Simple Neuron Models:

FitzHugh-Nagumo and Hindmarsh-Rose

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- Reduction of the Hodgkin-Huxley model
- The FitzHugh-Nagumo model
- Phase plane analysis
- Excitability (threshold-like behavior), periodic spiking (Hopf bifurcation)
- The Hindmarsh-Rose model for bursting neurons

Neuron models (sketch)

Single Neurons

experiments

Hodgkin–Huxley, 1952
– current based

detailed, specific models
– compartmental (structure)
– more currents
– adaptive (state–dep. prop.)

reduction

low–dimensional models

– FitzHugh–Nagumo, 1960's
– Hindmarsh–Rose, 1980's

Networks

simplification

* effective numerical simulation
* allow for most common features
  – excitability
  – spiking, different time scales

integrate–and–fire models
stochastic models

abstraction

Hopfield network, 1980's
– on–off neuron, learning, stat. physics

Hodgkin-Huxley model

- neuronal signals are short electrical pulses: **spikes** or **action potentials** on msec scale
- intracellular: incoming spike modifies **membran potential**

Hodgkin-Huxley (1952): Semirealistic **4-dimensional** model for the dynamics of the membran potential by taking into account Na+, K+, and a leak current. Dynamics of ion channels highly nonlinear $\Rightarrow$ emergence of chaotic evolution.

membran potential: $$\frac{dV}{dt} = C_{Na} m^3 h (E_{Na} - V) + C_K n^4 (E_K - V) + C_{leak} (V_{rest} - V) + I_{inj}(t)$$

sodium I_{Na}, fast: $$\frac{dm}{dt} = \alpha_m(V)(1-m) - \beta_m(V)m$$

slow: $$\frac{dh}{dt} = \alpha_h(V)(1-h) - \beta_h(V)h$$

potassium I_K, slow: $$\frac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V)n$$

Dynamics of currents m, h, n

General form:

$$\frac{dx}{dt} = -\frac{1}{\tau(V)}[x - x_s(V)]$$

Solution for constant V: $x(t) = (x_0 - x_s)\exp(-t/\tau) + x_s$

$\Rightarrow$ exponential relaxation to steady state value x_s

For varying $V(t)$: $x(t)$ follows varying steady state value $x_s(t)$

small τ: fast relaxation $\Rightarrow$ $x(t) \approx x_s(t)$

large τ: slow dynamics

Reduction to two-dimensional model

fast sodium dynamics:

approximate by steady state value: $m(t) \approx m_s(V)$

similar dynamics of slow sodium and potassium:

replace $h(t)\,, n(t)$ by one effective current $w(t)$

$\Rightarrow$ two equations for temporal evolution of $V(t)$ and $w(t)$

FitzHugh-Nagumo model

FitzHugh (1961) and Nagumo, Arimoto, Yoshizawa (1962) derived 2-dimensional model for an **excitable** neuron:

$$\text{membran potential:} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = v - \frac{v^3}{3} - w + I$$

$$\text{current variable:} \quad \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{\tau}(v + a - bw)$$

typical values: $a = 0.7\,,\, b = 0.8\,,\, \tau = 13$

$$\Rightarrow \quad \frac{\dot{v}}{\dot{w}} \sim 10 \quad \Rightarrow \quad w \text{ slow}\,,\, v \text{ fast}$$

For constant input $I = const$ **no** chaotic evolution

Phase plane analysis

Two-dimensional flow field:

$$\vec{F}(v,w) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v - \frac{v^3}{3} - w + I \\ \frac{1}{\tau}(v + a - bw) \end{pmatrix}$$

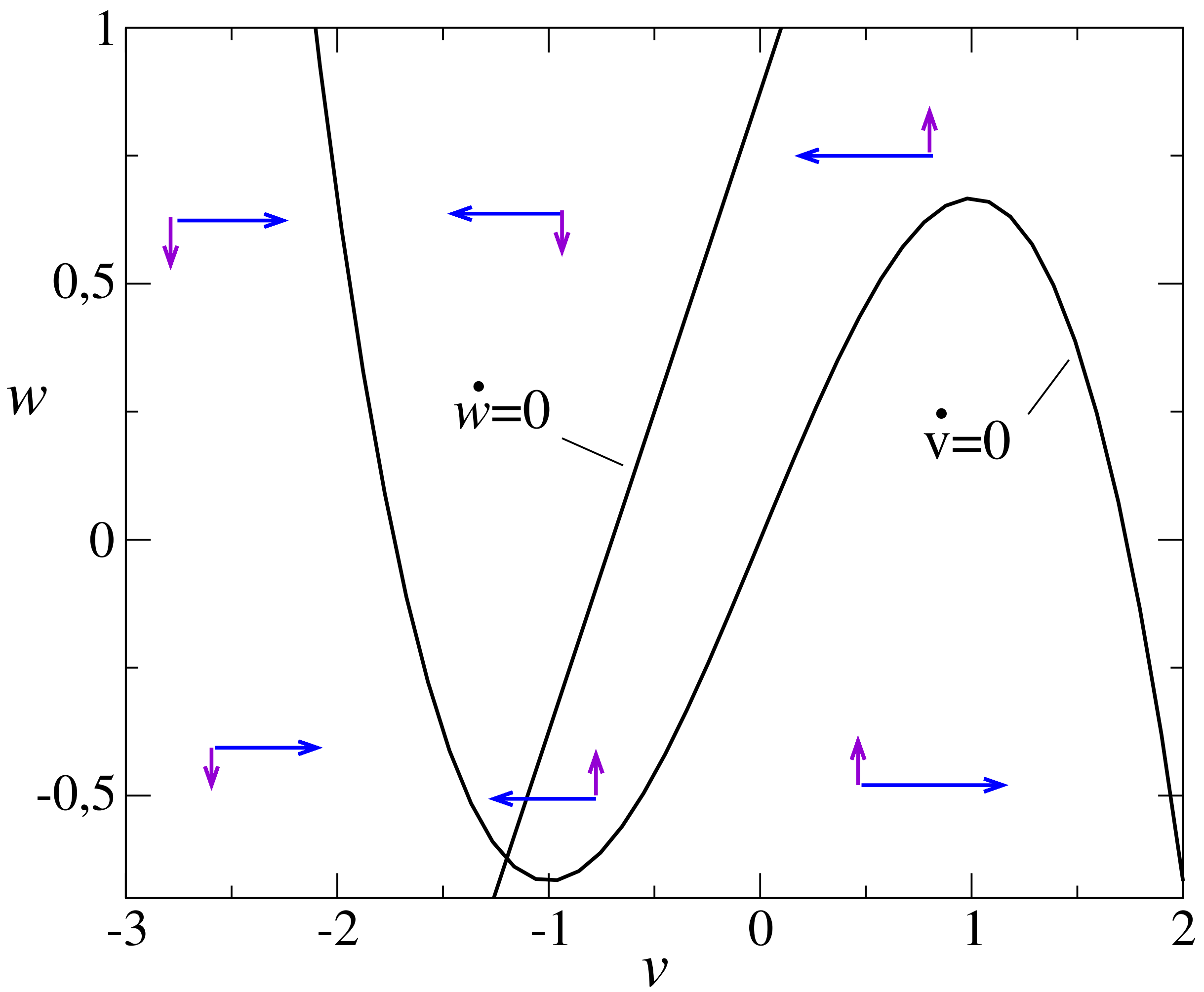
(numerical) solution: $\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \Rightarrow$ trajectory in 2-D plane

Characteristics:

- trajectories cannot cross (uniqueness of solutions)
- **nullclines** define lines in the 2-D plane:
  $\dot{v} = 0 \quad \Rightarrow \quad w = v - \frac{v^3}{3} + I$
  $\dot{w} = 0 \quad \Rightarrow \quad w = (v + a)/b$
- crossings of the nullclines correspond to fixed points (stable for $I = 0$)

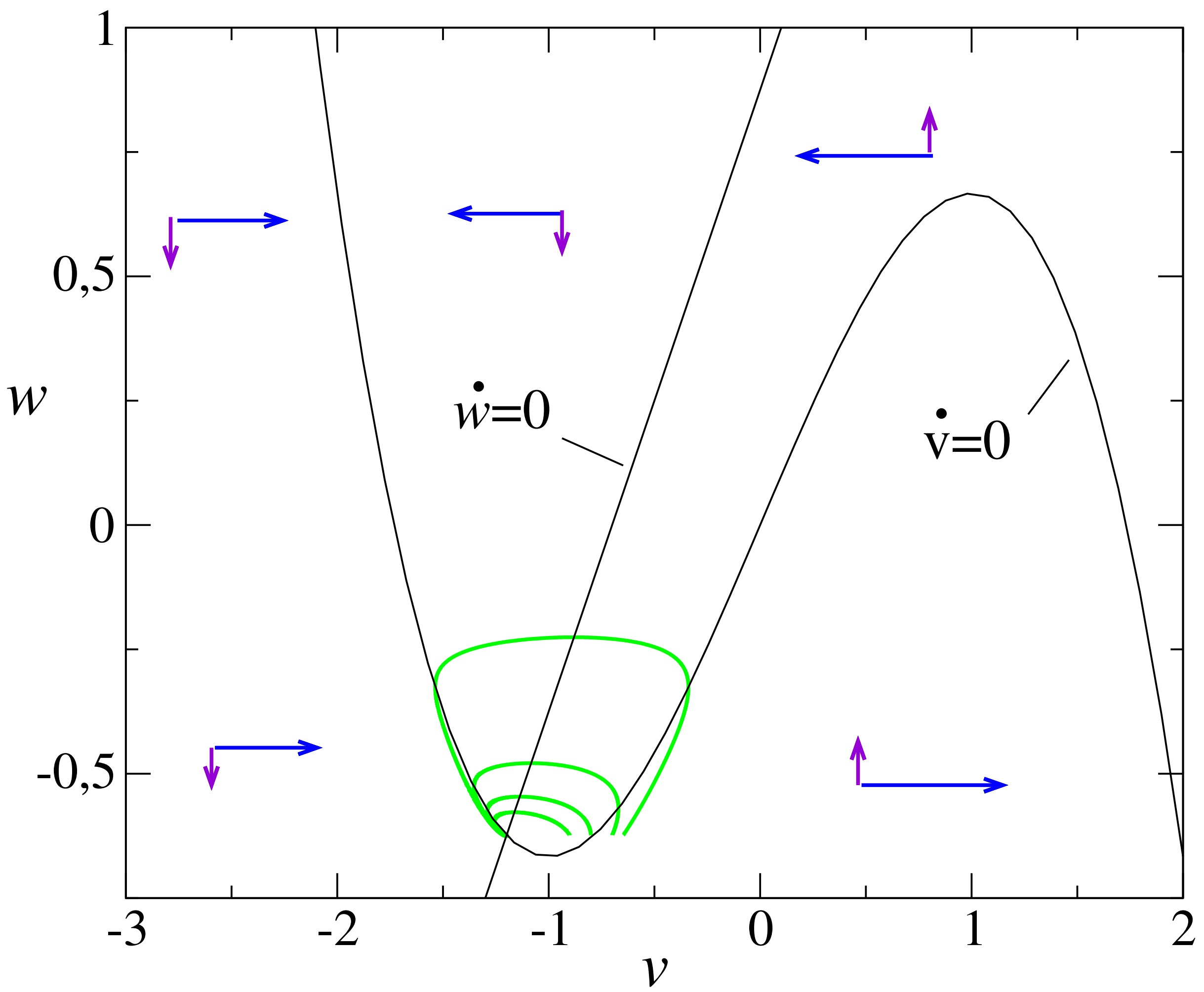
Phase plane portrait of FitzHugh-Nagumo model for $I = 0$

arrows indicate flow field $(\dot{v}, \dot{w})$



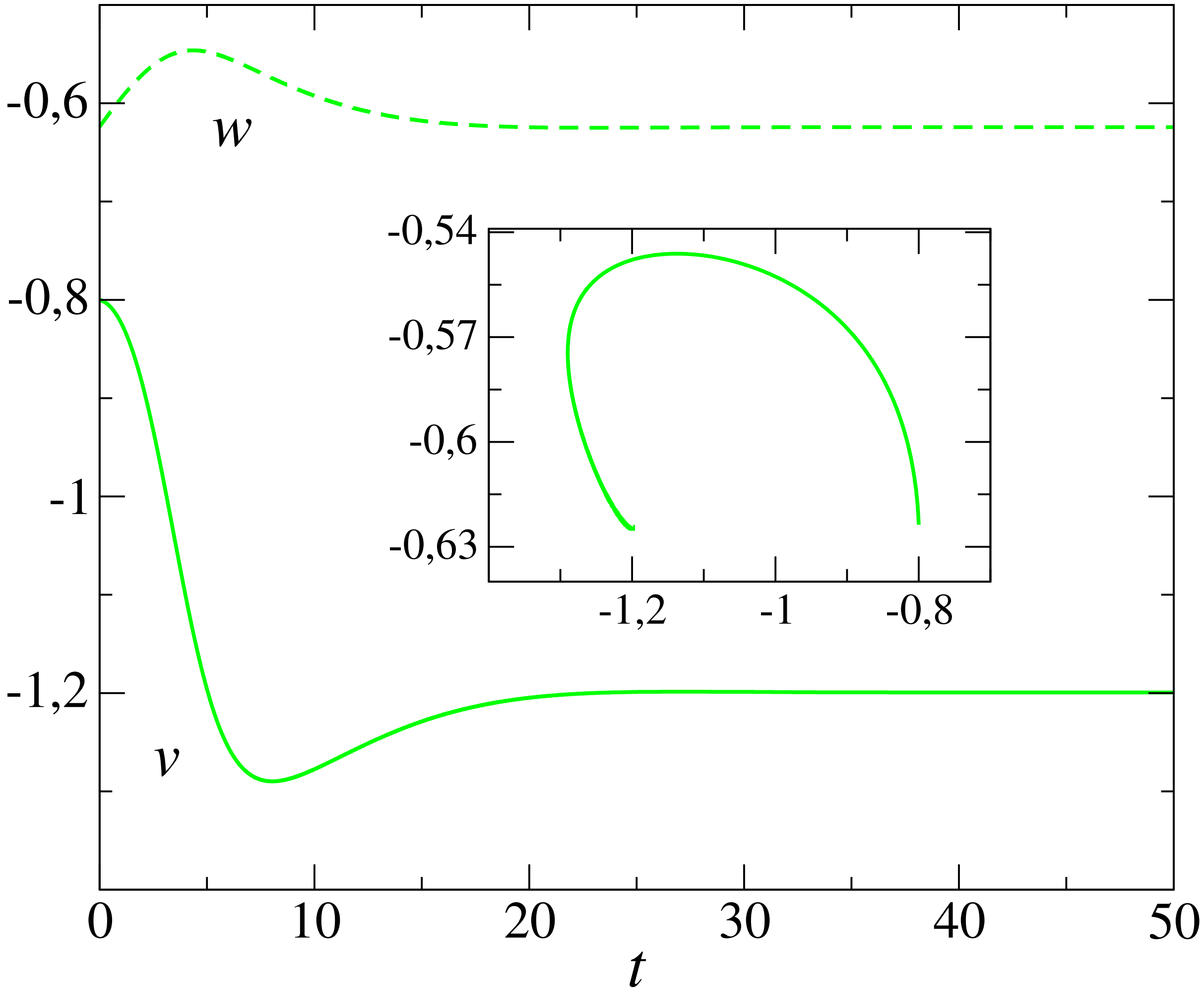
Subthreshold pulse injection

injection of weak pulse $I(t) = I_0 \delta(t - t_0)$: fast return to FP



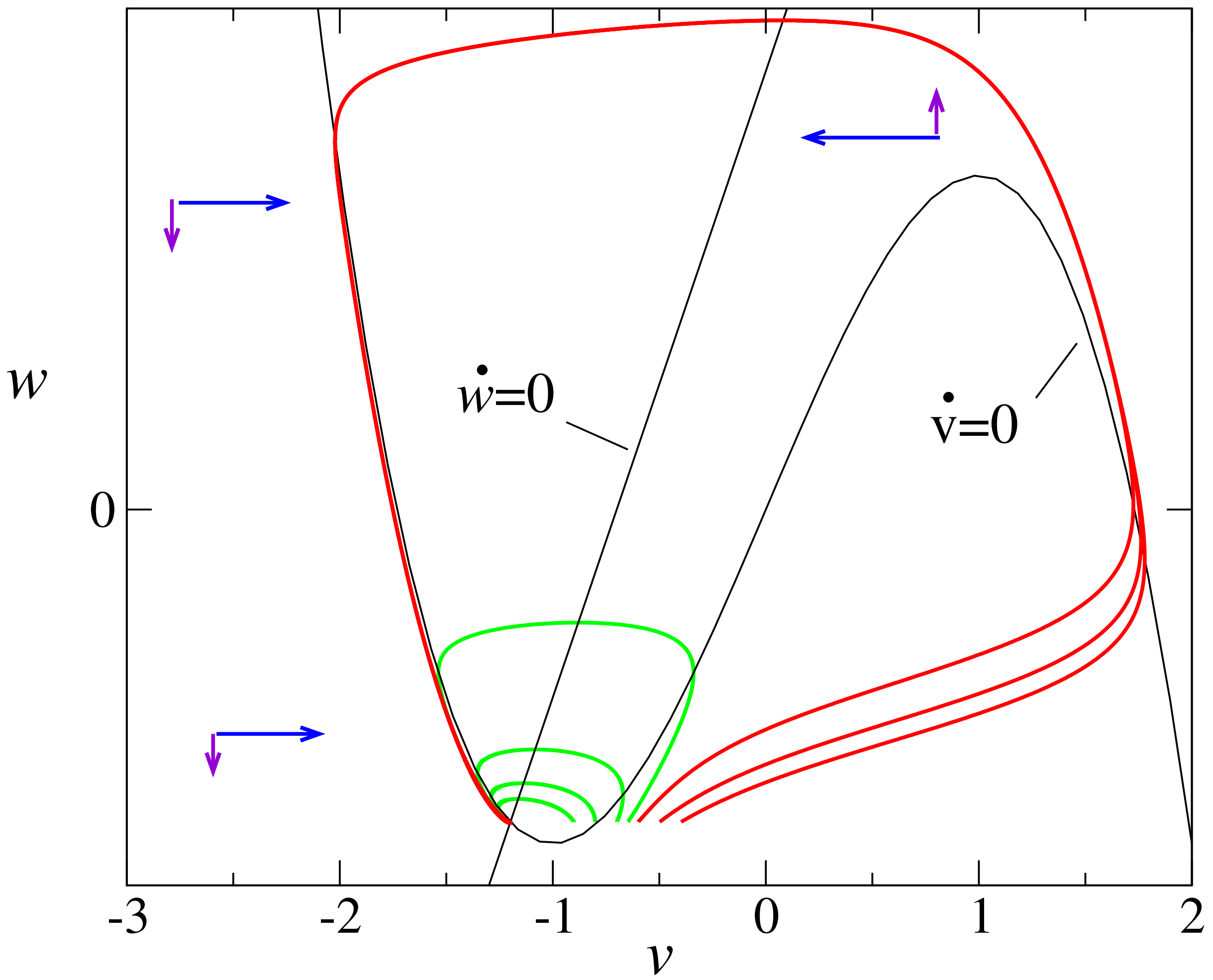
Subthreshold pulse injection

no action potential



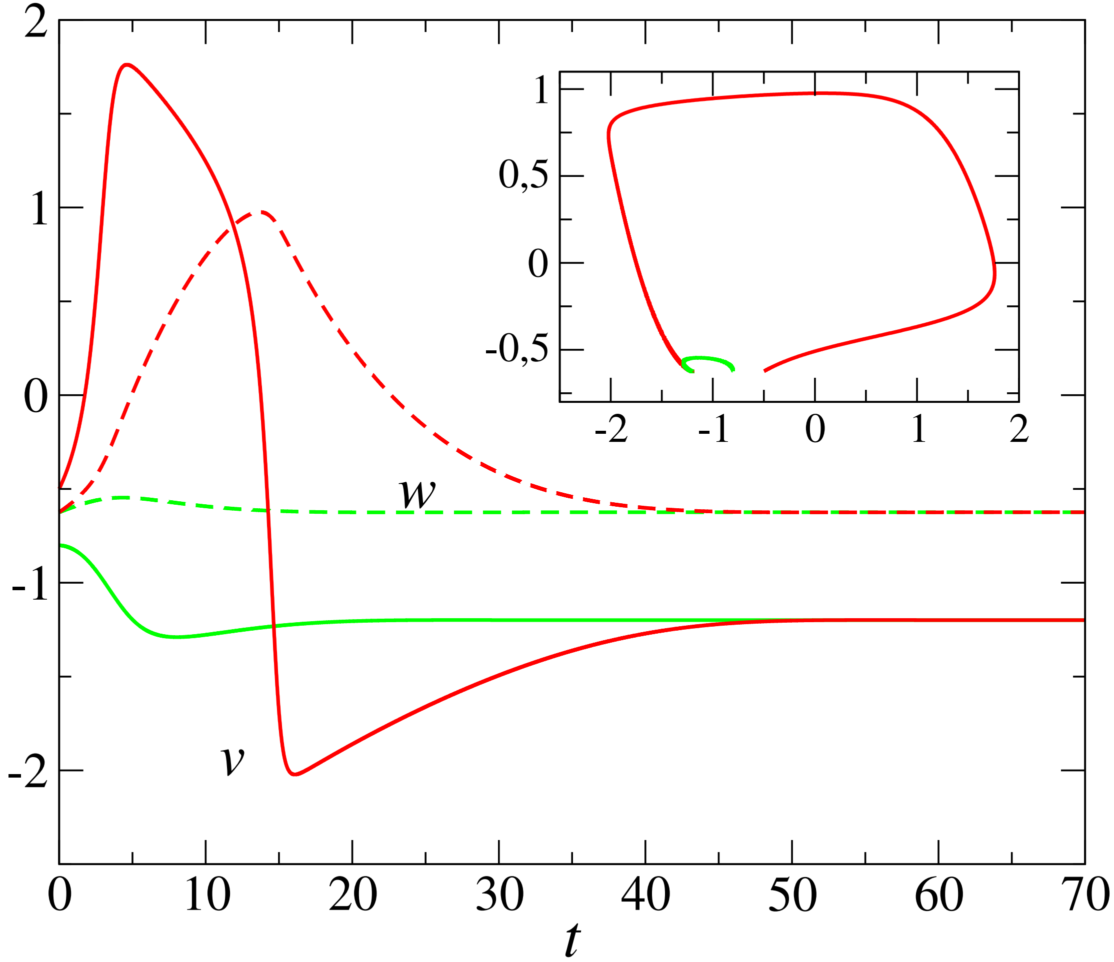
Suprathreshold pulse injection

stronger pulses: large excursion in phase plane



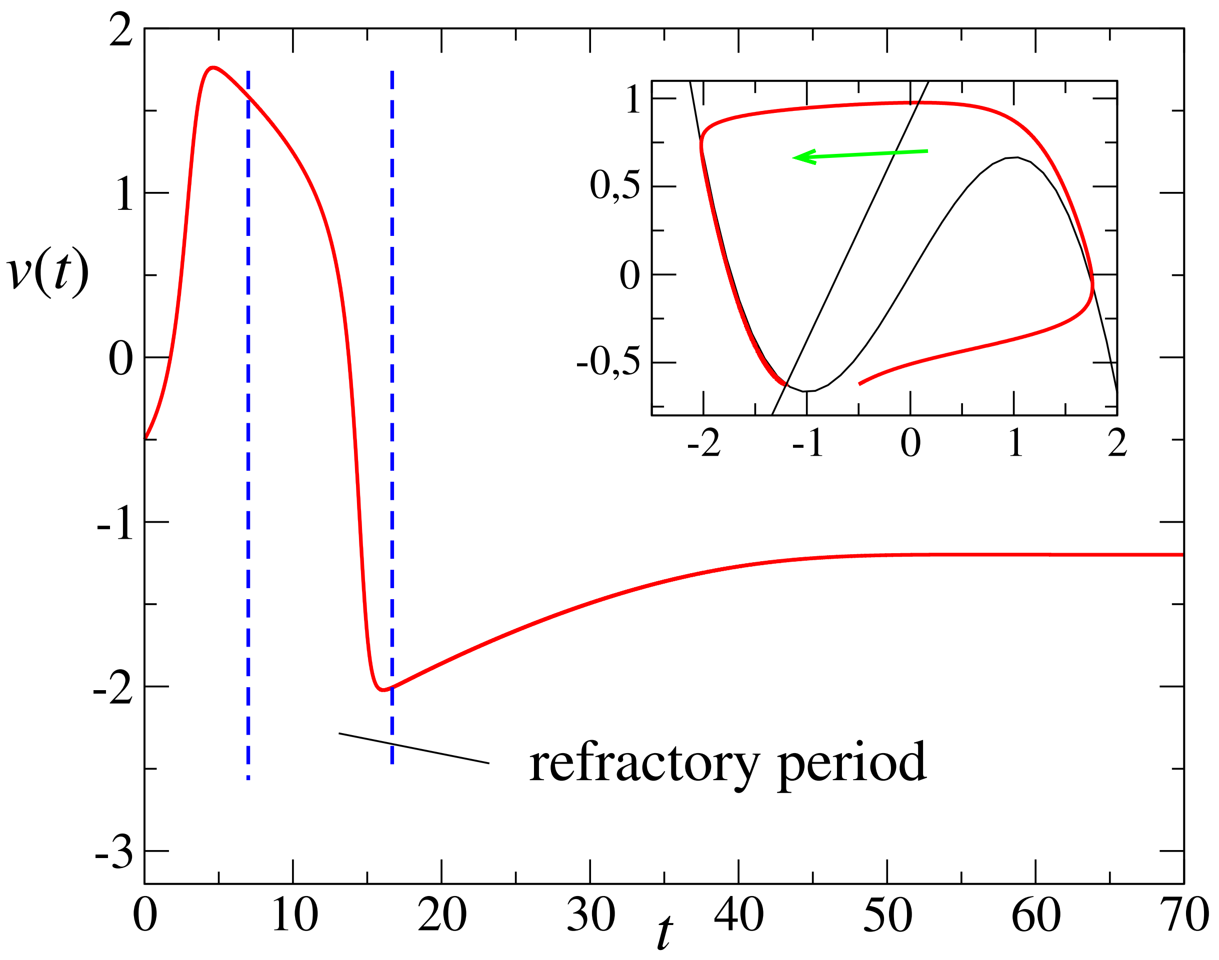
Suprathreshold pulse injection

spike response – action potential generation



Refractory period

Immediately after spike the neuron is indifferent to further input



FitzHugh-Nagumo model for constant $I > 0$

Phase plane analysis:

I shifts nullcline of v, nullcline of w unaffected

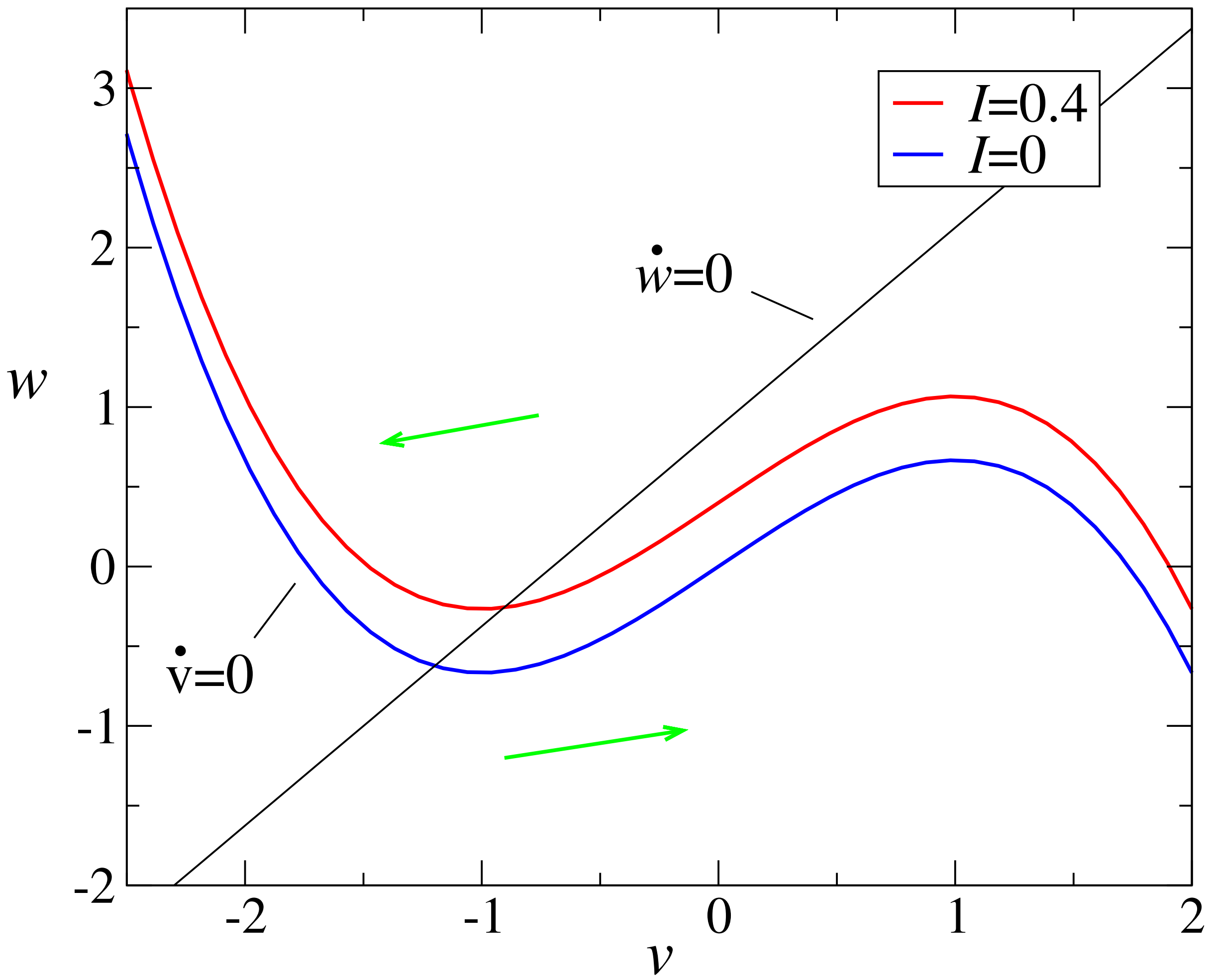
$$\dot{v} = 0: \quad w = v - \frac{v^3}{3} + I, \qquad \dot{w} = 0: \quad w = (v + a)/b$$

$\Rightarrow$ for large enough $I > 0.33$ the fixed point, $\dot{v} = \dot{w} = 0$, becomes unstable

$\Rightarrow$ Onset of **sustained oscillations** (Hopf-bifurcation)

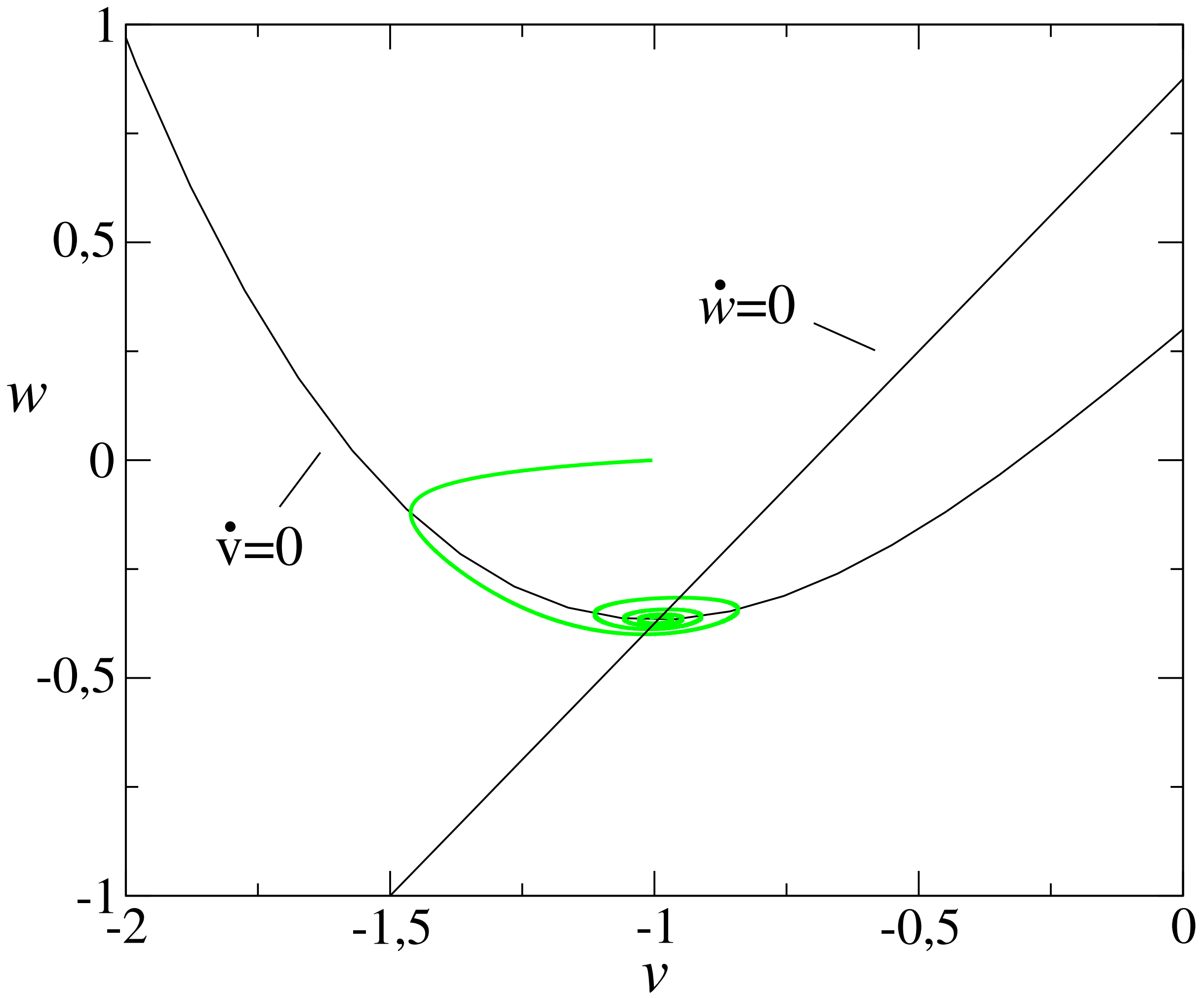
Nullclines for constant $I > 0$

v - nullcline shifted $\Rightarrow$ for $I > 0.33$ the fixed point becomes unstable



Below the bifurcation, $I = 0.3$

Fixed point remains stable $\Rightarrow$ small damped oscillations



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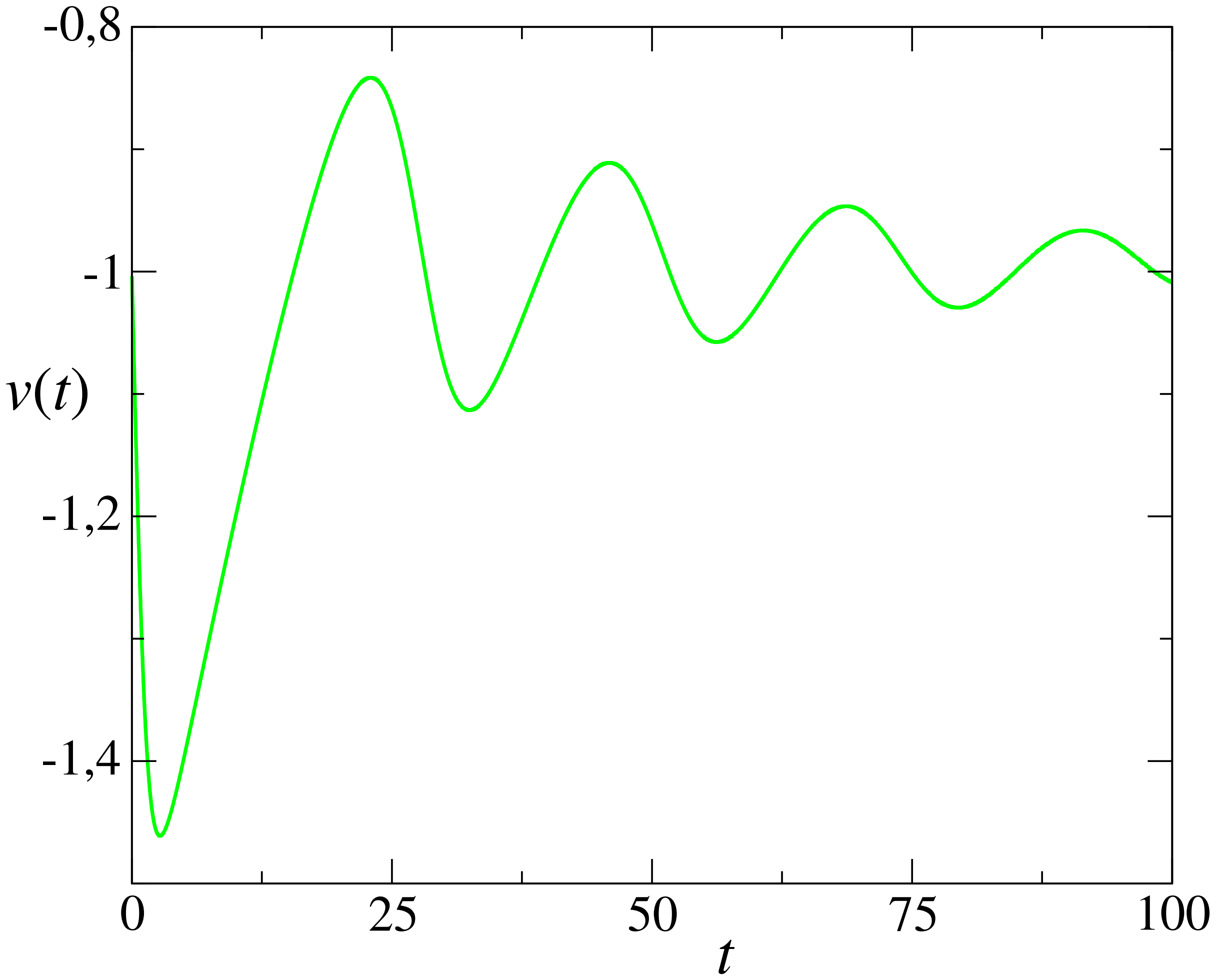
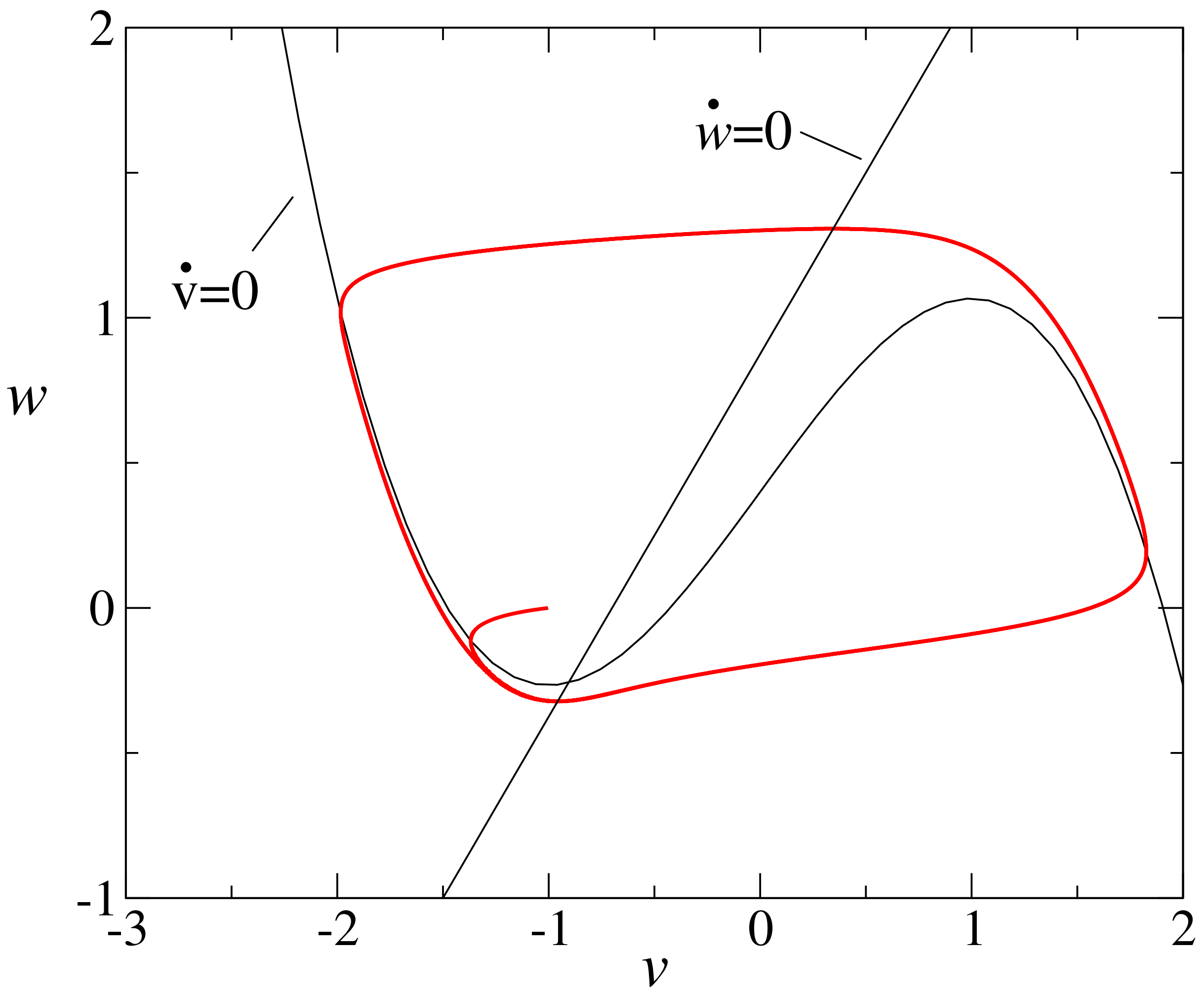


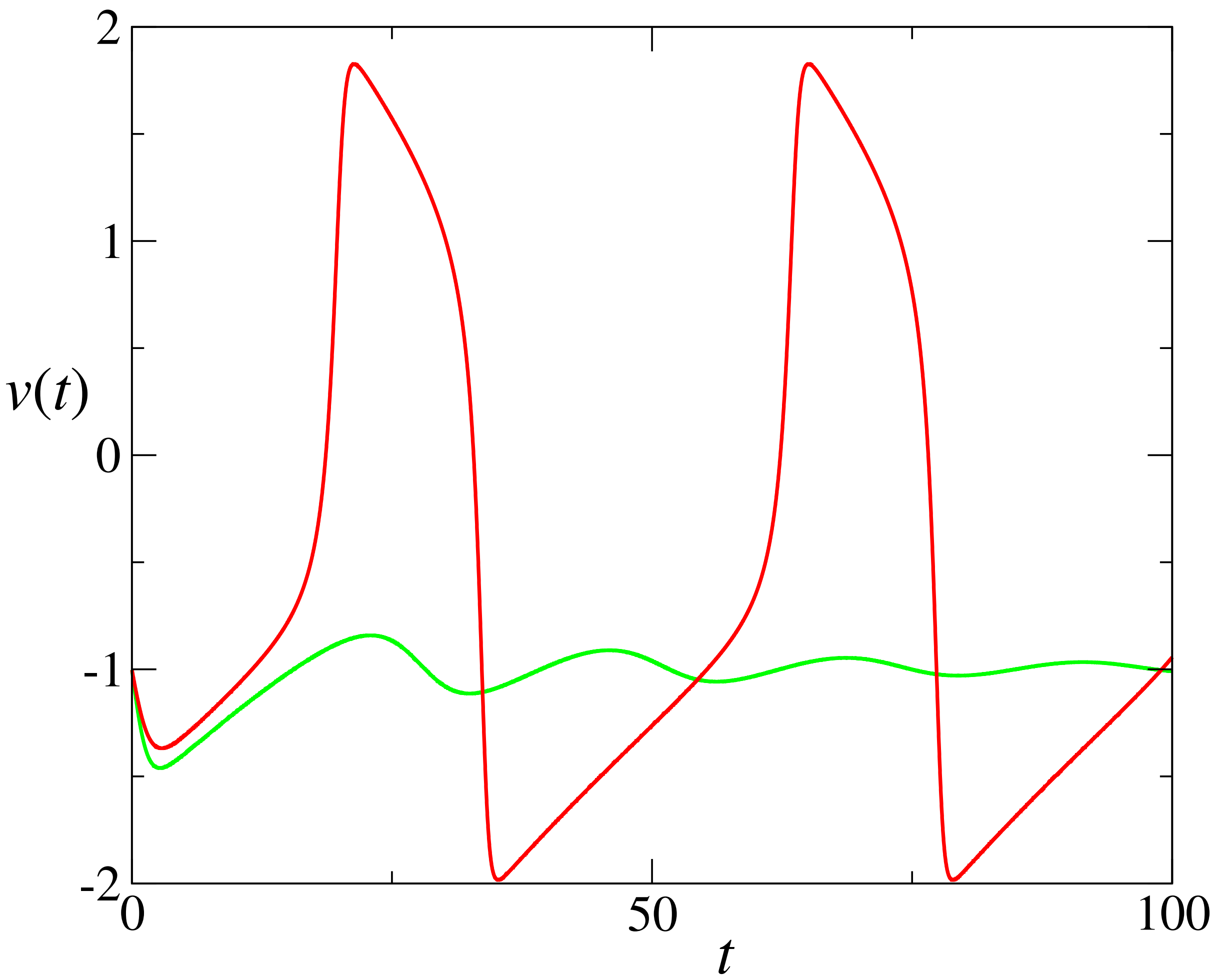
Above the bifurcation, $I = 0.4$

Fixed point unstable $\Rightarrow$ Hopf-bifurcation to sustained oscillations on limit cycle



Above the bifurcation, $I = 0.4$

Fixed point unstable $\Rightarrow$ periodic spiking



FitzHugh-Nagumo model for varying $I(t)$

Recapitulation:

- For $I = const > 0.33$ onset of stable oscillations with Frequency $\Omega(I)$
- Refractory period where system is rather indifferent to external signals

Time dependent input:

- periodic signals: resonance effects
- noisy signals: **coherence resonance**

Summary FitzHugh-Nagumo

- two dimensional model that can be derived from Hodgkin-Huxley via reduction of variables
- allows effective phase plane analysis
- ecitable: spike response to suprathreshold input pulse
- refractory period
- with increasing input current Hopf-bifurcation to sustained periodic spiking

- reduction of complexity: no self-sustained chaotic dynamics
- no bursting
- few parameters: difficult to adapt to neurons with specific properties

The Hindmarsh-Rose model

Developed 1982-1984 by J. L. Hindmarsh and R. M. Rose to allow for rapid firing or **bursting**

Idea:
Allow for triggered firing, i.e., switch between a stable rest state and a stable limit cycle (rapid periodic firing)
$\Rightarrow$ more than one fixed points required: can be achieved by deformation of the null-clines (nonlinear "current" equation)

Basic equations:

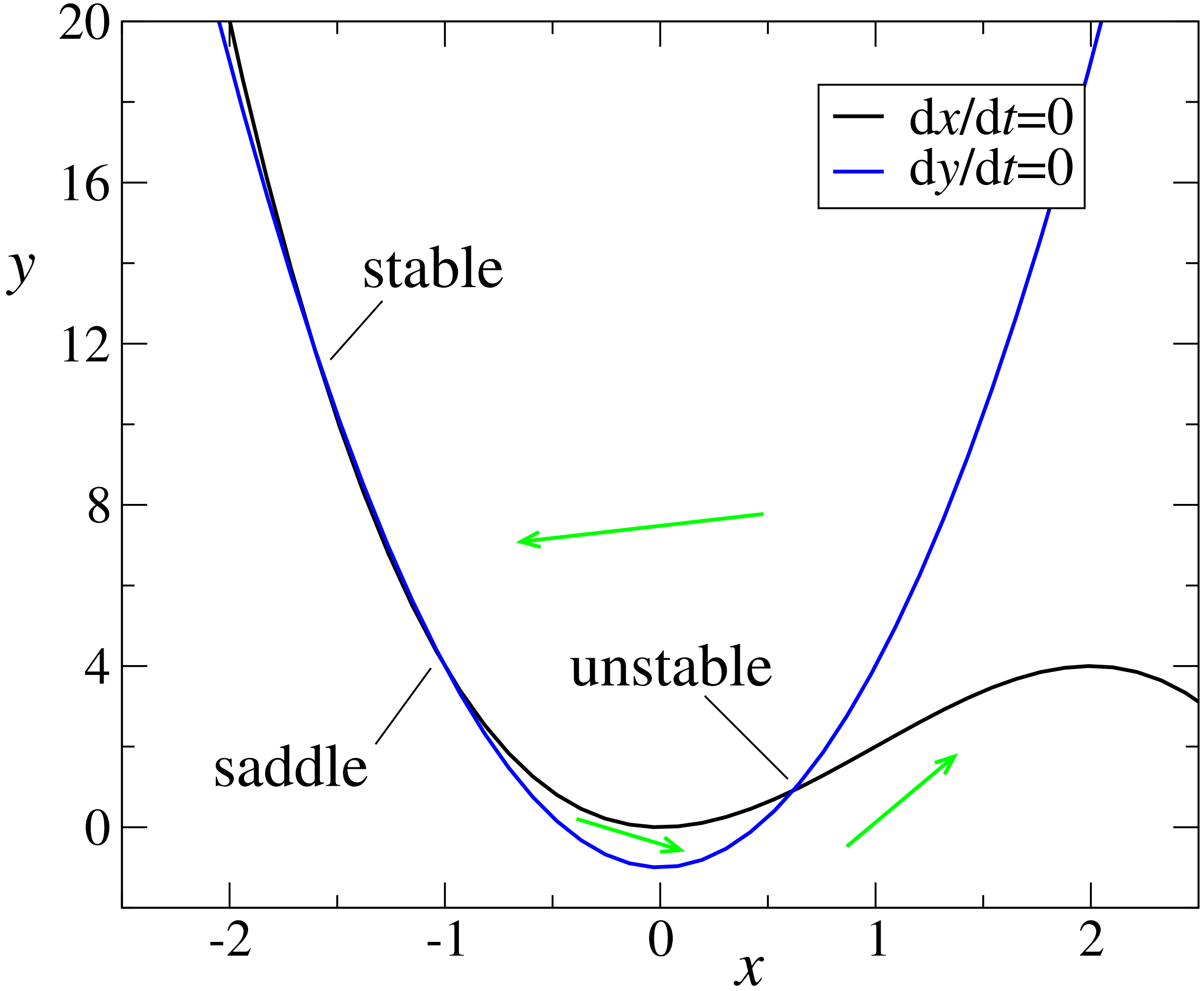
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x^2 - x^3 - y + I \,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 5x^2 - 1 - y$$

Nullclines:

$$\dot{x} = 0 : \quad y = 3x^2 - x^3 + I \,, \qquad \dot{y} = 0 : \quad y = 5x^2 - 1$$

Phase portrait of Hindmarsh-Rose model

3 Fixed points $\Rightarrow$ coexistence of rest state and limit cycle



Adaption variable

Termination of firing via additional adaption variable z that should:

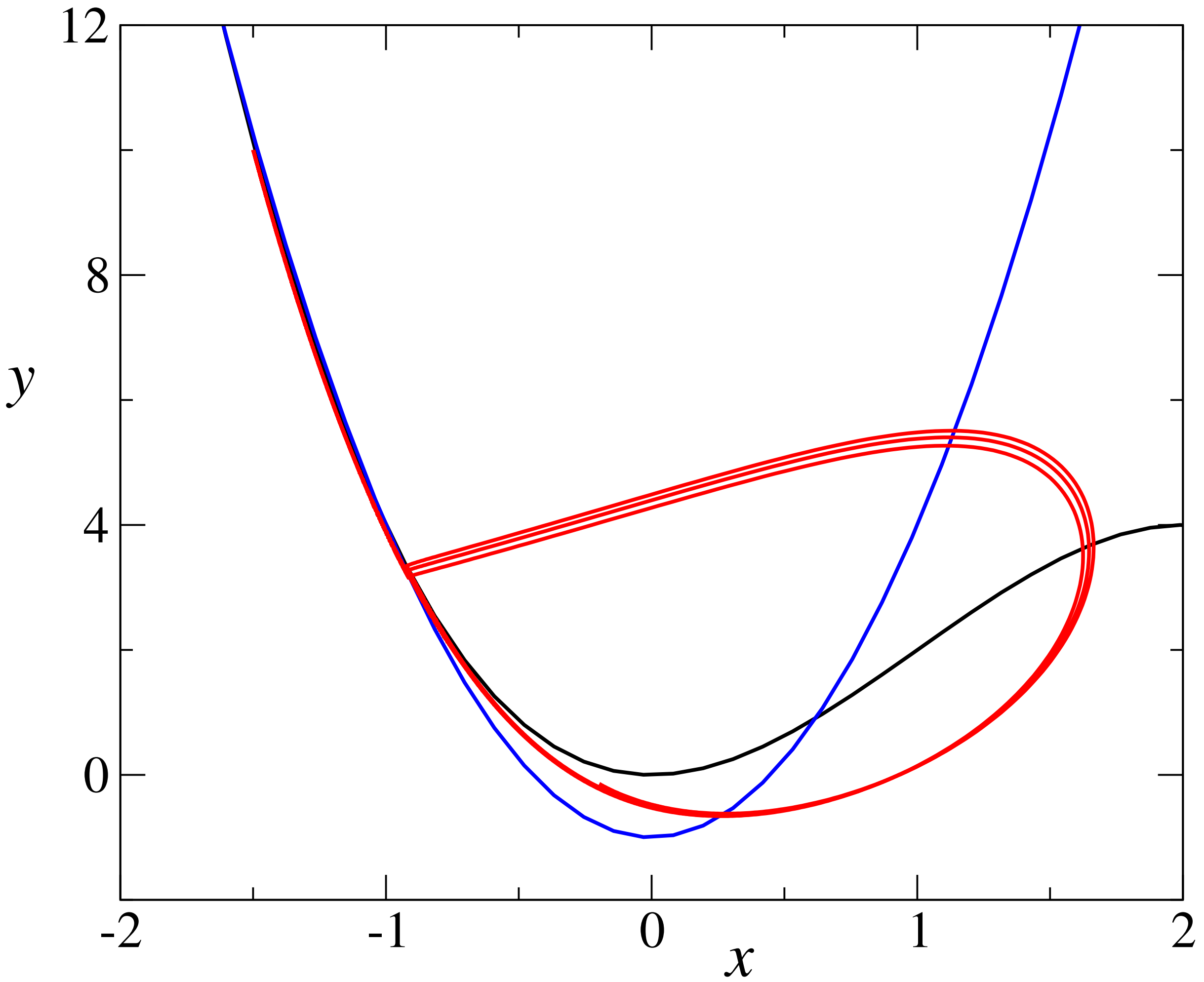
- lower the effective current when neuron is firing
- return to zero when x has reached its rest state value x_r

Complete equations:

$$\frac{\mathrm{dx}}{\mathrm{d}t} = 3x^2 - x^3 - y + I - z \ , \quad \frac{\mathrm{dy}}{\mathrm{d}t} = 5x^2 - 1 - y \ , \quad \frac{\mathrm{dz}}{\mathrm{d}t} = r\left[s\left(x - x_r\right) - z\right]$$

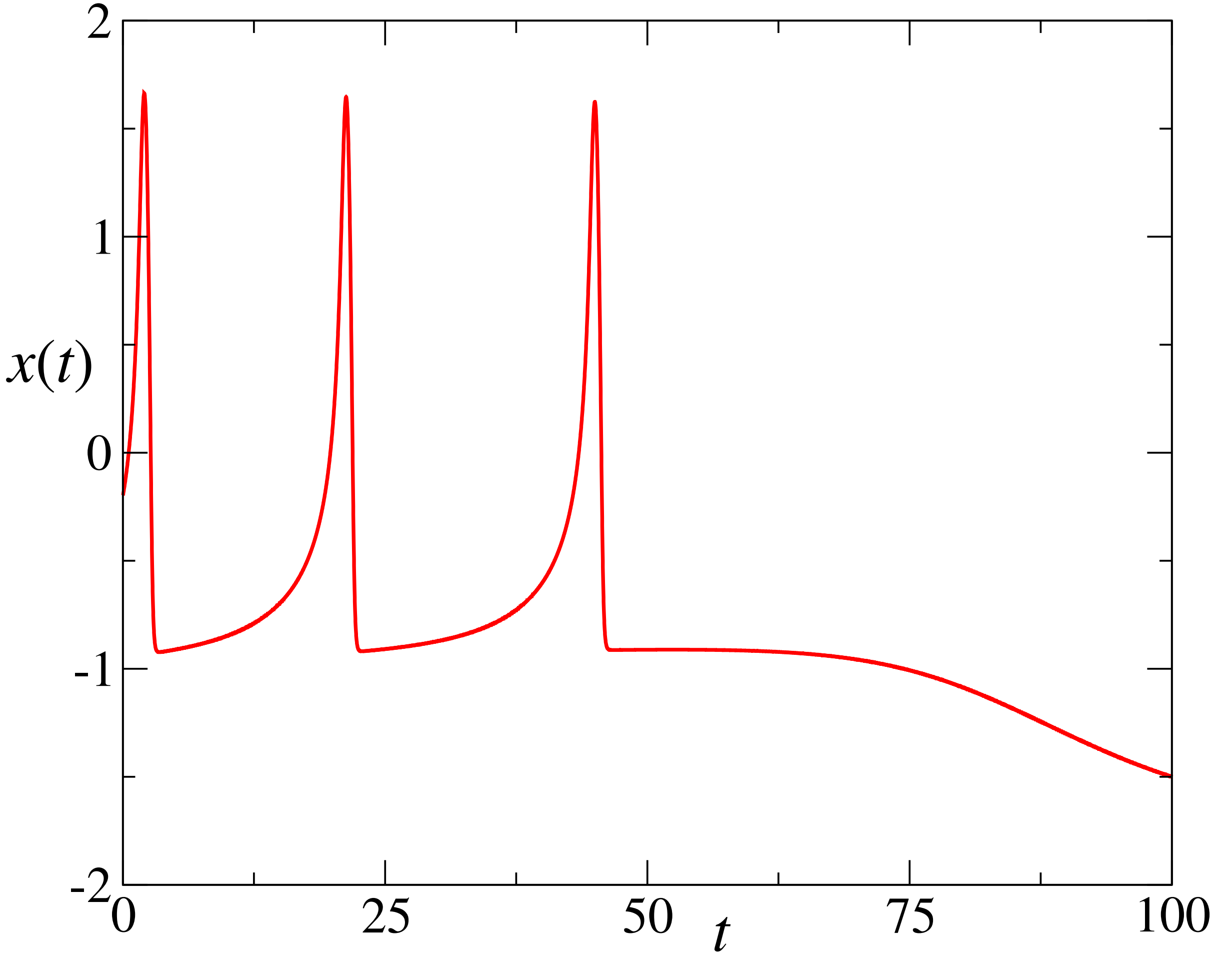
Bursting of Hindmarsh-Rose model

After repeated firing the dynamics returns to the stable fixed point



Bursting of Hindmarsh-Rose model

Several spikes with varying interspike-interval (ISI)



Features of the Hindmarsh-Rose model

3-D model for neuron with rapid firing

Suitable choice of parameters allows for

- regular bursting
- chaotic bursting

Suitable choice of parameters ? $\Longleftrightarrow$? real neurons

Further reading

- W. Gerstner and W. M. Kistler, Spiking Neuron Models: Single Neurons, Populations, Plasticity,
  http://diwww.epfl.ch/ gerstner/SPNM/SPNM.html .
- C. Koch, Biophysics of Computation: Information Processing in Single Neurons (Computational Neuroscience), Oxford University Press.
- J. Hindmarsh and P. Cornelius, The Development of the Hindmarsh-Rose model for bursting,
  www.worldscibooks.com/lifesci/etextbook/5944/5944_chap1.pdf .