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Towards a Statistical Mechanics of Spatiotemporal Chaos

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Coupled Hénon maps are introduced to model in a more appropriate way chaos in extended systems. An effective technique allows the extraction of spatiotemporal periodic orbits, which are then used to approximate the invariant measure. A further implementation of the ζ-function formalism reveals the extensive character of entropies and dimensions, and allows the computation of the associated multifractal spectra. Finally, the analysis of short chains indicates the existence of distinct phases in the invariant measure, characterized by a different number of positive Lyapunov exponents.

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Powerful methods have been introduced in the past years to investigate low-dimensional strange attractors, while less is known about spatially extended systems when many (infinite) degrees of freedom are switched on. Direct integration of partial differential equations often requires a large amount of computer time, thus limiting the accessible parameter region. Coupled map lattices (CML) [1], characterized by discrete space and time variables, are better suited for simulation and still reproduce many of the interesting features exhibited by more realistic systems [2]. However, the CML model most studied in the literature (a chain of 1D maps) has the drawback of being characterized by a noninvertible dynamics. To overcome such a difficulty, here we introduce a lattice of coupled Hénon maps,

$$x_{n+1}^j = a - (\tilde{x}_n^j)^2 + b x_{n-1}^j , \qquad (1a)$$

with

$$\tilde{x}_n^j = (1-\varepsilon) x_n^j + \tfrac{1}{2}\varepsilon (x_n^{j+1} + x_n^{j-1}) , \qquad (1b)$$

where n and j denote time and space variables, respectively; a and b are the usual parameters of the Hénon map and ε is the diffusive coupling parameter. For $b=0$, model (1) reduces to the familiar lattice of logistic maps [1]; for $b=1$, a conservative dynamics is generated; for $\varepsilon=0$, uncoupled Hénon maps are obtained.

In the present Letter, we consider model (1) as a paradigmatic example for developing and applying a thermodynamic formalism to extended systems, in strict analogy with what has already been done for 2D maps. Spatially and temporally periodic orbits are first extracted, by extending a technique invented for the single Hénon map [3]. An analytic procedure is then developed to determine the Lyapunov spectrum of a periodic orbit embedded in an infinite chain. An appropriate ζ-function formalism is eventually applied to evaluate the multifractal spectra of entropy and dimension densities.

A statistical-mechanics description of chaotic attractors can be achieved by encoding all trajectories as symbol sequences, through the introduction of a generating partition. However, the construction of such a partition becomes practically unfeasible already in a 4D phase space, when it requires the accurate definition of a 3D hypersurface. An alternative method to describe a strange attractor is based on the identification of periodic orbits of increasing period. It exploits the well-known fact that periodic orbits are dense in the invariant measure. Here, we can determine simultaneously periodic orbits and symbol sequences, by introducing a fictitious dynamics along the continuous "time" axis t,

$$\dot{x}_n^j(t) = (-1)^{s(n,j)} \{ x_{n+1}^j(t) - a + [\tilde{x}_n^j(t)]^2 - b x_{n-1}^j(t) \} , \qquad (2)$$

where $s(n,j) \in \{0,1\}$. A cycle (J,N) of period J in space and N in time is, by definition, a fixed point of Eq. (2), once suitable periodic boundary conditions have been settled. Our numerical experiments appear to establish that the dynamical system (2) is characterized by the following properties: (i) The vast majority of (J,N) orbits are stable for only one choice of the pattern of the $J\times N$ coefficients $s(n,j)$; (ii) different periodic orbits are stable for different $s(n,j)$ configurations. Accordingly, the $s(n,j)$'s provide a good symbolic representation of the dynamics generated by model (1). These results generalize what was numerically found by Biham and Wenzel for the single Hénon map [3] and rigorously proved for the logistic map in Ref. [4]. In our case, for $a=1.4$, $b=0.3$, and $\varepsilon=0.055$, there is less than 1% of failures due to orbits which are unstable for all $s(n,j)$ patterns.

Careful numerical simulations [5] suggest that the effect of a small coupling on the topology of the attractor is limited to the pruning of some of the orbits existing for $\varepsilon=0$. Therefore, the search of periodic trajectories can be further simplified by testing only those symbol patterns which are allowed in the uncoupled case. The few orbits missed by this technique have been identified through the implementation of Newton's method. Moreover, we must discard all the cycles which do not belong to the attractor. For the single Hénon map, this set of orbits reduces to the fixed point identified by a sequence of all 0's. In our case, the iteration of Eq. (1) indicates that all periodic solutions characterized by a sequence of all 0's at least in one site do not belong to the attractor.

Application of the ζ-function formalism to periodic orbits allows an accurate statistical analysis of the hyperbolic "phase" of strange attractors. In fact, once their Lyapunov exponents $\lambda^{(p)}$ (where $\lambda^{(p)}\geq\lambda^{(p')}$ if $p>p'$, p labeling the different exponents of a given orbit) are known, the coefficients of the power-series expansion of ζ^{-1} can be computed [6]. In spatially extended systems, in the limit of infinite chain length L, the Lyapunov spectrum $\Lambda(\rho)=\lambda^{(p)}$ becomes a function of the continuous variable $\rho=p/L$ (where $1\leq p\leq 2L$, for our model). In order to estimate $\Lambda(\rho)$, we start from the linearization of Eq. (1),

$$\delta_{n+1}^j=-\tilde{x}_n^j[2(1-\varepsilon)\delta_n^j+\varepsilon(\delta_n^{j-1}+\delta_n^{j+1})]+b\delta_{n-1}^j\,, \quad (3)$$

which, in vector notations, reads as $(\boldsymbol{\delta}_{n+1},\boldsymbol{\delta}_n)=\Gamma_n(\boldsymbol{\delta}_n,\boldsymbol{\delta}_{n-1})$. Γ_n being the Jacobian of a spatially periodic orbit, a periodic operator, it is natural to invoke the Bloch theorem in order to determine its spectrum and, in turn, the Lyapunov exponents. In the case of a stationary solution, Γ is independent of n and the stability of the orbit is deduced from its eigenvalues. According to the Bloch theorem, the eigenvectors of Γ can be expressed as

$$u^j(l,k)=e^{ikj}w^j(l,k),\quad v^j(l,k)=e^{ikj}z^j(l,k)\,, \quad (4)$$

where the wave number k is equal to $2\pi\phi/J$ $(0\leq\phi\leq1)$, and l is the band index $(1\leq l\leq 2J)$; $w^j(l,k)$ and $z^j(l,k)$ are periodic functions of j. By substituting Eq. (4) in the eigenvalue problem associated with Γ, we find

$$\begin{aligned}\mu w^j&=-\tilde{x}^j[2(1-\varepsilon)w^j+\varepsilon(e^{-ikj}w^{j-1}+e^{+ikj}w^{j+1})]+bz^j\,,\\ \mu z^j&=w^j\,,\end{aligned} \quad (5)$$

where μ is an eigenvalue, and the dependence on l and k has been dropped for the sake of simplicity. For each choice of the wave number k, $2J$ eigenvalues are found by solving the linear system (5). The Lyapunov exponents are then given by $\ln|\mu|$ ($|\cdot|$ denotes the modulus operation), and the spectrum $\Lambda(\rho)$ is determined after ordering the various bands, and taking into account possible overlaps. The extension to orbits also periodic in time is straightforward. We must consider the product $\tilde{\Gamma}_N=\prod_{n=1}^N\Gamma_n(k)$, where $\Gamma_n(k)$ is implicitly defined on the right-hand side of Eq. (5) as the operator acting on the vector $(\mathbf{w},\mathbf{z})$. The Lyapunov exponents are then obviously given by $\ln|\mu_N|/N$, where the μ_N's are the eigenvalues of $\tilde{\Gamma}_N(k)$. Accordingly, the determination of the Lyapunov spectrum of a spatio temporal periodic orbit is reduced to finding the eigenvalues of $2J\times 2J$ matrices, i.e., to a finite-dimensional problem.

From the theory of low-dimensional chaos, we know that the probability $\mathcal{P}_i$ to observe the symbol sequence (of length n) corresponding to a given periodic orbit scales as $\mathcal{P}_i\simeq e^{-H_in}$ [7], where H_i is the sum of all positive Lyapunov exponents for periodic orbit i. Assuming that the dynamical entropy H_i is an extensive quantity, it is convenient to introduce the density $h_i=H_i/L$. In the thermodynamic limit $(L\to\infty)$, h_i is expressed in terms of an integral, $h_i=\int\Lambda_i(\rho)d\rho$, where the integration extends over positive Λ_i's. For instance, for our choice of the parameter values, the local entropy of the stationary orbit $x_n^j=c=0.883896\ldots$ is $h_M=0.606074$.

Analogously, a dimension density α_i can be defined from the local version of Kaplan-Yorke formula,

$$\alpha_iL=P-\sum_{p=1}^{P}\lambda^{(p)}/\lambda^{(p+1)}\,, \quad (6)$$

where P is the largest integer such that the sum of the Lyapunov exponents is still positive. Rigorously speaking, relation (6) represents an upper bound to the exact local dimension [8]. Nevertheless, we are not aware of a single nongeneric example where the bound is not saturated. The extension of Eq. (6) to the continuum reads as

$$\int_0^{\alpha_i}\Lambda_i(\rho)d\rho=0\,. \quad (7)$$

The dimension density for the stationary solution $x_n^j=c$ and for the above mentioned parameter values is $\alpha_M=1.34240$.

Let us now briefly recall the basics of the thermodynamic formalism. We start from the study of the scaling behavior of a generic sum over all cycles of length n (i.e., over the elements of a suitable partition of the phase space) such as

$$\sum_i b_i \simeq e^{\beta n}, \tag{8}$$

where the b_i's are physical quantities which scale exponentially with n. The exponent β can be estimated by applying the grand-canonical formalism and introducing the inverse ζ function,

$$\zeta^{-1}(z) = \prod_i (1 - z^n b_i)^{m_i}, \tag{9}$$

where n denotes the period and where the product, at variance with the sum in Eq. (8), is extended over distinct (under temporal rotations) orbits only. The extra exponent m_i has been introduced to take into account the possibility that different orbits are characterized by the same b value. Finally, the exponent β is given by $\ln z_0$, where z_0 is the first zero of ζ^{-1}. While the exact evaluation of ζ requires the knowledge of an infinity of orbits, Cvitanović showed that the estimate of β, obtained by suitably truncating the Taylor expansion of ζ^{-1}, is often sufficiently accurate [6]. Since a formalism implicitly including the limit $L \to \infty$ is not available, we limit ourselves to consider finite-length chains (namely, $L=2$, 3, and 4). We will see that interesting and new phenomena appear already for such short chains.

Let us start from the generalized entropy densities $l(q)$, defined by

$$\sum_i e^{-h_i q n L} \simeq e^{l(q) n L}, \tag{10}$$

where the sum is extended over the orbits of period n in time and L in space (and their submultiples) and h_i is the local entropy density estimated for a chain of length L. Equation (10) has the same structure as Eq. (8), so that we can construct the corresponding ζ function. Equation (1) being invariant under both spatial translations and reflections, each orbit belongs to the class defined by these symmetry operations. The number of elements in a class represents the multiplicity m_i introduced in Eq. (9). As it is well known, we can pass from $l(q)$ to the more informative multifractal spectrum $g(h)$, via the Legendre transform [9]

$$g(h) = hq - l(q), \quad h = l'(q). \tag{11}$$

The results of numerical simulations for the above mentioned parameter values are reported in Fig. 1(a), the numbers denoting the chain lengths. The spectrum of entropy densities reduces, for $L=1$, to the standard spectrum of entropies of the single Hénon map. The spectrum for two coupled maps has been obtained by truncating the expansion of $\zeta^{-1}(z)$ after 18 terms, i.e., considering all orbits up to period $N=18$. The comparison of the two spectra shows that the topological entropy density

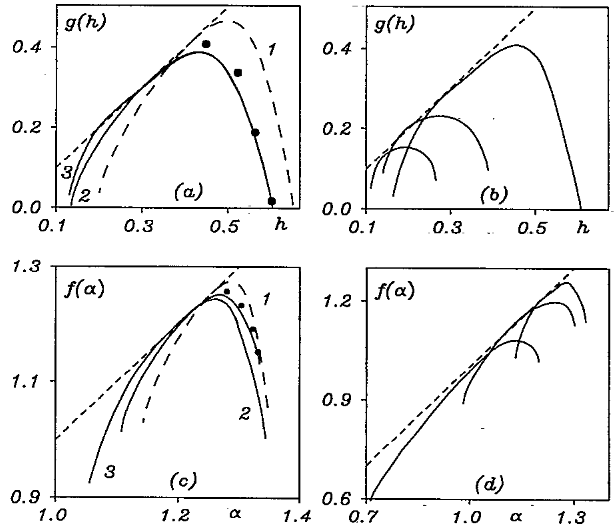

FIG. 1. Multifractal spectra of entropy density $g(h)$ [(a), (b)] and of dimension density $f(\alpha)$ [(c),(d)], for $a=1.4$, $b=0.3$, and $\varepsilon=0.055$. The numbers reported in (a) and (c) indicate the chain length, while solid circles refer to $L=4$. The curves obtained for one map coincide with the standard spectra of the Hénon map. In (b) and (d), the outcomes of simulations with four maps are reported. The curves, from left to right, correspond to the spectra of periodic orbits with 2, 3, and 4 positive Lyapunov exponents, respectively.

(i.e., the maximum of g) is definitely smaller for $L=2$. This clearly reveals the stabilizing effect of the diffusive coupling. Moreover, the spectra for $L=2$ and 3 (obtained with $N=15$) agree very well for $h>0.3$, confirming the conjecture that g and h are truly intensive variables. The agreement is less pronounced when the curves are compared with the results for $L=4$ and $N=11$ [see solid circles in Fig. 1(a)]. However, the discrepancy can be presumably attributed to a nonperfect convergence of the last values, due to the shorter period therein used (the explosion of the number of cycles prevents a significant improvement). A further evidence of the fast convergence for increasing L arises from the observation that the maximum of h is extremely close to the asymptotic ($L=\infty$) exponent h_M, previously computed for the stationary solution $x_n^j = c$.

The much larger fluctuations observed in the small h range require a different explanation. First of all, the nonhyperbolicity of the model allows the existence of weakly unstable orbits of long period, which slow down the convergence. However, the difficulties encountered with four maps are not apparently related to such a phenomenon. We conjecture that the slow convergence is caused by the existence of distinct "phases." In fact, the spectra $g(h)$ estimated separately from the orbits characterized by 2, 3, and 4 positive Lyapunov exponents, respectively, exhibit a much better convergence [see Fig. 1(b)]. This is in complete analogy with Ref. [10], where the same approach allowed us to show the existence of two phases in a strange repeller.

Let us recall that whenever different phases exist (e.g., homoclinic tangencies and hyperbolic points in nonhyperbolic attractors [7]), characterized by different spectra $g_p(h)$, the global spectrum is determined as the lowest concave curve such that $g(h) > g_p(h)$. If there are at least two different phases prevailing into distinct h regions, then a phase transition can be detected as a discontinuity in a derivative of $l(q)$. This seems to be our case, by observing the three curves reported in Fig. 1(b). A direct iteration of model (1) rules out the trivial hypothesis that the three curves are associated with three distinct attractors.

Now, we apply the same formalism to the estimation of fractal dimensions. The key expression is represented by the implicit equation [7]

$$\sum_i e^{-h_i nL} \exp\left[-\sum_p \lambda_i^{(p)} \tau^{(p)}(q) n\right] = 1\,, \qquad (12)$$

where $\tau^{(p)}(q) = (q-1) d^{(p)}(q)$ and $d^{(p)}(q)$ is the partial dimension along the pth direction. Such an equation has been successfully applied to 2D maps where, being $d^{(1)} = 1$, there is only one unknown quantity, namely, $d^{(2)}(q)$. It is still a meaningful relation whenever the number P of directions characterized by $d^{(p)} = 1$ is the same for all orbits. In the remaining cases, the single Eq. (12) is no longer sufficient to determine the partial dimensions. We claim that, whenever only the global dimension $d(q) = \sum_p d^{(p)}(q)$ is required, this difficulty can be overcome by rewriting Eq. (12) as follows:

$$\sum_i e^{-H_i n} \exp\{(q-1)\lambda_i^{(P+1)}[\alpha_i - d(q)] nL\} = 1\,. \qquad (13)$$

This relation is fully equivalent to Eq. (12) whenever the latter one can be meaningfully applied. Moreover, it can be implemented also when the number of integer dimensions is not the same for all the periodic orbits. As Eq. (13) is an implicit relation of type (8), we can again resort to the computation of an appropriate ζ function to compute $\tau(q) = (q-1) d(q)$. The Legendre transform $f(\alpha)$ of $\tau(q)$ is plotted in Fig. 1(c) for L from 1 to 3. At variance with entropies, the agreement between the dimension spectra for $L=2$ and 3 is not equally good. However, the spectrum for $L=4$ [solid circles in Fig. 1(c)] is definitely closer to that one for $L=3$ in the large α range, suggesting that the convergence is only slightly worse than for entropies.

However, the most impressive result is found at small dimensions, where the existence of three phases is again confirmed. In fact, the slow convergence observed when computing the global spectrum for $L=4$ is sped up by analyzing separately the sets of orbits characterized by a different number of positive Lyapunov exponents. This phenomenon is, in a sense, complementary to the phase transition observed in filtered chaotic signals [11], where the competing phases corresponded to distinct stable directions in phase space: here, it is the unstable directions which distinguish the phases.

For the sake of completeness, let us also recall that the actual procedure, entirely based on periodic orbits, is not able to capture the relevant features of the nonhyperbolic phase arising from the scaling behavior around the homoclinic tangencies [12].

In summary, the technique introduced in Eq. (2) opens the possibility of an effective determination of periodic orbits in spatially extended systems (in particular, in the case of coupled logistic maps). For comparison, the direct estimation of the spectra from periodic orbits can exhibit a comparable accuracy to that achieved by the implementation of the ζ-function formalism only in a limited range of small q values, and after some fitting work and extrapolations have been performed.

Our approach has to be considered as only a first, though important, step towards the development of an accurate statistical treatment of extended systems. In fact, further improvements and ideas are required to allow reliable investigations of longer chains. This is especially required at larger coupling values, where a slower convergence is found [5].

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