Intermittent Chaotic Chimeras

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Kuramoto at MPIPKS

On october, 11th 1997 I gave the following talk at MPIPKS

Transition from phase to amplitude turbulence in the CGLE

Seminar on Patterns and Dynamics in Complex Fluids and Biological Systems
Scientific Director: W. Zimmermann

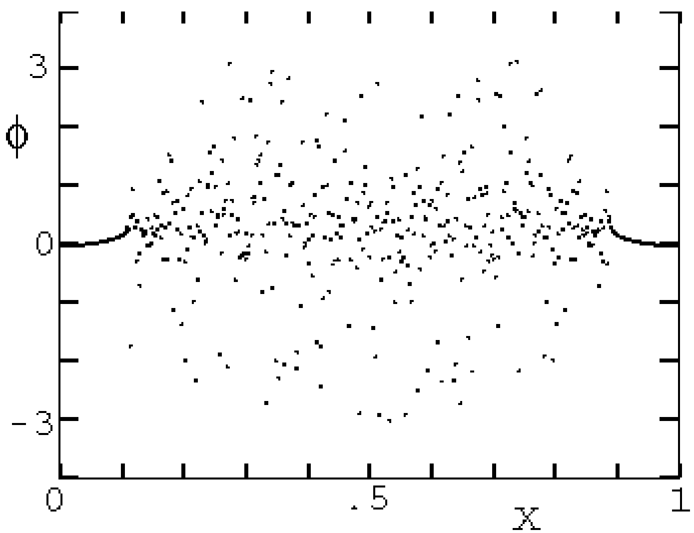
Kuramoto was sitting in the first row and after the talk came by me . . .

AT, H. Frauenkron, P. Grassberger, PRE 55 (1997) 5073

My 15 Minutes of Fame

On october, 12th 1997 at 8.30 AM, we met and Kuramoto says

- I am interested in non-locally coupled CGLE
- My collaborator, Dorjsuren Battogtokh, has found this numerical result



- It seems you developed a new integration scheme for CGLE
- May you please send it to Battogtokh ?

Y Kuramoto & D Battogtokh, Nonlinear Phenomena in Complex Systems (2002)

Pteroptix Malaccae

Usually, entrainment results in a constant phase angle equal to the difference between pacing frequency and free-running period as it does in P. Cribellata. The mechanism of attaining synchrony by Malaysian firefly Pteroptyx Malaccae is quite different. When the pacer changes, this firefly requires several cycles to reach a steady state. Once this steady state is achieved, the phase angle difference is near zero irrespective of the pacer period. This can be explained only by the animal adjusting the period of its oscillator to equal that of the driving oscillator.

`Experiments on Fireflies by Hanson, 1982`

Kuramoto Model with inertia

- Ermentrout developed a pulse-coupled (Winfree) model with inertia to deal with this kind of synchronization [B. Ermentrout, Journal of Mathematical Biology (1991)]
  - A phase model with inertia allows for adaptation of oscillator frequency to the forcing one
- Tanaka, Lichtenberg, Oishi [PRL, Physica D 1997] developed a generalization of Kuramoto model by including an inertia term

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i)$$

  - Hysteretic first order synchronization transition
  - Self-consistent mean field equation the macroscopic order parameter
  - For sufficiently large inertia $\rightarrow$ Clusters of drifting oscillators appear
  - Peculiar finite size and finite mass scaling

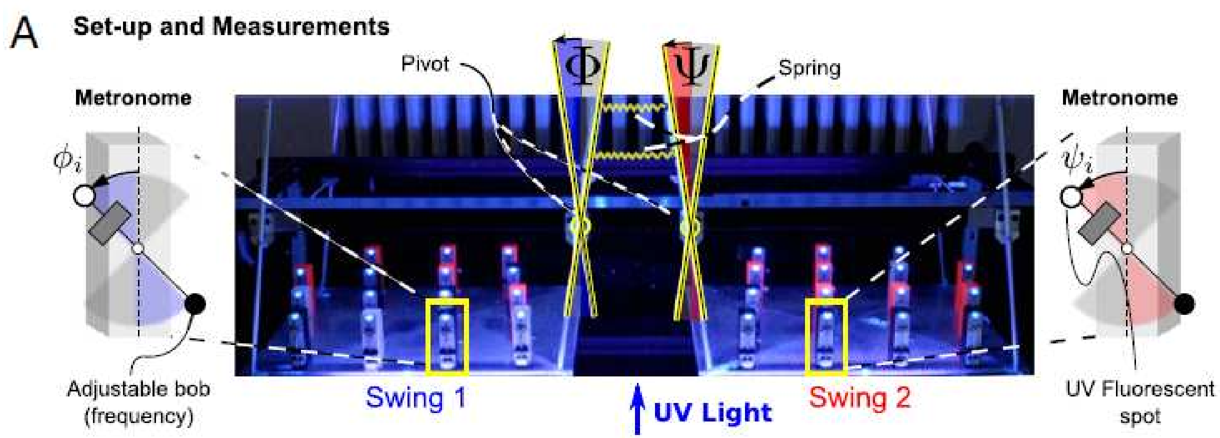
Acebrón et al PRE (2000); Gupta et al PRE (2014); Komarov et al PRE (2014); Olmi et al PRE (2014)

Plan of the Talk

Dynamics of two symmetrically coupled populations of rotators

- A brief description of the experiment
- Introduction of the 2 population model
  - Kuramoto model with inertia
- Emergence of chaotic broken symmetry states
  - Intermittent Chaotic Chimeras (ICC)
  - Chaotic Two Populations state (C2P)
- Linear Stability (Lyapunov) of an Intermittent State
  - Theoretical estimation for globally coupled systems
- ICCs are Transient States
- Some considerations on
  - The role of topology
  - The thermodynamic limit

The experiment (WOW !!!)



- Two populations of metronomes (self-sustained oscillator)
- Each population: $N = 15$ identical metronomes (same frequencies) on an alluminium swing (strong coupling)
- The two swings are coupled via 2 tunable springs (weak coupling)
- UV fluorescent spots on metronomes and swings

THE VIDEO !

E.A.Martens *et al.* PNAS , 2013

The Model

Two symmetrically coupled populations of N oscillators with inertia (rotators)

$$m\ddot{\theta}_i^{(\sigma)} + \dot{\theta}_i^{(\sigma)} = \Omega + \sum_{\sigma'=1}^{2} \frac{K_{\sigma\sigma'}}{N} \sum_{j=1}^{N} \sin(\theta_j^{(\sigma')} - \theta_i^{(\sigma)} - \gamma)$$

- $\sigma = 1, \ 2$ identifies the population
- $\theta_i^{(\sigma)}$ is the phase of the ith oscillator in population σ
- Ω is the natural frequency
- $\gamma = \pi - 0.02$ is the fixed frequency lag
- $K_{\sigma,\sigma} > K_{\sigma,\sigma'}$

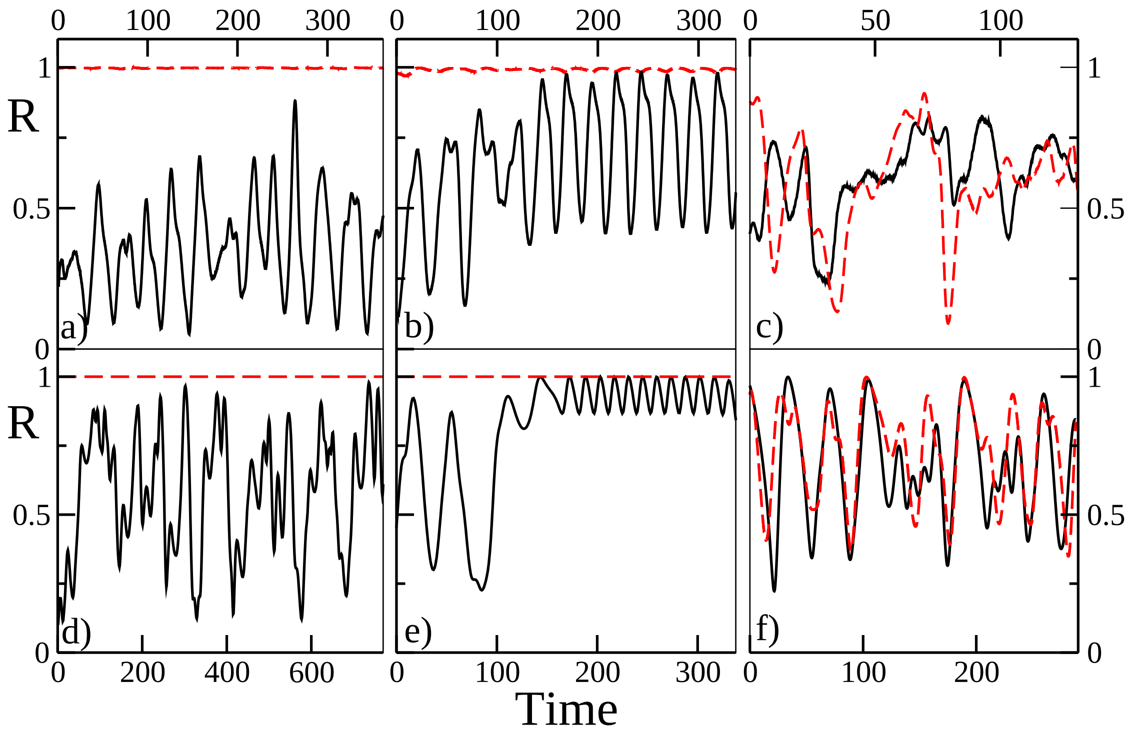
The collective evolution of each population is characterized in terms of the macroscopic fields

$$\rho^{(\sigma)}(t) = R^{(\sigma)}(t) \exp\left[i\Psi(t)\right] = N^{-1} \sum_{j=1}^{N} \exp\left[i\theta_j^{(\sigma)}(t)\right]$$

where $R^{(\sigma)}$ is the order parameter for the synchronization transition

In analogy with Abrams, Mirollo, Strogatz and Wiley, PRL (2008).

Experiment vs Model



- (a)-(d) Chaotic Chimeras
- (b)-(e) Intermittency
- (c)-(f) C2P states

- $R^{(1)}$ and $R^{(2)}$ for the two populations, $N = 15$
- Different Initial conditions
  - Broken Simmetry Conditions in (a,b) (d,e)
  - Uniform Conditions in (c,f)

Chaotic Chimeras

or Regular States ?

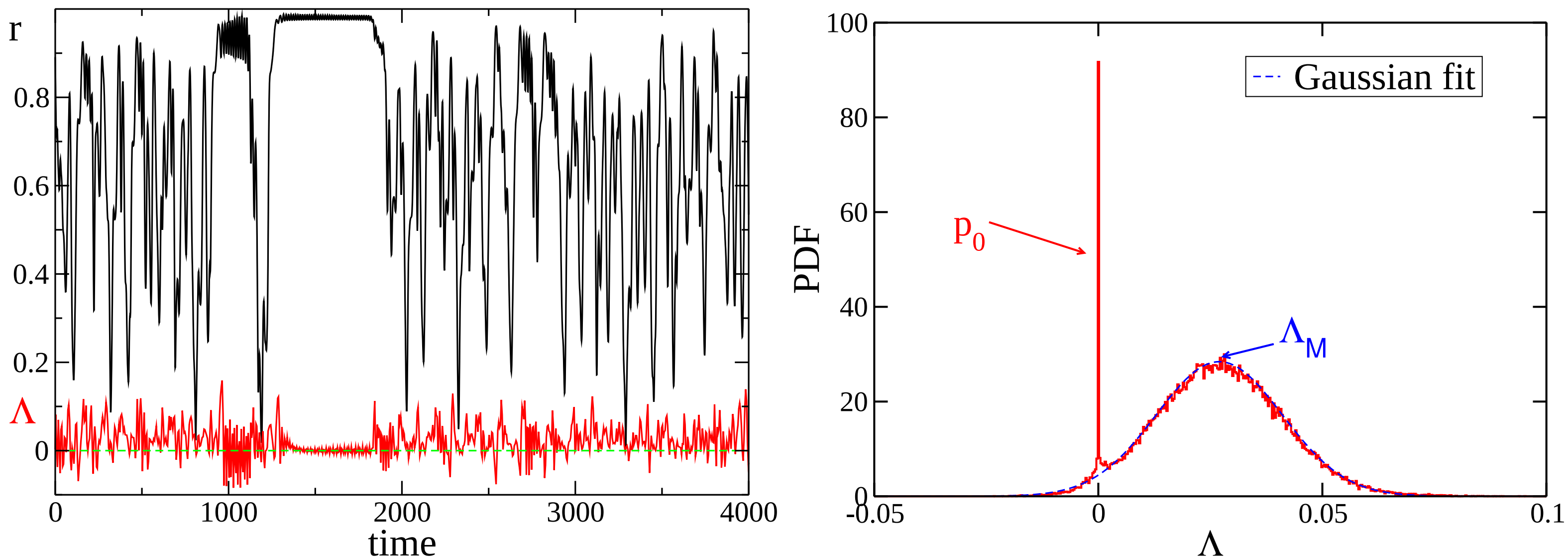
Chaotic Chimeras

- Spatio-Temporally Chaotic Chimeras observed in ring of coupled oscillators
  Bordyugov et al PRE (2010); Wolfrum and Omelchenko (2011); Sethia and Sen, PRL (2014)
- Kuramoto model on a ring with finite-range interactions
  - Chimeras are transient
  - The transient time diverges exponentially with the size
  - Chimeras are weakly chaotic, with features of spatially extended systems

  Wolfrum et al Chaos (2011)
- Chaotic Chimeras (CCs) reported in two population of pulse-coupled oscillators
  Pazó & Montbrió, PRX (2014)
- Our aim: describe dynamical features of CCs for 2 coupled populations of Kuramoto model with inertia

Intermittent Chaotic Chimera

The chaotic population exhibits clear intermittent behavior, displaying a laminar phase where the two populations tend to synchronize and a turbulent phase.

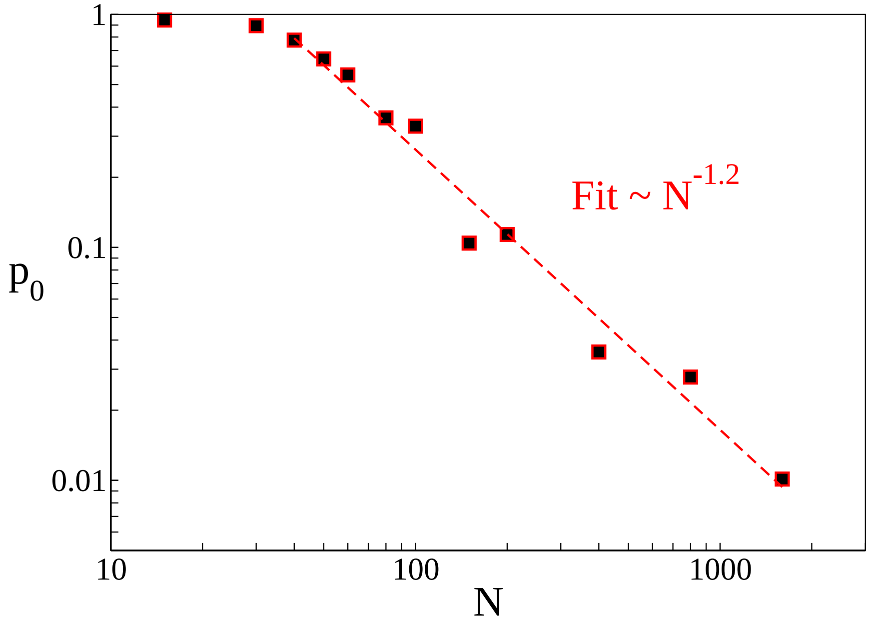


The finite time Lyapunov exponent (FTLE) $\Lambda(t) = \frac{1}{\Delta t} \ln[\sqrt{\sum_{i=0}^{4N} \mathcal{T}_i(\Delta t)\mathcal{T}_i(\Delta t)}]$

is calculated by performing a short time average of the magnitude of the tangent vector

$$\mathcal{T} = (\delta\dot{\theta}_1^{(1)}, ..., \delta\dot{\theta}_N^{(1)}, \delta\dot{\theta}_1^{(2)}, ..., \delta\dot{\theta}_N^{(2)}, \delta\theta_1^{(1)}, ..., \delta\theta_N^{(1)}, \delta\theta_1^{(2)}, ..., \delta\theta_N^{(2)})$$

Laminar Phase

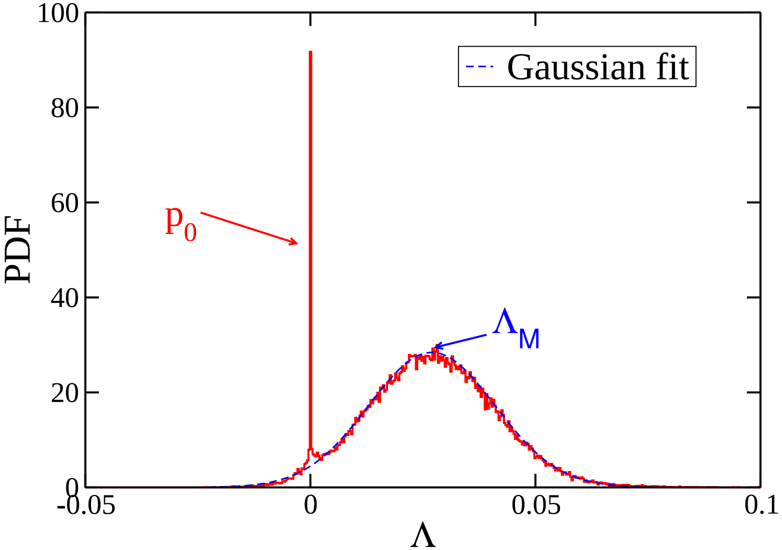


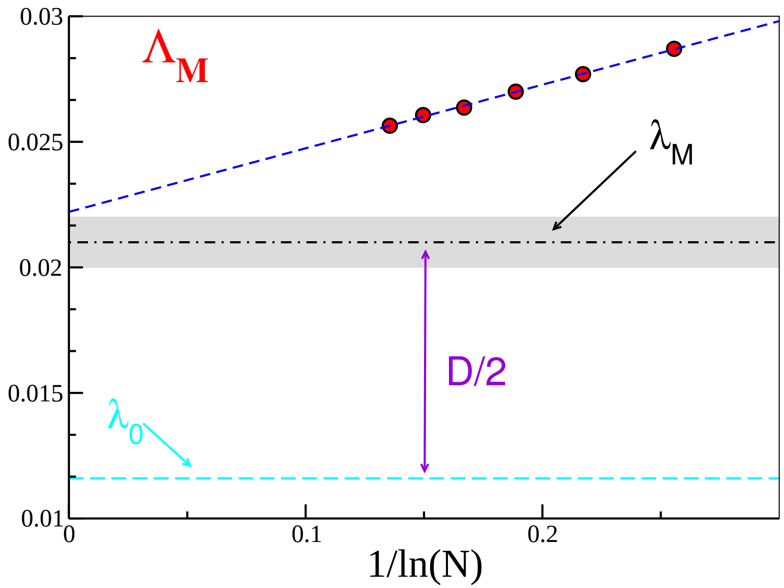
- The laminar phase, whose probability of occurrence is measured by p_0 vanishes in the thermodynamic limit;
- In the limit $N \to \infty$ only the turbulent regime is present,

Maximal Lyapunov exponent

The maximal Lyapunov exponent Λ_M restricted to the turbulent phase

- Λ_M remains positive in the limit $N \to \infty$
- Λ_M scales as $1/lnN$ with the system size





$$\Lambda_M(N \to \infty) > \lambda_0 > 0$$

λ_0 is the mean field LE for the chaotic population

Mean Field Result

The mean field evolution of a single rotator forced by the complex fields $\rho^{(\sigma')} = R^{(\sigma')}(t)e^{i\Psi(t)}$ is

$$m\ddot{\phi}^{(\sigma)} + \dot{\phi}^{(\sigma)} = \Omega + \sum_{\sigma'=1}^{2} K_{\sigma\sigma'} \left[\Im\rho^{(\sigma')} \cos(\phi^{(\sigma)} + \gamma) - \Re\rho^{(\sigma')} \sin(\phi^{(\sigma)} + \gamma)\right] .$$

the growth rate of the infinitesimal perturbation $d(t) = \sqrt{|\delta\dot{\phi}^{(1)}(t)|^2 + |\delta\phi^{(1)}(t)|^2}$ is the mean field Lyapunov exponent (LE) λ_0 for the chaotic population.

- The evolution of $\ln d(t)$ in the tangent space can be seen as a drifting Brownian particle with average velocity λ_0 and a diffusion coefficient D .

$$\ln d(t) \simeq \lambda_0 t \qquad [\ln d(t) - \lambda_0 t]^2 = Dt$$

- From the mean field analysis one gets $\lambda_M = \lambda_0$, this is correct for the bulk part of the spectrum, but wrong for the maximal LE
- The interaction with the other rotators should be taken in account

$$\dot{d}_j(t) = e^{\lambda_0 t}[d_j(t) + \frac{1}{N}\sum_k \mathcal{A}_{jk}(t) d_k(t)]$$

Same analysis as in the talk by Edward Ott

Beyond mean field

The particles $\ln|d_j(t)|$ are indeed interacting Brownian particles

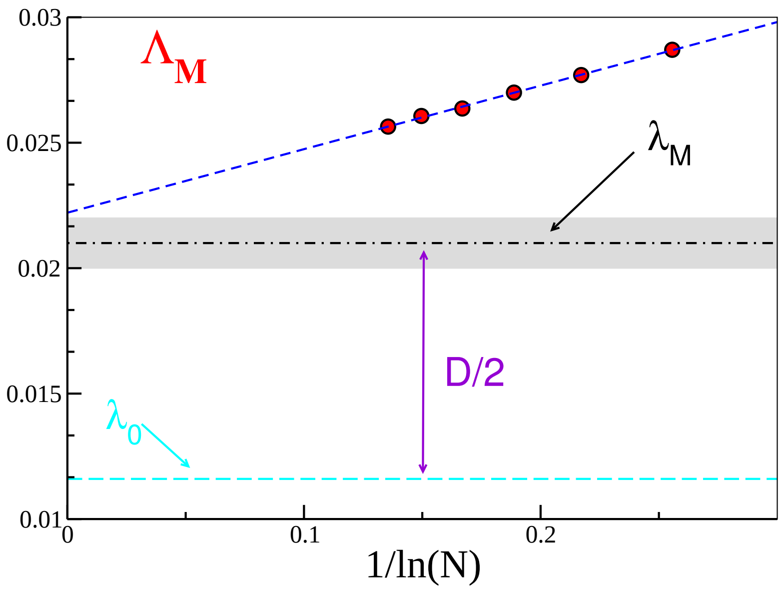
$$\dot{d}_j(t) = \mathrm{e}^{\lambda_0 t}[d_j(t) + \frac{1}{N}\sum_k \mathcal{A}_{jk}(t) d_k(t)]$$

Assuming a localized Lyapunov vector the modulus is dominated by the largest component $d_M(t)$ therefore

- $\ln|d_j(t)|$ diffuses freely if $|d_j(t)| >> |d_M(t)|/N$
- otherwise the coupling term enters in the game and the growth is dominated by $|d_M(t)|/N$
  - the particles can move only within a box of size $\simeq \ln(N)$: the slowest particles are pulled and the fastest particle pushes the box;
  - The maximal LE λ_M is the average velocity of the box.

Finite Size Scaling

One can write a Fokker-Planck equation for the evolution of these pseudo-particles within the box and from the stationary solution one gets the correct scaling for the maximal LE



$$\lambda_M(N) = \lambda_0 + \frac{D}{2} + \frac{a}{\ln(N)} + \mathcal{O}\left(\frac{1}{\ln^2(N)}\right)$$

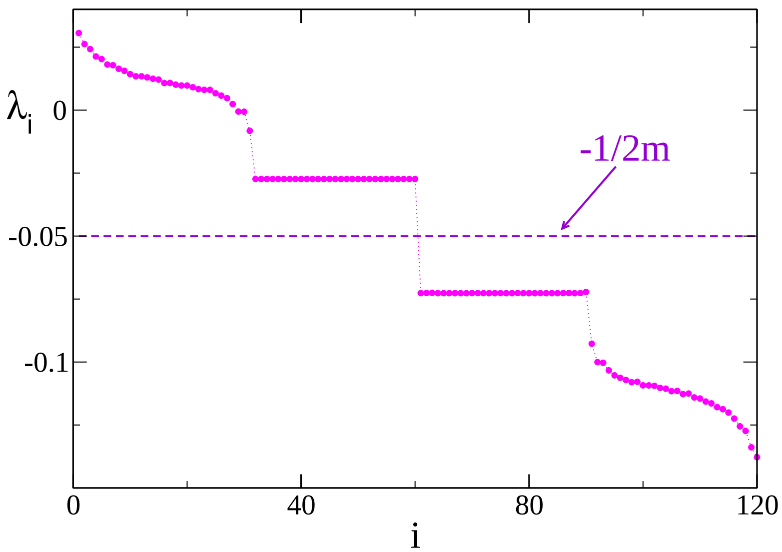
- λ_0 is the mean field LE value
- D is the diffusion coefficient associated to the fluctuations of λ_0

First quantitative verification for a continuous time system of the Takeuchi et al theory, with the extra difficulties to have intermittent dynamics and two interacting populations

Takeuchi, Chaté, Ginelli, Politi, AT, PRL (2011)

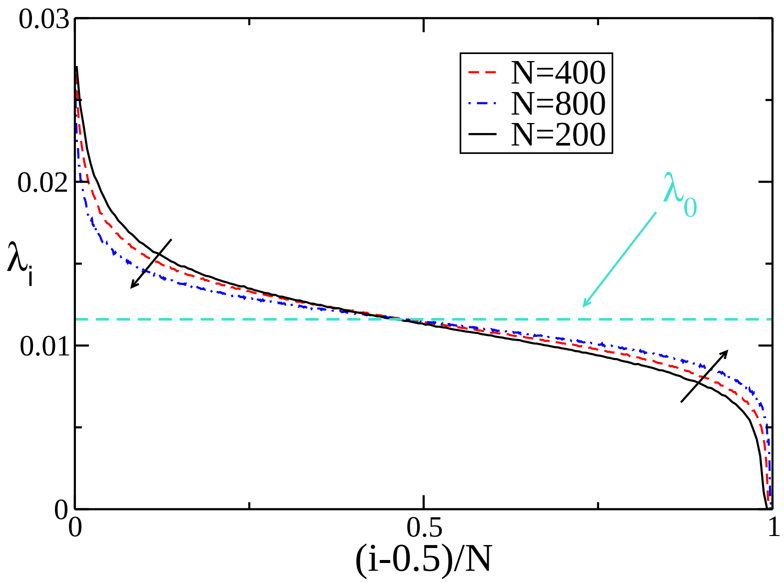
Lyapunov Spectrum

- The Lyapunov spectrum is made of $4N$ exponents, but it exhibits a pairing rule
  $\lambda_i + \lambda_{4N-i+1} = -\frac{1}{m} \quad i = 1, \ldots, 2N$ U Dressler PRA (1988)
- The spectrum is composed of a positive part and a constant negative part.
- The negative part is associated to the synchronized population and coincides with the mean field value for this population
- The positive part is associated to the chaotic population



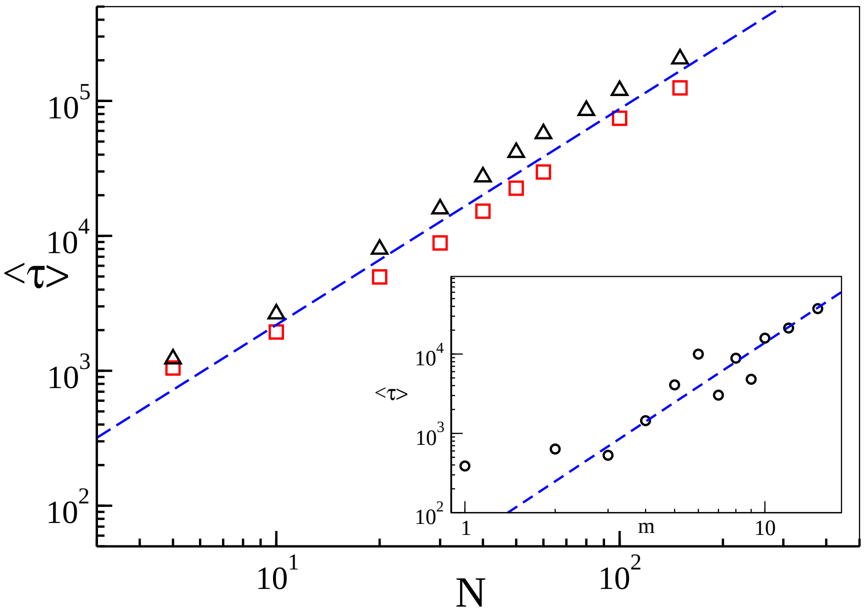
Positive Lyapunov Spectrum

- The chaos is high dimensional $N-2$ positive Lyapunov exponents
- The most part of the positive spectrum tends to flatten to the mean field LE value λ_0, trivially extensive
- the Lyapunov in two sub-extensive bands $\mathcal{O}(\ln N)$ take different values with respect to λ_0, similarly to what shown for λ_M



Same behavior demonstrated for one population of globally coupled dissipative units
Takeuchi, Chaté, Ginelli, Politi, AT, PRL (2011)

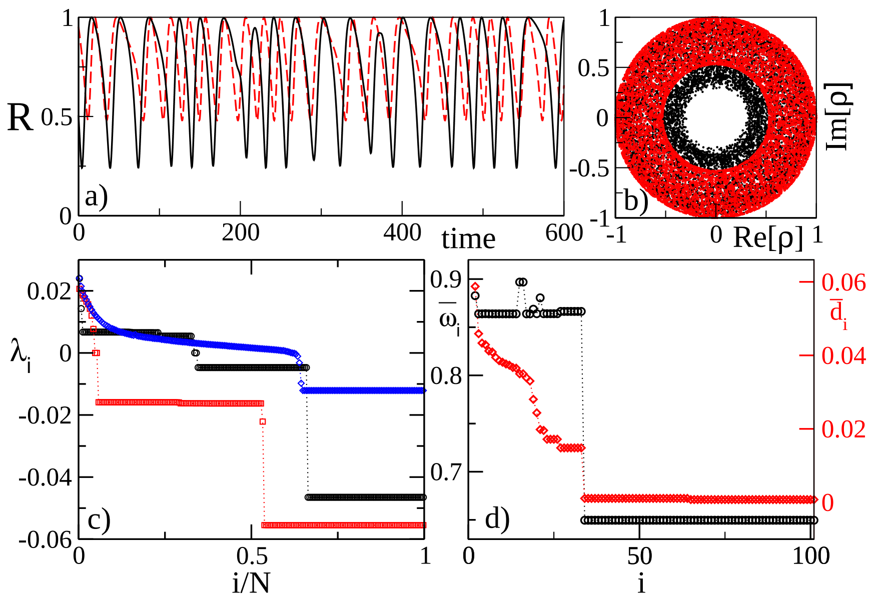
Life Times of the ICCs



- The chaotic chimeras converge to a regular non chaotic state after a transient time τ for all investigated masses ($m = 6$, 8 and 10)
- The final, stable, state can be either the fully synchronized solutions or even a broken symmetry state, corresponding to a breathing chimera.

- Chaotic transients diverge with N as a power law with an exponent $\alpha \approx 1.60$.
  - This result is in contrast with the observed exponential growth of the transient time found by Wolfrum and Omel'chencko in PRE (R) (2011), typical of spatially extended systems (with Kuramoto-Sakaguchi oscillators).

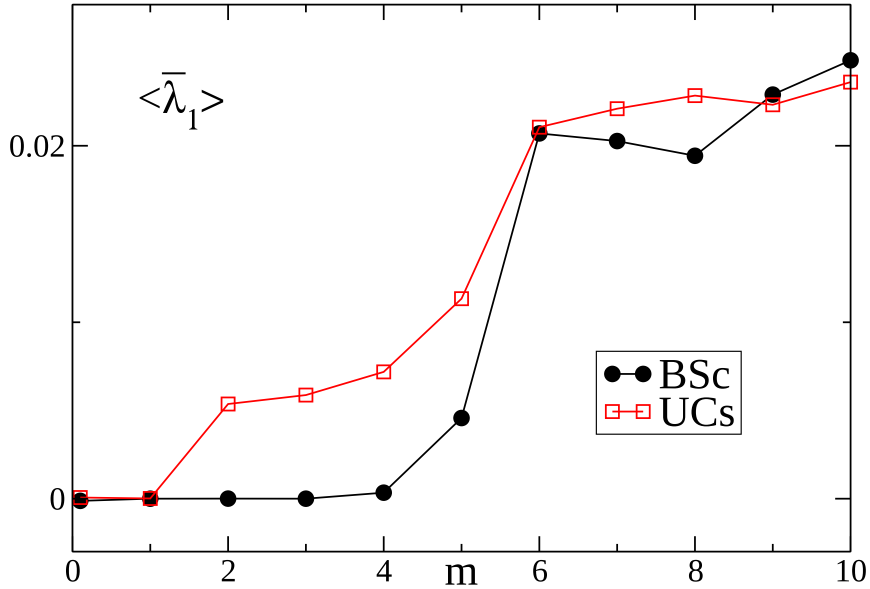
Chaotic Two Population States



- $\bar{\omega}_i = \frac{d\bar{\theta}_i}{dt}$ average velocities
- $\bar{d}_i$ average contribution of the i-th oscillator to the maximal LE

- C2P are states with broken symmetry
- C2P are not transient states
- C2P are Multistable
- Most of the oscillators of the two populations form a common cluster
- Only the isolated oscillators contribute to chaos

Mass Dependence



- At small m coexisting breathing chimeras and quasi-periodic chimeras, no chaos
- At sufficiently large m chaotic states emerge
- We never observed a stationary chimera state

Main Results

Wolfrum et al. (CHAOS 2011), analyzing a ring of identical Kuramoto oscillators with a finite range interaction, have shown that

- Chimera states are chaotic transients diverging exponentially with N;
- Chaos is weak
- Lyapunov spectra scales as in spatially extended systems.

We considered a network of two fully coupled populations; in this case

- The life-times of the ICCs diverge as a power-law with N and m
- The maximal Lyapunov exponent remains positive in the infinite size limit and tends to split from the rest of the spectrum.
- The spectrum becomes asymptotically flat (thus trivially extensive), but this part is sandwiched between subextensive bands (as for fully coupled systems) [Takeuchi et al. PRL 2011]

S. Olmi, E. Martens, S. Thutupalli, AT, arXiv:1507.07685

Conclusions and Outlook

- Topology matters for the stability properties of chaotic chimera states;
- Furthermore, the presence of inertia is responsible of the fact that ICCs become a stationary chaotic state in the thermodynamic limit

These systems are akin to Hamiltonian models , namely Hamiltonian Mean Field (HMF) Model (Antoni & Ruffo, 1995) -- 20 years of the HMF Model

- The Lyapunov spectrum satisfies the following a pairing rule
  $$\lambda_i + \lambda_{4N-i+1} = -\frac{1}{m} \quad i = 1, \dots, 2N$$ [U Dressler, PRA, 1988]
- Transient times diverging as $N^{1.7}$ have been reported for the metastable states observed in the HMF model [YY Yamaguchi, PRE, 2003]

Can the properties of ICCs be related to Hamiltonian features ?

Fokker-Planck Equation

The distribution of the particles $u = \ln |d|$ in the box $u \in [0; u_{max}]$ in the reference frame moving with velocity $\lambda_M(N)$ is ruled by the following Fokker-Planck equation

$$\frac{\partial P(u,t)}{\partial u} = -\frac{\partial}{\partial u}[\Delta\lambda]P + \frac{D}{2}\frac{\partial^2 P(u,t)}{\partial u^2}$$

where $\Delta\lambda = \lambda_0 - \lambda_M(N)$

For sufficiently large N, since $u_{max} \propto \ln N$, the stationary solution is given by

$$P_s(u) = \frac{2\Delta\lambda}{D}\mathrm{e}^{-\frac{2\Delta\lambda u}{D}}$$

Furthermore, the following normalization condition should hold

$$\int_{u_max}^{\infty} du P_s(u) = \frac{\mathcal{O}(1)}{N}$$

since only 1 particle should be nearby u_{max}, this leads to the finite size scaling

$$\lambda_M(N) = \lambda_0 + \frac{D}{2} + \frac{a}{\ln(N)} + \mathcal{O}\left(\frac{1}{\ln^2(N)}\right)$$

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