac{\partial w_0}{\partial V} + \frac{p_2}{2}\frac{\partial w_0}{\partial a} + \dots\right) \mathrm{d}V \\ &= v - \frac{q_2}{2} \int_{-\infty}^{+\infty} \frac{\partial w_0}{\partial V} \mathrm{d}V + \frac{p_2}{2}\frac{\partial}{\partial a} \int_{-\infty}^{+\infty} w_0 \mathrm{d}V + \dots \\ &= v - \frac{q_2}{2} w_0|_{-\infty}^{+\infty} + \frac{p_2}{2}\frac{\partial\, 1}{\partial a} + \dots = v \ . \end{aligned} \tag{18}$$

All the higher-order corrections entering in $w(V,t)$, denoted by $(\dots)$ in (18), have the form of higher-order derivatives of w_0 with respect to V and a; therefore they yield a zero contribution to the estimation of $\langle V\rangle$. Thus, Eq. (18) is correct not only to the 2nd order, but also for higher orders of accuracy. We can see that the interpretation of the macroscopic variables $a=\pi r$ and $v=\langle V\rangle$ in terms of the firing rate and of the mean membrane potential entering in Eq. (12) or Eqs. (15a)–(15d) remains exact even away from the Lorentzian distribution.

SMALLNESS HIERARCHY OF THE PSEUDO-CUMULANTS

Eq. (12) for $m>1$ can be recast in the following form

$$\dot{W}_{m>1} = 2m(v+i\pi r)W_m + 2(\mathcal{N}_R + i\mathcal{N}_I)\delta_{2m} + im\Big(-mW_{m+1} + \sum\nolimits_{n=2}^{m-1} W_n W_{m+1-n}\Big) , \tag{19}$$

where W_m is present only in the first term of the right-hand side of the latter equation.

Let us now understand the average evolution of W_m, $m>1$. In particular, by dividing Eq. (15a) by r and averaging over time, one finds that

$$\langle v\rangle_t = -\frac{1}{2\pi}\left(\Delta_\eta\langle r^{-1}\rangle_t + \langle p_2 r^{-1}\rangle_t + \Delta_J\right) , \tag{20}$$

where $\langle\dots\rangle_t$ denotes the average over time and where we have employed the fact that the time-average of the time-derivative of a bounded process is zero, i.e. $\langle\frac{\mathrm{d}}{\mathrm{d}t}\ln r\rangle_t = 0$. Since $r(t)$ can be only positive, $\langle v\rangle_t$ will be strictly negative for a heterogeneous population (with $\Delta_\eta\neq 0$ and/or $\Delta_J\neq 0$) in the case of nonlarge deviations from the Lorentzian distribution, i.e., when p_2 is sufficiently small. In particular, for asynchronous states $v=\langle v\rangle_t$, hence, Eq. (20) yields a relaxation dynamics for W_m under forcing by W_{m+1} and $W_1,\dots,W_{m-1}$; by continuity, this dissipative dynamics holds also for oscillatory regimes which are not far from the stationary states.

Let us explicitly consider the dynamics of the equations (19) for $m=2,3,4$, namely:

$$\dot{W}_2 = 4(v+i\pi r)W_2 - i4W_3 + 2(\mathcal{N}_R + i\mathcal{N}_I) , \tag{21}$$
$$\dot{W}_3 = 6(v+i\pi r)W_3 + i3W_2^2 - i9W_4 , \tag{22}$$
$$\dot{W}_4 = 8(v+i\pi r)W_4 + i8W_2W_3 - i16W_5 , \tag{23}$$
$$\dots .$$

Let us first see how the attractivity of the Lorentzian distribution in the absence of noise follows from these equations. For $\mathcal{N}_R=\mathcal{N}_I=0$, we consider a small deviation from the Lorentzian distribution such that $|W_n|<C\varepsilon^{n-1}$, where C is some positive constant and $\varepsilon\ll 1$ is a smallness parameter. In this case, from Eq. (21) one observes that W_2 tends to $\sim W_3$, while from Eq. (22), $W_3\to\sim W_2^2$. Here W_4 is neglected as for the initial conditions one finds $|W_2^2|\sim\varepsilon^2\gg|W_4|\sim\varepsilon^3$ and below we will see that similar relation remains valid over time. Therefore, $W_2\to\sim W_2^2$, which means that $W_2(t\to+\infty)\to 0$. Further, from Eq. (23), $W_4\to\sim W_2W_3\sim W_2^3\to 0$. Here W_5 is neglected as for the initial conditions one finds $|W_2W_3|\sim\varepsilon^3\gg|W_5|\sim\varepsilon^4$. In the course of evolution W_4 tends to $\sim W_2^3$, while W_3 tends to $\sim W_2^2$, one can similarly show that $W_5\to\sim W_2^4$, and so forth. Hierarchy $W_n\sim W_2^{n-1}$ is similar to $W_n\sim\varepsilon^{n-1}$ and allows us to neglect W_4 in $\dot{W}_3$, W_5 in $\dot{W}_4$, and so forth, not only for the initial stage of the dynamics, but also at a later time. Thus, in the absence of noise, the system tends to a state $W_1\neq 0$, $W_{m>1}=0$ (at least from a small but finite vicinity of this state). This tells us that the Lorentzian distribution is an attractive solution in this case.

In the presence of noise, by assuming that $|\mathcal{N}_R+i\mathcal{N}_I|\sim\sigma^2$, a similar analysis of Eqs. (21)–(23) yields $|W_2|\to\sim\sigma^2$, $|W_3|\to\sim|W_2^2|\sim\sigma^4$, ...,

$$|W_m|\to\sim\sigma^{2(m-1)} .$$

The above scaling is well confirmed by the data reported in Fig. 1. Therefore, a two-element truncation (13)–(14) of the infinite equation chain (12) is well justified as a first significant correction to the Lorentzian distribution dynamics. Presumably, this might also hold for some regimes in homogeneous populations (where $\Delta_\eta=\Delta_J=0$), even thought the heuristic explanation we provide here heavily relays on the positivity of $(\Delta_\eta\langle r^{-1}\rangle_t+\Delta_J)$ in Eq. (20).

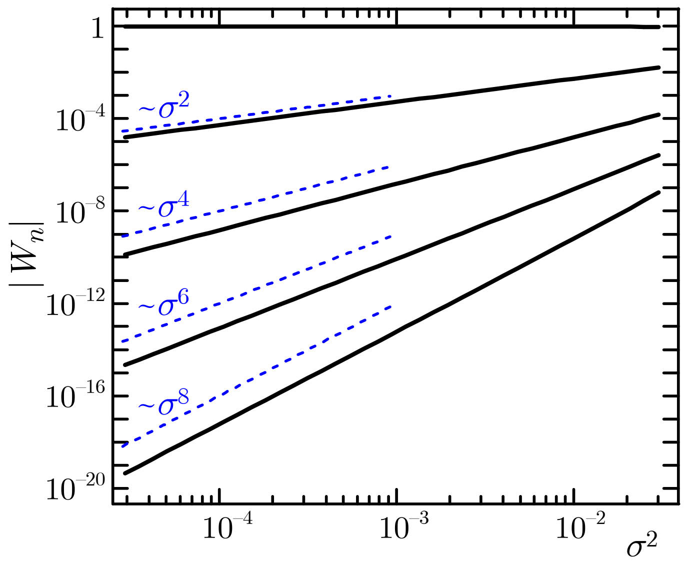

FIG. 1. Modulus of the pseudo-cumulants $|W_n|$ versus the noise variance σ^2 for n ranging from 1 to 5 from top to bottom. The pseudo-cumulants are estimated by integrating Eq. (10) in the Letter with extended precision (30 digits) and by limiting the sum to the first 100 elements. Other parameters: $I_0 = 0.1$, $\eta_0 = -1$, $J_0 = 1$, $\Delta_\eta = 0.1$, and $\Delta_J = 0.1$.

CONVENTIONAL AND PSEUDO- CUMULANTS

The relationships between conventional and pseudo- cumulants

Let us discuss the relationships existing between conventional and pseudo-cumulant representations. The characteristic function $F(k)$ and its logarithm $\Phi(k)$ of any *real-valued* random variable V must obey the symmetry properties $F(k) = F^*(-k)$ and $\Phi(k) = \Phi^*(-k)$. Hence, the expression (7) for the function $\Phi(k)$ in terms of pseudo-cumulants is the most general one respecting such symmetry

$$\Phi(k) = -\sum_{n=1}^{\infty} \frac{q_n|k|^n + ip_n|k|^{n-1}k}{n} \quad ;$$

where we set $q_1 = a$ and $p_1 = -v$.

Thus, the pseudo-cumulants can be expressed as derivative of $\Phi(k)$, as follows :

$$W_n = \frac{-1}{(n-1)!}\frac{d^n\Phi(k=0)}{dk^n} \ \text{ for } \ k > 0, \qquad W_n^* = \frac{-(-1)^n}{(n-1)!}\frac{d^n\Phi(k=0)}{dk^n} \ \text{ for } \ k < 0, \tag{24}$$

where $W_n^* = q_n - ip_n$ is the complex conjugate of W_n.

However, the expression (7) does not guarantee the existence of conventional cumulants and moments. The existence of the n-th moment $M_n = \langle V^n \rangle$ requires that the PDF $w(V)$ will decay faster than $1/|V|^{n+1}$ for $V \to \pm\infty$. If the n-th moment is finite, the derivative $\frac{\mathrm{d}^n}{\mathrm{d}k^n}F(k=0) = i^n M_n$ exists and is continuous (as well as all lower-order derivatives). The finiteness of $\frac{\mathrm{d}^n}{\mathrm{d}k^n}F(k=0)$ forbids the existence of all the terms $|k|^{2m-1}$ with odd $(2m-1) \le n$ and $|k|^{2m-1}k$ with even $2m \le n$, meaning that only smooth terms k^{2m-1} and k^{2m} are present up to the n-th power (the k^n-term included). Therefore, in this case we will have

$$K_l = \begin{cases} -\frac{(l-1)!p_l}{i^l}\ , & \text{for odd } l \le n\,; \\ -\frac{(l-1)!q_l}{i^l}\ , & \text{for even } l \le n\,; \end{cases} \tag{25}$$

where K_l is the l-th order *conventional cumulant*, defined from the expansion $\Phi(k) = \sum_{n=1}^{\infty} K_n \frac{(ik)^n}{n!}$.

To summarize, if the distribution $w(V)$ decays as $1/|V|^{n+1+\alpha}$ with $0 < \alpha \le 1$ then we have the following situation. The expansion of $\Phi(k)$ contains for the orders up to $l \le n$ only the conventional-cumulant part, $K_1 + ... + K_l(ik)^l/l! + ...$, and the pseudo-cumulants are purely real (imaginary) for even (odd) l in agreement with (25). For the orders $l > n$, conventional moments and cumulants diverge and the pseudo-cumulants W_l have generally both the real q_l and imaginary p_l parts given by (24).

In particular, if the distribution $w(V)$ decays faster that any power law (e.g., exponentially fast) then all the moments are finite and all pseudo-cumulants have only the conventional-cumulant part.

Geometric interpretation of the second pseudo-cumulant

The small-k behavior of the characteristic function $F(k)$, which is the Fourier transform of the PDF $w(V)$, represents the large-scale properties of $w(V)$. In particular, the leading part of the expansion reads as

$$\Phi(k) = \ln F(k) = -a|k| + ikv - q_2\frac{k^2}{2} - ip_2\frac{k|k|}{2} + \dots\,,$$

where the terms $\sim k$ characterize the 'Lorentzian' profile of the distribution; while the second order terms $\sim k^2$ represent the leading corrections to the distribution for large V.

In particular, $q_2 = \mathrm{Re}(W_2)$ can be interpreted as an analogue of the *kurtosis* for distorted Gaussian distributions. Indeed, Eq. (17) reads as $w = w_0 + q_2\left(\frac{3a}{\pi[a^2+(V-v)^2]^2} - \frac{4a}{\pi[a^2+(V-v)^2]^3}\right) - \frac{p_2}{2}\frac{\partial^2 w_0}{\partial V\partial a} + \dots$ and $q_2 > 0$ implies an increase of the deviations from w_0 at large $|V|$.

The p_2-term is odd in k and therefore represents the asymmetry in $w(V)$ between V and $-V$; this is derived from the fact that $F(-k) = \langle e^{ik(-V)}\rangle$. Moreover, from Eq. (16) one can see, that the median of the distribution $w(V)$ is not affected by the p_2-term. This can be easily shown by evoking the expression for $w(V)$ reported in (17) and by noticing that, according to the definition of a median, the integral of the Lorentzian distribution $w_0(V)$ over the half-axis $V \in [v; +\infty)$ equals 1/2. Indeed, the following integral vanishes

$$\int_v^{+\infty}\frac{\partial^2}{\partial V\partial a}w_0(V)\,\mathrm{d}V = \frac{\partial}{\partial a}w_0(V)\bigg|_v^{+\infty} = \frac{\partial}{\partial a}\frac{1}{2} = 0\,.$$

Hence, the integral of $w(V)$ over the half-axis $V \in [v; +\infty)$ is still 1/2 and not modified by the p_2-term. Similarly the median is not affected by the q_2-term, as $\int_v^{+\infty}\frac{\partial^2}{\partial V^2}w_0(V)\,\mathrm{d}V = \frac{\partial}{\partial V}w_0(V)\big|_v^{+\infty} = 0$. Therefore, v remains the median of the distribution. Thus, $p_2 = \mathrm{Im}(W_2)$ measures the asymmetry of $w(V)$ for a fixed median value given by $v = -\mathrm{Im}(W_1)$ and therefore it can be interpreted as the *skewness* of the distorted distribution.

In the specific case of QIFs, the integrals over V are defined as the principal value ones and the mean value of V can be calculated also for a distribution with Lorentzian tails. Thus, here one can speak not only of the median—this interpretation will be valid universally—but also of the mean value; as one can see from Eq. (18), the p_2-term does not shift the population-mean value $\langle V\rangle$.

A REFERENCE SCALE FOR THE NOISE

Let us reconsider the MF equations (15), in particular equation (15a) divided by r and averaged over time yields

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\ln r\right\rangle_t = \left\langle\frac{\Delta_\eta}{\pi r}\right\rangle_t + \frac{\Delta_J}{\pi} + 2\langle v\rangle_t + \left\langle\frac{p_2}{\pi r}\right\rangle_t\,.$$

As we have already mentioned $\langle\frac{\mathrm{d}}{\mathrm{d}t}\ln r\rangle_t = 0$ and r is always positive, therefore for $p_2 \ll 1$, one finds $\langle v\rangle_t < 0$.

To clarify the relevance of the noise term, one can rewrite Eqs. (15c)–(15d) as

$$\dot{q}_2 = -4(-v)q_2 - 4\pi r p_2 + 2\mathcal{N}_R + 4p_3\,, \tag{26a}$$

$$\dot{p}_2 = +4\pi r q_2 - 4(-v)p_2 + 2\mathcal{N}_I - 4q_3\,. \tag{26b}$$

The latter system yields on average a linearly decaying dynamics with average decay rate $-4\langle v\rangle_t$ plus a counterclockwise rotation on the plane (q_2, p_2) with instantaneous angular velocity $4\pi r$ and with constant driving terms given by $2\mathcal{N}_R$ and $2\mathcal{N}_I$ plus small (q_3, p_3) corrections. Therefore, the effect of noise is fundamental in order to obtain non-vanishing values for the terms q_2 and p_2.

Indeed, this becomes evident in the stationary case, where by neglecting (q_3, p_3) in Eqs. (26), one obtains

$$\bar{q}_2 = -\frac{\mathcal{N}_R \bar{v} + \mathcal{N}_I \pi \bar{r}}{2(\bar{v}^2 + \pi^2 \bar{r}^2)} , \qquad \bar{p}_2 = \frac{\mathcal{N}_R \pi \bar{r} - \mathcal{N}_I \bar{v}}{2(\bar{v}^2 + \pi^2 \bar{r}^2)} . \tag{27}$$

In the case of globally coupled network with additive noise of amplitude σ, where $\mathcal{N}_R = \sigma^2$ and $\mathcal{N}_I = 0$, the terms $(\bar{q}_2, \bar{p}_2)$ are directly proportional to the noise variance σ^2, as already shown in Fig. 1 (c-d) in the Letter.

An estimation of a reference value for the noise strength can be obtained by considering the effect of the stationary terms (27) on the evolution of the firing rate and of the mean membrane potential given by Eqs. (15a)–(15b). In this case we can give a clear physical interpretation of the stationary corrections $\bar{q}_2$ and $\bar{p}_2$. They can be interpreted as a measure of an additional source of heterogeneity in the system induced by the noise. To be more precise, $\bar{q}_2$ $(\bar{p}_2)$ can be considered as a modification of the mean input current in (15b) (of the width of the distribution of the heterogeneities in (15c)) and therefore it should be compared with the median of the effective input current $I_0 + \eta_0 + J_0 r$ (with the HWHM of the effective input currents $\Delta_\eta + \Delta_J r$) appearing in the same equation.

A first reference value for the noise strenght $(\mathcal{N}_R^*, \mathcal{N}_I^*)$ can be estimated from (15a) by setting $|\bar{p}_2| = \Delta_\eta + \Delta_J \bar{r}_0$, thus obtaining

$$|\mathcal{N}_R^* \pi \bar{r}_0 - \mathcal{N}_I^* \bar{v}_0| = 4\pi |\bar{v}_0| \bar{r}_0 (\bar{v}_0^2 + \pi^2 \bar{r}_0^2) , \tag{28}$$

where $\bar{v}_0$ and $\bar{r}_0$ are the time-independent solution of Eqs. (15a)–(15b) for $\mathcal{N}_R = \mathcal{N}_I = 0$.

From (15b) a second reference scale for the noise $(\mathcal{N}_R^{**}, \mathcal{N}_I^{**})$ can be derived by setting $|\bar{q}_2| = |I_0 + \eta_0 + J_0 \bar{r}_0|$, thus obtaining

$$|\mathcal{N}_R^{**} \bar{v}_0 + \mathcal{N}_I^{**} \pi \bar{r}_0| = |\mathcal{N}_R^* \pi \bar{r}_0 - \mathcal{N}_I^* \bar{v}_0| \frac{|I_0 + \eta_0 + J_0 \bar{r}_0|}{2\pi |\bar{v}_0| \bar{r}_0} . \tag{29}$$

The solution of the above equation, which would define the second scale, is proportional to the solution of (28) which will set $(\mathcal{N}_R^*, \mathcal{N}_I^*)$, therefore it is justified to consider only the latter ones as the reference values for the noise amplitude. In the following we will estimate this reference noise for the specific cases considered in the Letter.

Globally coupled network with extrinsic noise

For the globally coupled network with addtive noise we have $(\mathcal{N}_R = \sigma^2, \mathcal{N}_I = 0)$, therefore the reference noise strenght σ_* is given by

$$\sigma_*^2 = 4|\bar{v}_0|(\bar{v}_0^2 + \pi^2 \bar{r}_0^2) , \tag{30}$$

and the second noise scale σ_{**} is obtained from (29)

$$\sigma_{**}^2 = \sigma_*^2 \frac{|I_0 + \eta_0 + J_0 \bar{r}_0|}{2\bar{v}_0^2} ; \tag{31}$$

which confirms that we can limit to consider as a scale for the noise σ_*, since the second noise variance value is proportional to the first one.

Let us restrict our analysis to $\Delta_\eta = 0$, where $\bar{v}_0 = -\frac{\Delta_J}{2\pi}$, and $\bar{r}_0 = \frac{J_0 + \sqrt{J_0^2 + 4\pi^2(I_0+\eta_0) + \Delta_J^2}}{2\pi^2}$. In this case we have an explicit expression for the reference noise scale, i.e.

$$\sigma_*^2\big|_{\Delta\eta=0} = \frac{\Delta_J}{\pi^3} \left(2\pi^2 (I_0 + \eta_0) + \Delta_J^2 + J_0^2 + J_0 \sqrt{J_0^2 + 4\pi^2 (I_0 + \eta_0) + \Delta_J^2} \right) . \tag{32}$$

In particular, for the parameters employed in Fig. 1 (a-e) in the Letter, i.e. $I_0 = 0.0001$, $\eta_0 = 0$, $J_0 = -0.1$, $\Delta_J = 0.1$, one obtains $\sigma_* \approx 0.00458$, while for those employed in Fig. 1 (f-g) in the Letter, i.e. $I_0 = 0.38$, $\eta_0 = 0$ $J_0 = -6.3$, $\Delta_J = 0.01$, one finds $\sigma_* \approx 0.014$. In Fig. 1 of the Letter we have employed the above values σ_* to rescale the noise amplitudes as $\tilde{\sigma} = \sigma/\sigma_*$. The correctness of the choice of this reference values is confirmed from the fact that in the asynchronous state, analysed in Fig. 1 (a-b), the deviations from the MPR results (magenta dashed lines) become evident for $\tilde{\sigma} \simeq \mathcal{O}(1)$.

Sparse networks exhibiting endogenous fluctuations

For the sparse deterministic network under the Poissonian approximation for the input spike trains, we can write $\mathcal{N}_R = \frac{J_0^2 r}{2K}$ and $\mathcal{N}_I = -\Delta_0 \mathcal{N}_R$, with $\Delta_J = \Delta_0 |J_0|$. In this case, the reference scale for the noise is simply given by

$$\mathcal{N}_R^* = \frac{4\pi \bar{r}_0 |\bar{v}_0| (\bar{v}_0^2 + \pi^2 \bar{r}_0^2)}{\pi \bar{r}_0 + \Delta_0 \bar{v}_0} \; , \tag{33}$$

since $\mathcal{N}_I$ is directly proportional to $\mathcal{N}_R$.

Once more we consider the case $\Delta_\eta = 0$, where we have an explicit expression for the mean membrane potential $\bar{v}_0 = -\frac{J_0 \Delta_0}{2\pi}$ and for the firing rate $\bar{r}_0 = \frac{J_0 + \sqrt{J_0^2 + 4\pi^2 (I_0 + \eta_0) + J_0^2 \Delta_0^2}}{2\pi^2}$. More specifically, for the parameters employed in Fig. 2 in the Letter, namely $K = 5000$, $\Delta_0 = 0.01$ and $\eta = 0$, we obtain for a large coupling value $J_0 = -5$ $\mathcal{N}_R^* \simeq 0.00039$ for $I_0 = 0.19$ as in panel (b) and $\mathcal{N}_R^* \simeq 0.002311$ for $I_0 = 0.50$ as in panel (c). In these two specific cases, we measured the corresponding average firing rates and from these values we have obtained an estimate of the average $\mathcal{N}_R$ and of the corresponding rescaled noise amplitude $\tilde{\mathcal{N}}_R = \mathcal{N}_R / \mathcal{N}_R^*$. In particular, for $J_0 = -5$ we found $\tilde{\mathcal{N}}_R \simeq 0.27$ ($\tilde{\mathcal{N}}_R \simeq 0.11$) for the case reported in panel (b) (panel (c)). This difference in the $\tilde{\mathcal{N}}_R$-values explains why for corresponding synaptic coupling the quantitative agreement between network simulations and MF results is worst in panel (b).

NOISY HOMOGENEOUS POPULATIONS

The Ott-Antonsen approach [4], as well as the MPR mean-field model [2], have been derived for heterogeneous deterministic systems and it is known that the corresponding reduced manifolds are no more attractive for homogeneous populations [5].

Our approach has been developed for heterogenous noisy populations: an important question is if and when it can be extended to the case of populations of identical element. This specific point goes beyond the scopes of this Letter and it will be addressed in a future publication [6]. However,here report preliminary analyses showing that there are situations where our MF model reproduces perfectly the network dynamics even in completely homogenous situations. Two examples are shown in Fig. 2 (a-d) for a fully coupled network with additive Gaussian noise for two different noise amplitudes. In these specific examples the external DC current is initially set to $I_0 = 2$, where the system reveals an asynchronous dynamics, corresponding to a stable focus in the MF. At a time $t = 20$ current I_0 is increased to a value 4, where the system displays COs, and maintained at such value for a certain time interval and then restored to the initial value. As evident from the figures, the MF evolution is in perfect agreement with the network dynamics, apart finite size fluctuations, and it is even able to capture the relaxation oscillations towards the stable focus at times $t > 50$.

In Fig. 2 (e) we consider a cut at constant noise amplitude $\sigma = 0.006$ in the phase diagram reported in Fig. 1 (e) of the Letter. In particular, we examine the evolution of the network and MF dynamics by decreasing adiabatically ΔJ from an initial finite value ($\Delta J = 0.1$) to a vanishingly small value of ΔJ, then we increase again adiabatically the parameter back to the initial value. For the network we can reach $\Delta J = 0$ (the homogeneous case), while the MF exhibits diverging solutions for $\Delta J \to 0$. However, the MF captures the Hopf sub-critical bifurcation from the asynchronous dynamics to COs at $\Delta J_{HB} \simeq 0.0089$ (black solid line in Fig. 1 (e) of the Letter) as well as the saddle-node of limit cycles at $\Delta J_{SN} \simeq 0.06255$ (red solid line in Fig. 1 (e) of the Letter) displayed also by the network, apart finite size corrections (as shown in Fig. 2 (e)). Furthermore, the standard deviation of the mean membrane potential Σ_v are reasonably well reproduced down to $\Delta J \simeq 0.02$, i.e. for systems that we can consider *de facto* as homogenous due to the quite large value of the median of the synaptic coupling, namely $J_0 = -6.3$. Therefore, it is true that in this case the MF gives diverging solutions in the homogenous case, however the homogenous solutions are essentially indistinguishable from the heterogeneous one at $\Delta J = 0.02$, where the MF still gives reasonable results.

From our preliminary analyses [6], it emerges that the homogeneous case is better captured by a MF approach for not vanishingly small values of the firing rate. This seems consistent with our findings in the present case. Indeed, in Fig. 2 (c) and (d) the population firing rate is $r \simeq 0.5 - 1.0$, while for the case reported in Fig. 2 (e) $r \simeq 0.05 - 0.08$, i.e. much smaller. A *theoretical* criterion for the applicability of the MF formulation can be formulate as follows: if the population-mean firing rate (*or* other mean fields driving the macroscopic dynamics) as a function of a small parameter (e.g., the noise intensity) can be represented by a power series, the approach can be safely employed in for noisy homogenous populations. As an example, if the firing rate follows a law like $\sigma^n \exp(-A/\sigma^2)$ such a dependence

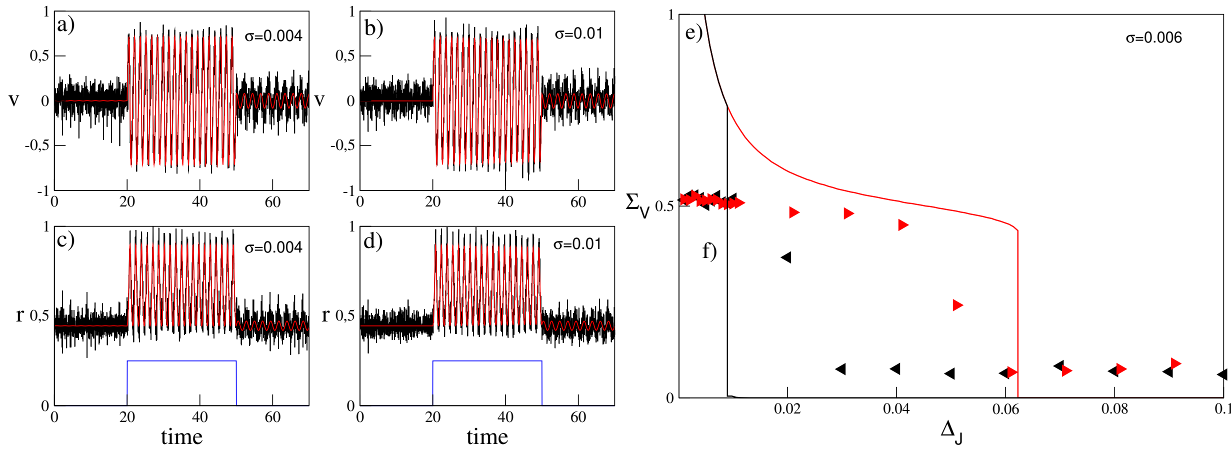

FIG. 2. **Homogeneous Globally Coupled Network subject to Additive Noise** (a-d) Red (black) solid line show the time course of v (a-c) and r (b-d) of the MF (of the network with $N = 10000$). Panels (a) and (c) refer to $\sigma = 0.004$, panels (b) and (d) to $\sigma = 0.01$. The external current is $I_0 = 4$ for time $\in [20, 50]$ and $I_0 = 2$ otherwise, I_0 is shown as blue solid line in panels (c) and (d) after a suitable rescaling. Other parameters: $J_0 = -0.1$ and $\eta_0 = \Delta_J = \Delta_\eta = 0$. (e) Standard deviation Σ_v versus Δ_J a noise amplitudes $\sigma = 0.006$. Lines (symbols) refer to MF (network) results: solid red (black) lines and right (left) triangles are obtained by increasing (decreasing) Δ_J. Other parameters are as in Fig.1 (f) of the Letter: namely, $I_0 = 0.38$, $J_0 = -6.3$ and $\eta_0 = \Delta_\eta = 0$.

cannot be represented by its power series, which formally reads as $\sigma^n(0 + 0 \cdot \sigma^2 + 0 \cdot \sigma^4 + \dots)$ [7]. Formal application of the pseudo-cumulant approach to this case will yield $r = 0$.

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[1] The Fourier transform of the Lorentzian distribution is P.V. $\int_{-\infty}^{+\infty} e^{ikV} \frac{a}{\pi[a^2+(V-v)^2]} \mathrm{d}V = e^{ikv-a|k|}$.
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[7] Function $\sigma^n \exp(-A/\sigma^2)$ possesses an essential singularity at $\sigma = 0$ even though it is infinitely smooth at this point.