Hysteretic Transitions in the Kuramoto Model with Inertia

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Pteroptix Malaccae

Usually, entrainment results in a constant phase angle equal to the difference between pacing frequency and free-running period as it does in P. cribellata. The mechanism of attaining synchrony by Malaysian firefly Pteroptyx malaccae is quite different. When the pacer changes, this firefly requires several cycles to reach a steady state. Once this steady state is achieved, the phase angle difference is near zero irrespective of the pacer period. This can be explained only by the animal adjusting the period of its oscillator to equal that of the driving oscillator. (experiments by Hanson, 1987)

- A phase model with inertia allows for adaptation of its frequency to the forcing one
  B. Ermentrout, Journal of Mathematical Biology 29 , 571 (1991)

Plan of the Talk

- Introduction of the Kuramoto model with inertia
- Analogy with the damped oscillator (coexistence of stable periodic and fixed point solutions)
- Mean field theory of the hysteretic transition (Tanaka, Lichtenberg, Oishi 1997)
- Fully coupled network of N oscillators
  - Existence of clusters of locked oscillators of any size between the hysteretic curves
  - Limits of stability of the coherent and incoherent solutions (dependence on the size N and on the mass m)
  - Emergence of drifting clusters
- Diluted network
- Italian high voltage power grid

The Model

Kuramoto model with inertia

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i)$$

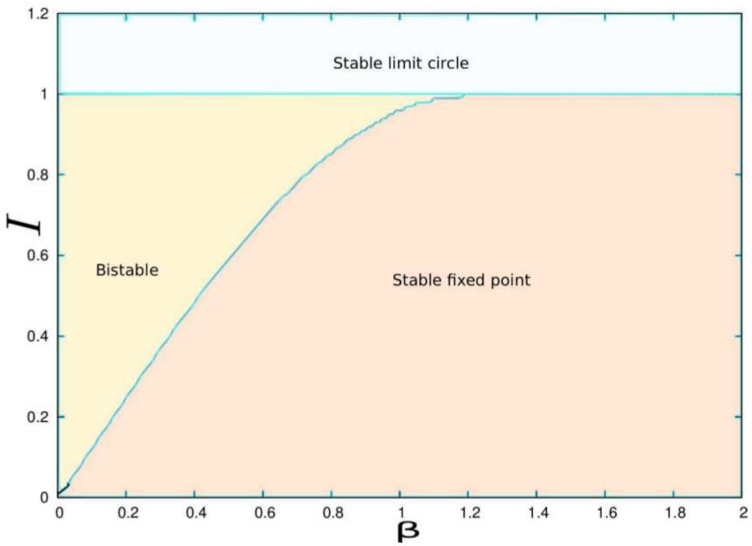
- θ_i is the instantaneous phase
- Ω_i is the natural frequency of the $i-$th oscillator with Gaussian distribution
- K is the coupling constant
- N is the number of oscillators

By introducing the complex order parameter $r(t)e^{i\phi(t)} = \frac{1}{N}\sum_j e^{i\theta_j}$

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i - Kr\sin(\theta_i - \phi)$$

$r = 0$ asynchronous state, $r = 1$ synchronized state

Damped Driven Pendulum



$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i - Kr\sin(\theta_i)$$

$$I = \frac{\Omega_i}{Kr}$$
$$\beta = \frac{1}{\sqrt{mKr}}$$

$$\ddot{\phi} + \beta\dot{\phi} = I - \sin(\phi)$$

For sufficiently large m (small β)

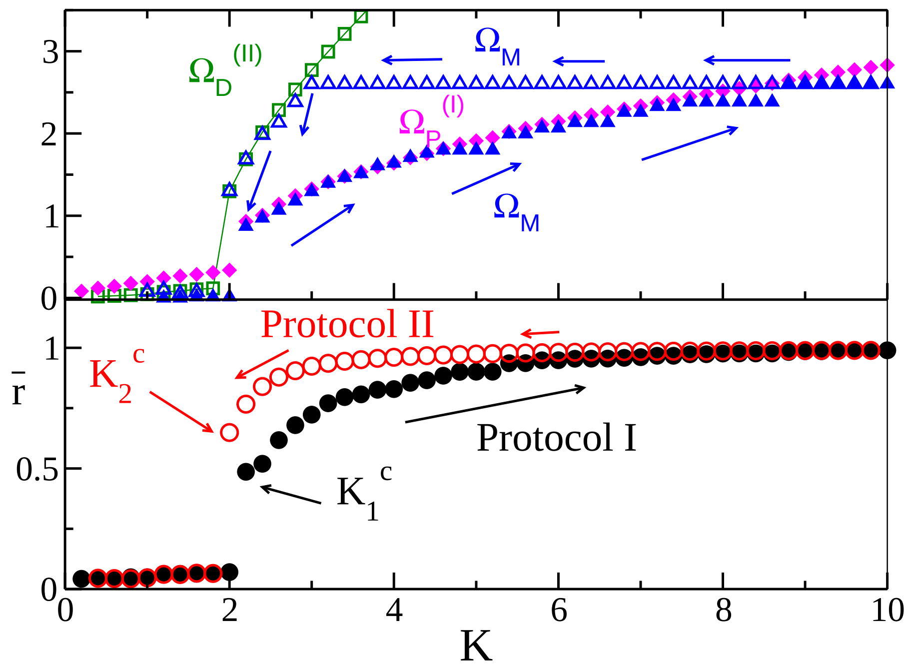
- For small Ω_i two fixed points are present: a stable node and a saddle.
- At larger frequencies $\Omega_i > \Omega_P = \frac{4}{\pi}\sqrt{\frac{Kr}{m}}$ a limit cycle emerges from the saddle via a homoclinic bifurcation
- Limit cycle and fixed point coexists until $\Omega_i \equiv \Omega_D = Kr$, where a saddle node bifurcation leads to the disappearence of the two fixed points
- For $\Omega_i > \Omega_D$ only the oscillating solution is present

For small mass (large β), there is no more coexistence. (Levi et al. 1978)

Simulation Protocols

Dynamics of N oscillators

- Ω_M maximal natural frequency of the locked oscillators
- $\Omega_P^{(I)} = \frac{4}{\pi}\sqrt{\frac{Kr}{m}}$
- $\Omega_D^{(II)} = Kr$



$m = 2$

Protocol I: Increasing K

The system remains desynchronized until $K = K_c^1$ (filled black circles).
Ω_M increases with K following Ω_P^I.
Ω_i are grouped in small clusters (plateaus).

Protocol II: Decreasing K

The system remains synchronized until $K = K_c^2$ (empty black circles).
Ω_M remains stucked to the same value for a large K interval than it rapidly decreases to 0 following Ω_D^{II}.

Mean Field Theory I

Tanaka et al. (TLO) in [PRL, Physica D (1997)] examined the origin of the hysteretic transition finding that

- by following Protocol I and II there is a group of drifting oscillators and one of locked oscillators which act separately
  - locked oscillators are characterized by $<\dot{\theta}>=0$
  - drifting oscillators $<\dot{\theta}>\neq 0$
- Drifting and locked oscillators are separated by a frequency:
  - Following Protocol I the oscillators with $\Omega_i < \Omega_P$ are locked
  - Following Protocol II the oscillators with $\Omega_i < \Omega_D$ are locked
- These two groups contribute differently to the total level of synchronization in the system

$$r = r_L + r_D$$

Mean Field Theory II

Total level of synchronization in the system: $r = r_L + r_D$

For the locked population TLO derived the self-consistent equation

$$r_L^{I,II} = Kr \int_{-\theta_{P,D}}^{\theta_{P,D}} \cos^2\theta g(Kr\sin\theta) d\theta$$

where $\theta_P = \sin^{-1}(\frac{\Omega_P}{Kr})$, $\theta_D = \sin^{-1}(\frac{\Omega_D}{Kr}) = \pi/2$.

For the drifting population the self-consistent equation is

$$r_D^{I,II} \simeq -mKr \int_{-\Omega_{P,D}}^{\infty} \frac{1}{(m\Omega)^3} g(\Omega) d\Omega$$

The former equation are correct in the limit of sufficiently large masses

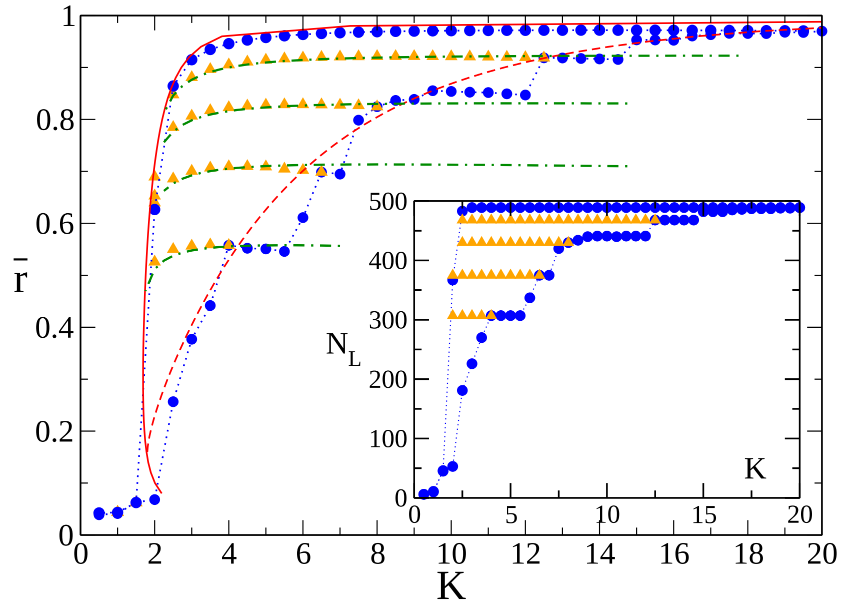
Hysteretic Behavior

Numerical Results for Fully Coupled Networks ($N = 500$)

- The data obtained by following protocol II are quite well reproduced by the mean field approximation r^{II}
- The mean field extimation r^{I} by TLO does not reproduce the stepwise structure in protocol I

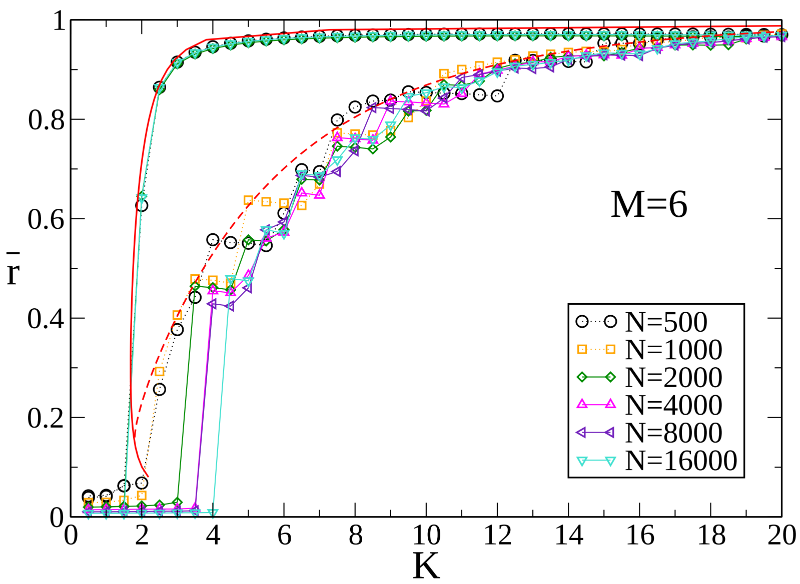
- Clusters of N_L locked oscillators of any size remain stable between r^{I} and r^{II}
- The level of synchronization of these clusters can be theoretically obtained by generalizing the theory of TLO to protocols where Ω_M remains constant

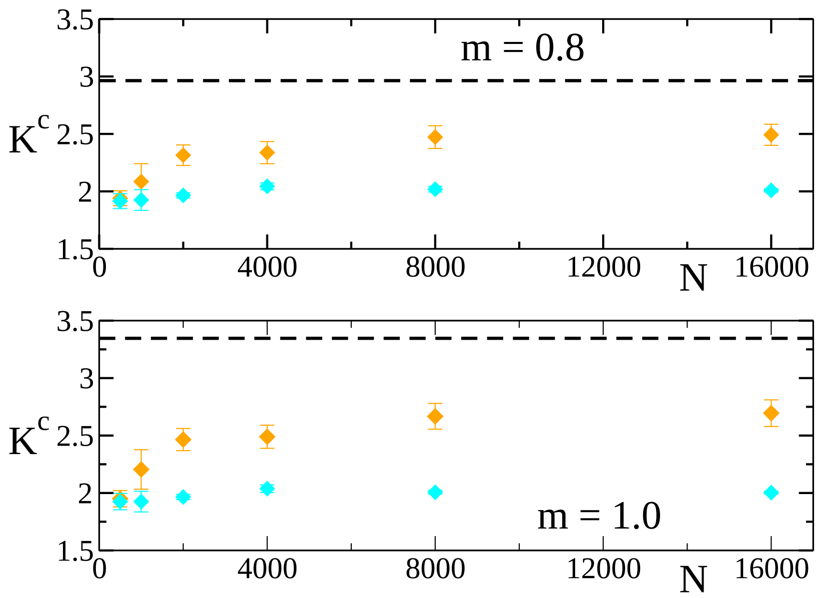
$m = 6$



Finite Size Effects

- K_1^c is the transition value from asynchronous to synchronous state (following Protocol I)
- K_2^c is the transition value from synchronous to asynchronous state (following Protocol II)
- K_1^c is strongly influenced by the size of the system
- K_2^c does not depend heavily on N





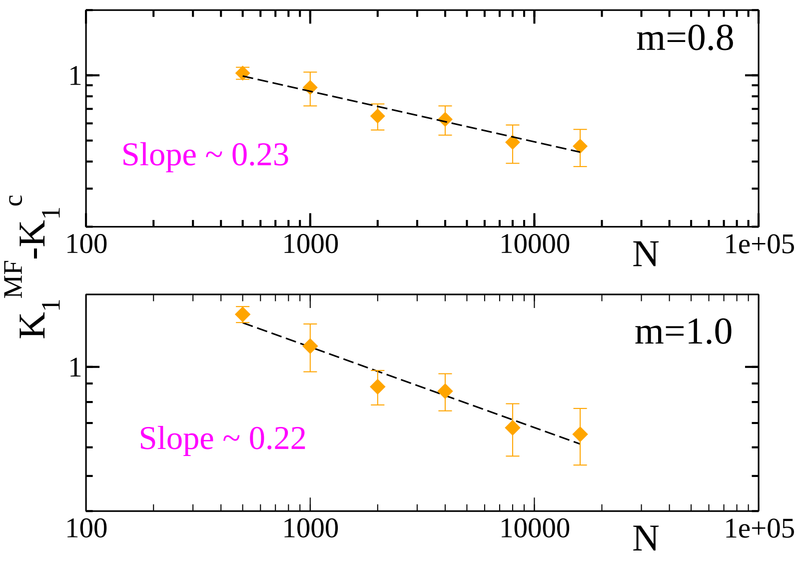
Dashed line → K_1^{MF} mean field value by Gupta et al (PRE 2014)

Finite Size Effects K_1^c

The mean field critical value has been estimated Gupta, Campa, Ruffo (PRE 2014) by employing a nonlinear Fokker-Planck formulation for the evolution of the single oscillator distribution $\rho(\theta, \dot{\theta}, \Omega, t)$ for coupled oscillators with inertia and noise

$$\frac{1}{K_1^{MF}} = \frac{\pi g(0)}{2} - \frac{m}{2} \int_{-\infty}^{\infty} \frac{g(\Omega) d\Omega}{1 + m^2 \Omega^2}$$

where $g(\Omega)$ is an unimodal distribution
Acebron et al, PRE (2000)



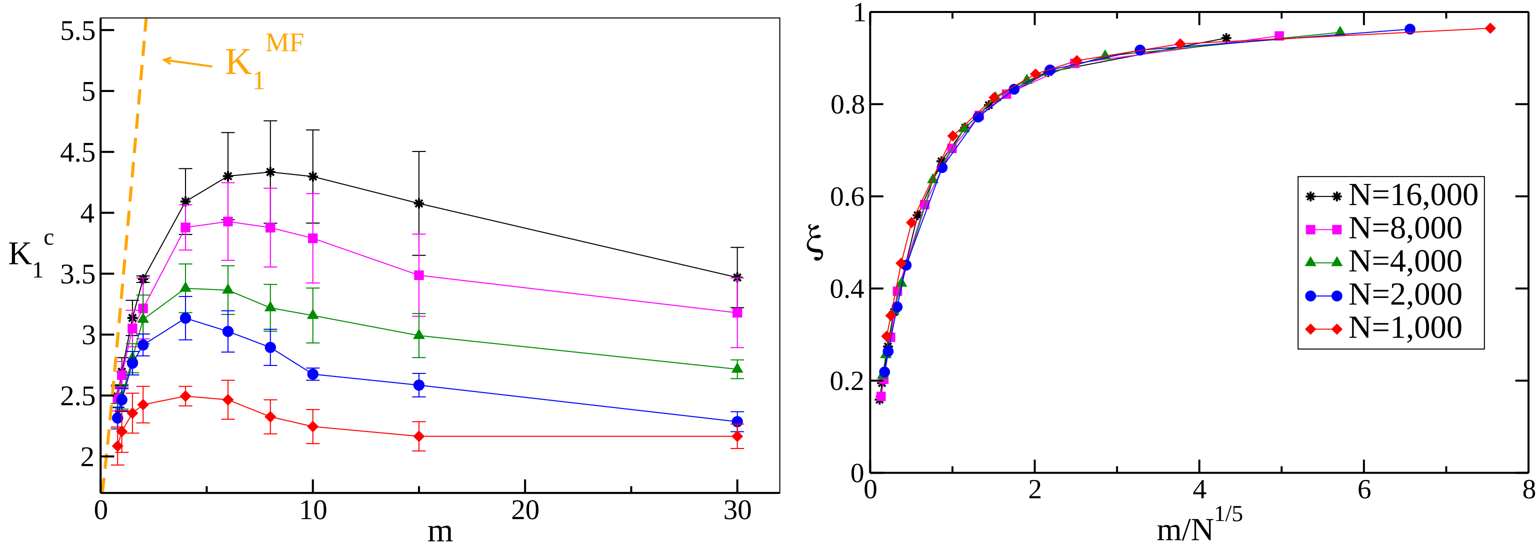
We observe the following scaling with the system size N for fixed mass

$$K_1^{MF} - K_1^c(N) \propto N^{-1/5}$$

this is true for sufficently low masses

Dependence On the Mass K_1^c

- K_1^c increases with m up to a maximal value and than decreases at larger masses
- by increasing N K_1^c increases and the position of the maximum shifts to larger masses (finite size effects)

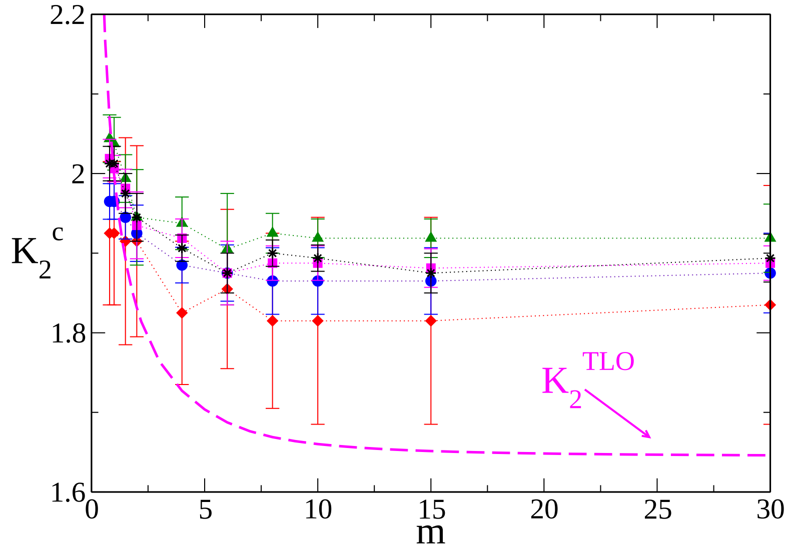


The following general scaling seems to apply

$$\xi \equiv \frac{K_1^{MF} - K_1^c(m,N)}{K_1^{MF}} = G\left(\frac{m}{N^{1/5}}\right) \quad \text{where } K_1^{MF} \propto 2m \text{ for } m > 1$$

Dependence On the Mass K_2^c

The TLO approach fails to reproduce the critical coupling for the transition from asynchronous to synchronous state (i.e., K_1^c), however it gives a good estimate of the return curve obtained with protocol II from the synchronized to the aynchronous regime



- K_2^c initially decreases with m then saturates, limited variations with the size N
- K_2^{TLO} is the minimal coupling associated to a partially synchronized state given by TLO approach for protocol II
- K_2^{TLO} exhibits the same behaviour as K_2^c, however it slightly understimates the asymptotic value (see the scale)

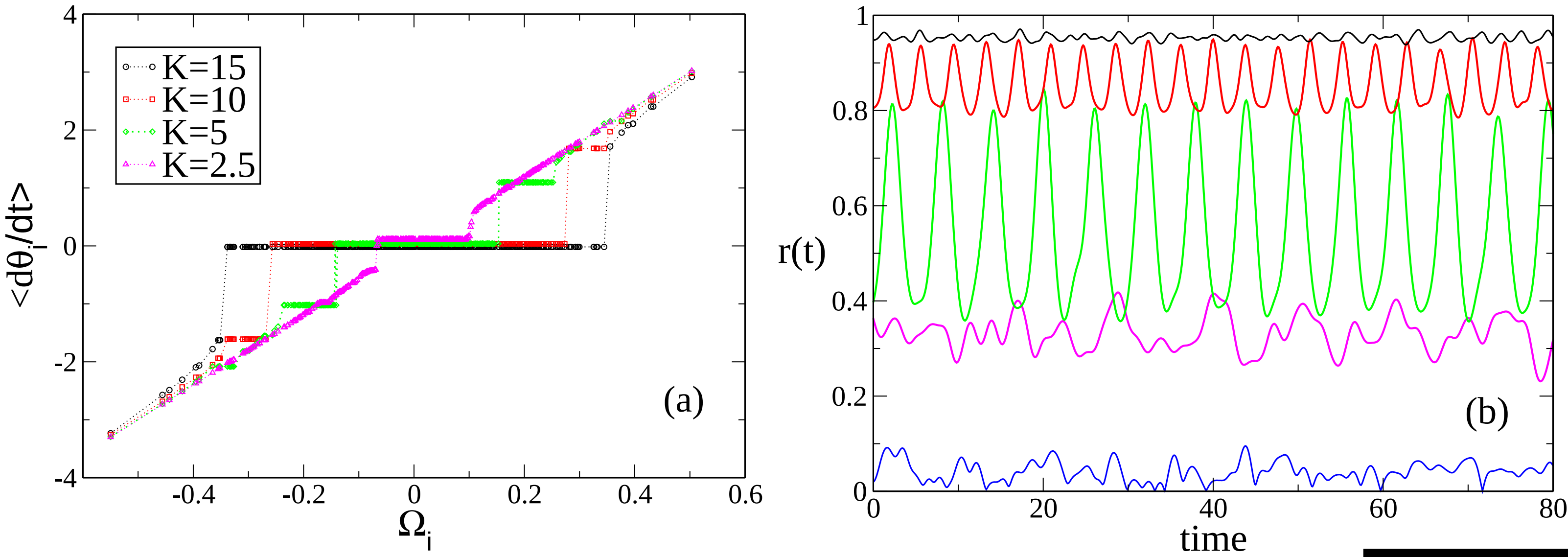
Drifting Clusters I

For larger masses (m=6), the synchronization transition becomes more complex, it occurs via the emergence of clusters of drifting oscillators.

The partially synchronized state is characterized by the coexistence of

- a cluster of locked oscillators with $< \dot{\theta} > \simeq 0$
- clusters composed by drifting oscillators with finite average velocities

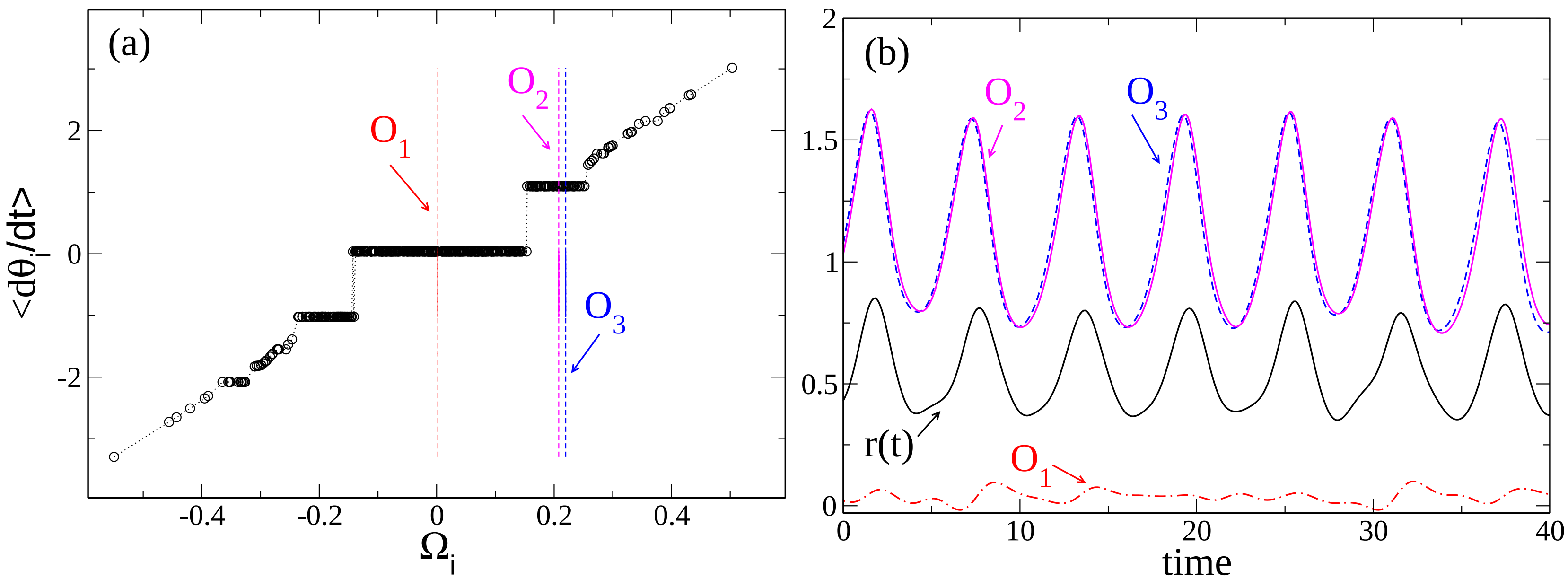
The effect of these extra clusters is to induce (periodic or quasi-periodic) oscillations in the temporal evolution of the order parameter.



Drifting Clusters II

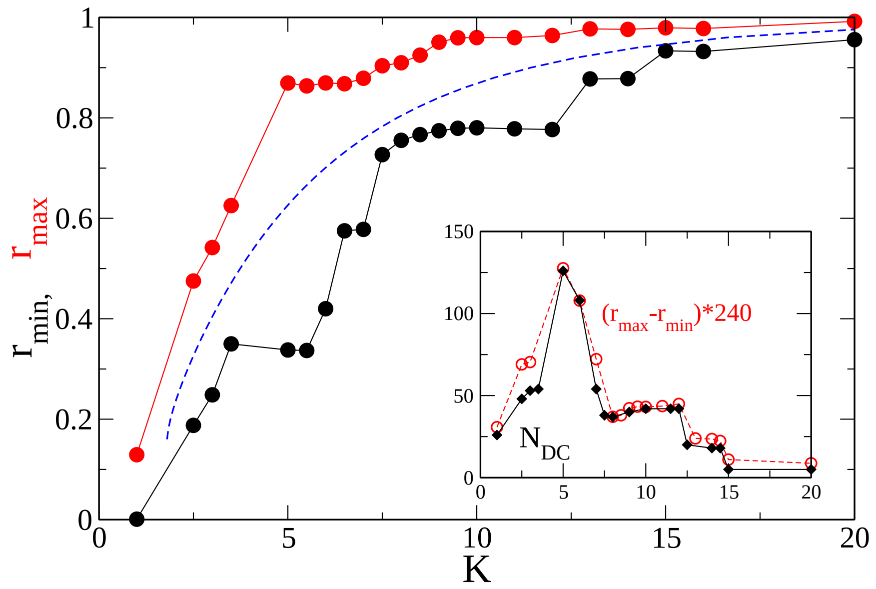
If we compare the evolution of the instantaneous velocities $\dot{\theta}_i$ for 3 oscillators and $r(t)$ we observe that

- the phase velocities of O_2 and O_3 display synchronized motion
- the phase velocity of O_1 oscillates irregularly around zero
- the almost periodic oscillations of $r(t)$ are driven by the periodic oscillations of O_2 and O_3



Drifting Clusters III

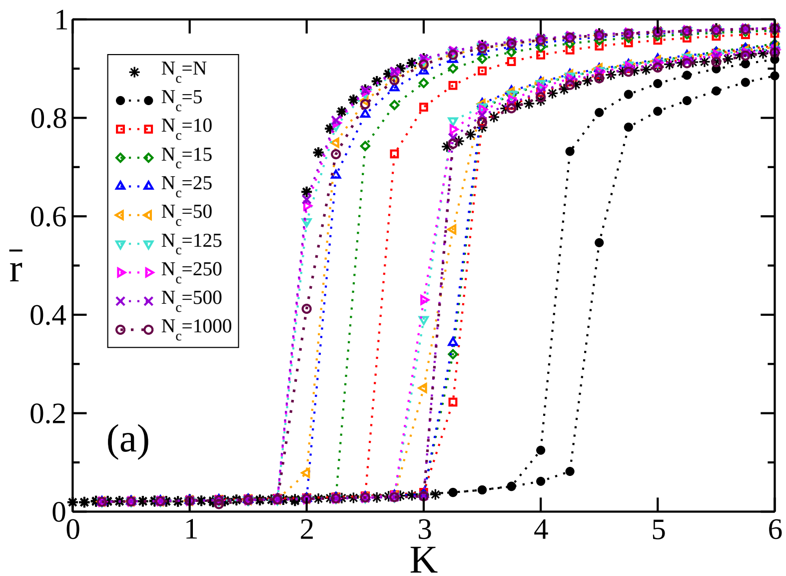
- The amplitude of the oscillations of $r(t)$ and the number of oscillators in the drifting clusters N_{DC} correlates in a linear manner
- The oscillations observable in the order parameter are induced by the presence of large secondary clusters characterized by finite whirling velocities
- At smaller masses oscillations in $r(t)$ are present, but reduced in amplitude. These oscillations are due to finite size effects since no clusters of drifting oscillators are observed

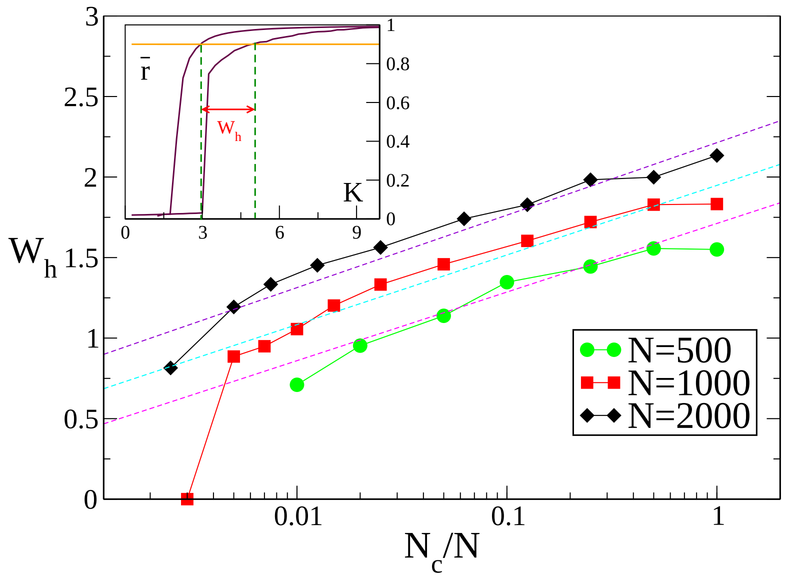


- Blue dashed line $\Rightarrow$ estimated mean field value r^I by TLO
- The mean field theory captures the average increase of the order parameter but it does not foresee the oscillations

Diluted Network

- Constraint 1 : the random matrix is symmetric
- Constraint 2 : the in-degree is constant and equal to N_c

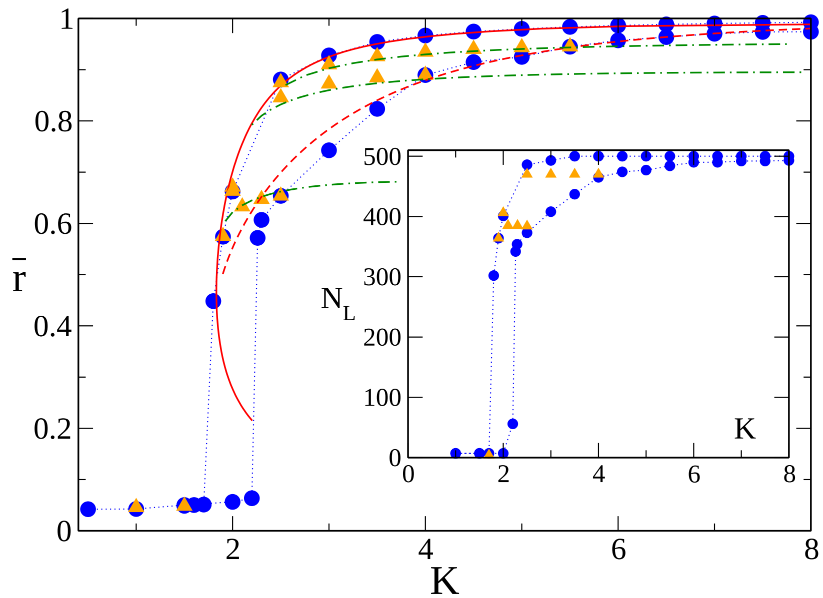




- Diluted or fully coupled systems (whenever the coupling is properly rescaled with the in-degree) display the same phase-diagram
- For very small connectivities the transition from hysteretic becomes continuous
- By increasing the system size the transition will stay hysteretic for extremely small percentages of connected (incoming) links

Diluted Network II

- The TLO mean field theory still gives reasonable results (70% of broken links)
- All the states between the synchronization curves obtained following Protocol I and II are reachable and stable



- These states, located in the region between the synchronization curves, are characterized by a frozen cluster structure, composed by a constant N_L
- The generalized mean-field solution $r^0(K, \Omega_0)$ is able to well reproduce the numerically obtained paths connecting the synchronization curves (I) and (II)

Italian High Voltage Power Grid

Each node is described by the phase:

$$\phi_i(t) = \omega_{AC} t + \theta_i(t)$$

where $\omega_{AC} = 2\pi \;\; 50$ Hz is the standard AC frequency and θ_i is the phase deviation from ω_{AC}.

Consumers and generators can be distinguished by the sign of parameter P_i:

$$P_i > 0 \; (P_i < 0)$$

corresponds to generated (consumed) power.

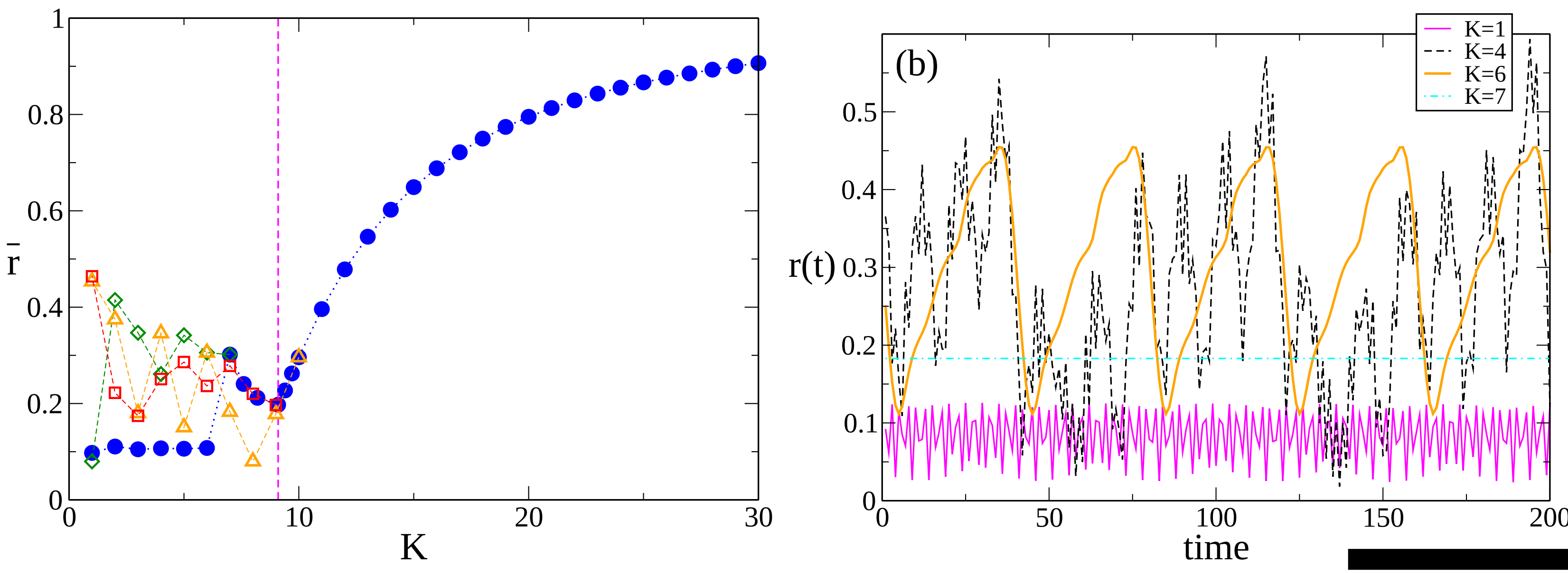
$$\ddot{\theta}_i = \alpha \left[-\dot{\theta}_i + P_i + K \sum_{ij} C_{i,j} \sin(\theta_j - \theta_i) \right]$$

Average connectivity $< N_c >= 2.865$

[Filatrella et al., The European Physical Journal B (2008)]

Italian High Voltage Power Grid

- We do not observe any hysteretic behavior or multistability down to $K = 9$
- For smaller coupling an intricate behavior is observable depending on initial conditions
- Generators and consumers compete in order to oscillates at different frequencies
- The local architecture favours a splitting based on the proximity of the oscillators
- Several small whirling clusters appear characterized by different phase velocities
- The irregular oscillations in $r(t)$ reflect quasi-periodic motions



Main Results

We have studied the synchronization transition for a globally coupled Kuramoto model with inertia for different system sizes and inertia values.

- The transition is hysteretic for sufficiently large masses
- Clusters of locked oscillators of any size coexist within the hysteretic region
- A generalization of TLO theory is capable to reproduce all the possible synchronization/desynchronization hysteretic loops
- The presence of clusters composed by drifting oscillators induces oscillatory behaviour in the order parameter

The properties of the hysteretic transition have been examined also for random diluted network.

- The main properties of the transition are not affected by the dilution
- The transition appears to become continuous only when the number of links per node becomes of the order of few units

`S. Olmi, A. Navas, S. Boccaletti, A. Torcini, submitted to PRE`

Extension of the Mean Field Theory

In principle one could fix the discriminating frequency to some arbitrary value Ω_0 and solve self-consistently

$$r = r_L + r_D$$

$$r_L^{I,II} = Kr \int_{-\theta_0}^{\theta_0} \cos^2\theta g(Kr\sin\theta) d\theta \quad r_D^{I,II} \simeq -mKr \int_{-\Omega_0}^{\infty} \frac{1}{(m\Omega)^3} g(\Omega) d\Omega$$

This amounts to obtain a solution $r^0 = r^0(K, \Omega_0)$ by solving

$$\int_{-\theta_0}^{\theta_0} \cos^2\theta g(Kr^0 \sin\theta) d\theta - m \int_{-\Omega_0}^{\infty} \frac{1}{(m\Omega)^3} g(\Omega) d\Omega = \frac{1}{K}$$

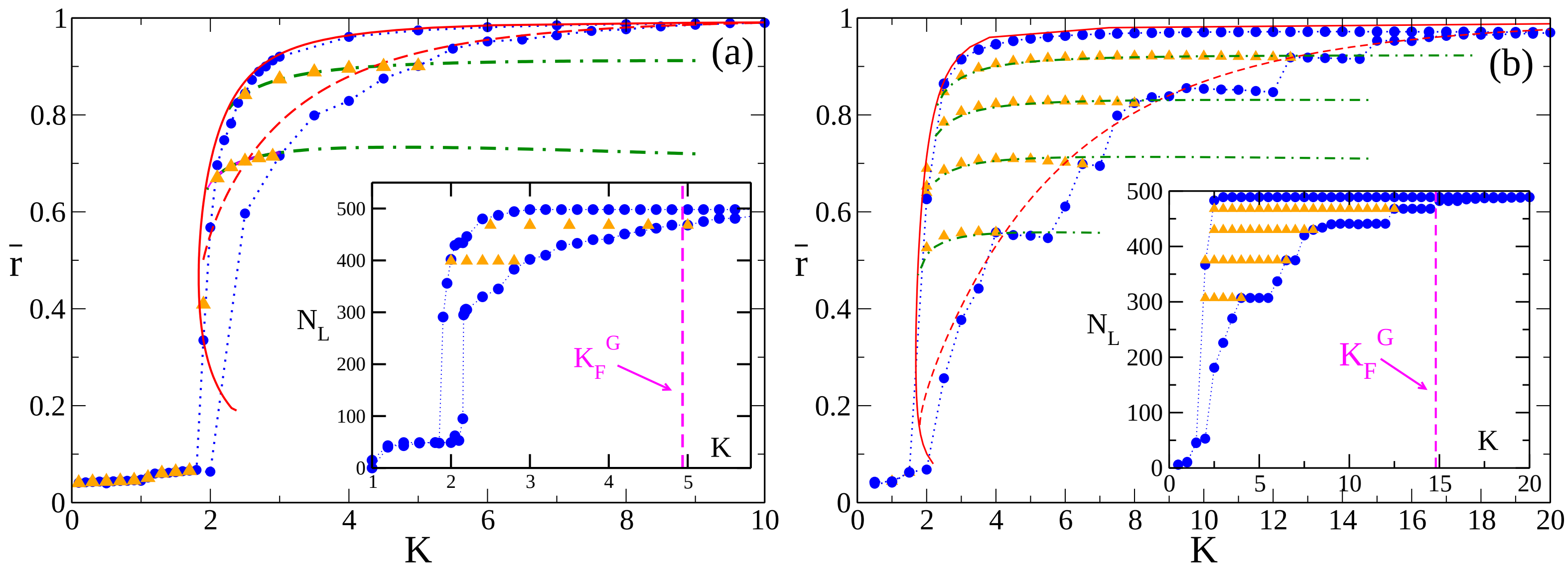
with $\theta_0 = \sin^{-1}(\Omega_0/Kr^0)$. The solution exists if $\Omega_0 < \Omega_D = Kr^0$.

$\Rightarrow$ A portion of the (K, r) plane delimited by the curve $r^{II}(K)$ is filled with the curves $r^0(K)$ obtained for different Ω_0 values.

Hysteretic Behavior

Fully Coupled Networks

- A step-wise structure emerges at larger masses do to the break down of the independence of the whirling oscillators
- The number of locked oscillators N_L follows the same step-wise structure
- N_L remains constant until it reaches the descending curve



Italian High Voltage Power Grid

By following Protocol II

- the system stays in one cluster up to $K = 7$
- at $K = 6$ wide oscillations emerge in $r(t)$ due to the locked clusters that have been splitted in two (is this also the origin for the emergent multistability?)
- By lowering further K several whirling small clusters appear and r becomes irregular