

## SPIRAL INSTABILITIES IN PERIODICALLY FORCED EXTENDED OSCILLATORY MEDIA

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We investigate two instabilities of spiral waves in oscillatory media subject to different types of forcing using the complex Ginzburg-Landau equation. First, the transition of spiral waves via so-called superspirals to spatio-temporal chaos is related to a coexistence of the Eckhaus instability of the wave field and the intrinsic oscillatory meandering instability of the spiral core. Second, resonantly forced oscillatory media are shown to possess a novel scenario of spiral breakup. Bifurcation analysis and linear stability analysis yield explanations for the phenomenology observed by direct simulations.

### 1. Introduction

Rotating spiral waves are well-known patterns in oscillatory media<sup>1</sup>. Recently, superspiral structures (see Fig. 1) have been observed experimentally in the Belousov-Zhabotinsky reaction<sup>2</sup> and have been treated by linear analysis<sup>3</sup>. In previous work, we have shown that moving sources emit modulated amplitude waves analogous to the way a stationary source emits a plane wave<sup>4</sup>. After introducing the model in Sec. 2 we use these results in Sec. 3 to conjecture bifurcation diagrams of two-dimensional superspirals. Additional resonant forcing of an oscillatory medium yields a plethora of

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\*Work partially supported by a Otto Hahn fellowship of the Max Planck Society.

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‡MB likes to thank S. Ruffo and Università di Firenze for hospitality and financial support

new patterns due to frequency locking (see <sup>5,6</sup> and references therein). We study in Sec. 4 how spiral waves destabilize under these conditions.

## 2. Complex Ginzburg-Landau Equation

A universal description of oscillatory media near a supercritical Hopf bifurcation is given by the complex Ginzburg-Landau equation (CGLE)<sup>1</sup>

$$\partial_t A = (\epsilon + i\nu)A + (1 + ic_1)\Delta A - (1 - ic_3)|A|^2 A + \gamma \bar{A}^{n-1}. \quad (1)$$

The complex field  $A(x, y, t)$  contains amplitude and phase of local oscillations,  $\epsilon$  measures the distance from the Hopf bifurcation,  $\nu$  measures the frequency detuning,  $c_1$  ( $c_3$ ) give the linear (nonlinear) dispersion and  $\gamma$  describes the strength of resonant forcing.

For a homogeneous system we assume that  $\epsilon = 1$  and  $\nu = 0$ . If forcing is absent ( $\gamma = 0$ ) then the CGLE exhibits spiral waves of the form  $A(r, \theta, t) = F(r)e^{i(\theta + f(r, t))}$  in polar coordinates  $(r, \theta)$ . For  $r \rightarrow \infty$  the radial dynamics approaches a travelling wave with  $F(r) \rightarrow \sqrt{1 - q_S^2}$ ,  $f(r, t) \rightarrow q_S r - \omega_S t$ , a selected wavenumber  $q_S$  uniquely determined by  $c_1, c_3$  and a frequency  $\omega_S = -c_3 + q_S^2(c_1 + c_3)$ <sup>7</sup>.

## 3. Superspirals

### 3.1. Simulations

We first perform numerical simulations of the CGLE (1) without forcing  $\gamma = 0$  and detuning  $\nu = 0$ . A non-saturating core instability has been observed for large values of  $c_1$ <sup>8</sup>. However, if a small and localized heterogeneity near the spiral core of the form  $\epsilon = 1 + b_0 e^{-r^2/\sigma}$  is added, meandering behavior similar to the one typically seen in reaction-diffusion systems is found<sup>9</sup>. A unique meandering period is selected which depends on the parameters  $c_1, c_3, b_0, \sigma$ .

One example of a meandering spiral that possesses saturated modulations in its wave field (superspiral) is presented in Fig. 1. The real part of  $A(x, y, t)$  (Fig. 1(a)) shows phase waves with non-uniform wave length that travel from left to right. The spiral-shaped amplitude modulation in Fig. 1(b) lead to the name superspiral. This amplitude modulation saturates away from the core (near  $x = 64$ ) as is evident from the profile of  $|A|$  in Fig. 1(c) along a one-dimensional cut.

In order to study the underlying mechanisms it is useful to control the meandering period and the properties of the wave field independently.

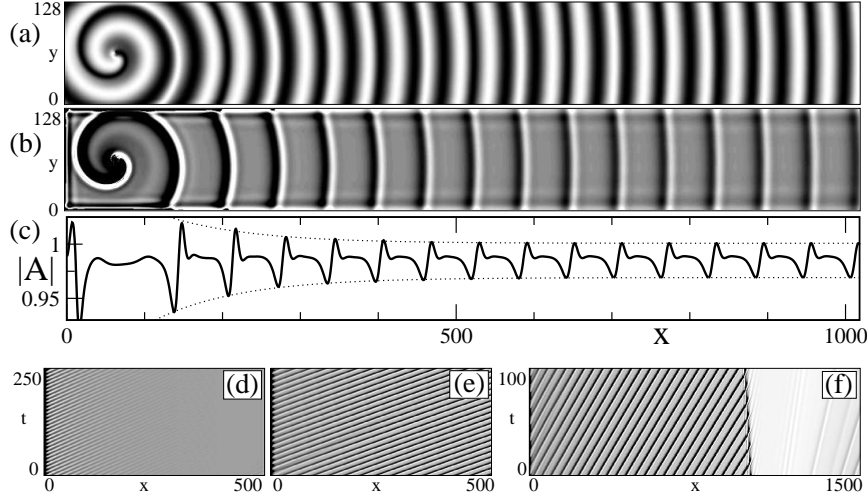


Figure 1. Numerical simulations of (meandering) superspirals in the CGLE (1). (a) Snapshot of the real part of  $A$ , (b)  $|A|$  and (c) profile of  $|A|$  along a horizontal cut through the 2D system. Simulations of the one-dimensional radial dynamics of  $|A|$  for meandering spirals reveal three regimes with (d) damped modulations for small meandering period, (e) saturated modulations for intermediate and (f) growing modulations with subsequent spiral breakup for large meandering period. Parameters are  $c_1 = 3.5, c_3 = 0.37, \gamma = 0, \epsilon = 1 + 0.7e^{-r^2/10}$  in (a-c) and  $c_1 = 3.5, c_3 = 0.4, \gamma = 0, \epsilon = 1$  and oscillating left boundary (Eq. (2)) with  $R_S = 0.5$  in (d) with  $T = 8$ , (e)  $T = 13$  and (f)  $T = 15$ . Grayscale between minimum (black) and maximum (white) values.

Hence, we studied a one-dimensional analogue of a meandering spiral, *i.e.* a 1D system with a moving Dirichlet boundary condition (BC) at one end and the usual zero-flux BC at the opposite end. For fixed Dirichlet source  $A(0, t) = 0$ , the selected “1D spiral” has the same general properties as the 2D spiral (see Sec. 2) and the selected wavenumber  $q_{S1} \sim q_S$  is known analytically<sup>7</sup>. To introduce meandering, we vary the source position  $x_S$  by

$$A(x \leq x_S) = 0 \quad \text{with} \quad x_S = R_S \cos(2\pi t/T). \quad (2)$$

In the 1D simulations we observe that the modulations possess the same temporal period as the forcing period  $T$  independent of  $R_S$ .

For different choices of the meandering period  $T$  we find at most three distinct regimes with damped (Fig. 1(d)), saturated (e) or growing (f) modulation. Below we summarize the underlying selection mechanism.

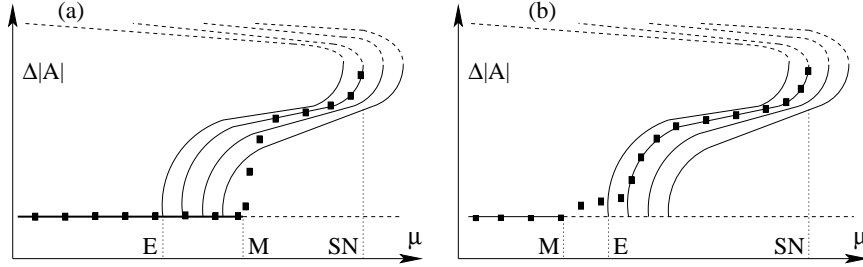


Figure 2. Two scenarios of superspiral formation (a) with the Eckhaus instability (E) occurring before the meandering instability (M) or (b) with the opposite order. Squares represent possible observations of  $\Delta|A| = |A|_{max} - |A|_{min}$  near the boundary of a finite system in simulations or experiments with control parameter  $\mu$ . Thin curves denote the family of MAWs from which a unique one is selected by the period  $T = T(\mu)$  of the meandering. The selected solution vanishes in a saddle-node bifurcation (SN).

### 3.2. Bifurcation Diagram

Previous work has shown that the asymptotic solution in the far field of the superspiral is a modulated amplitude wave (MAW)<sup>4</sup>. MAWs have the form  $A(x, t) = a(z)e^{i\phi(z)}e^{i(qx - \omega t)}$  with  $z = x - vt$  and periodic functions  $a(z), \phi(z)$ . Notice, the saturated profile in Fig. 1(c) corresponds to an exact solution  $a(z)$ . MAWs have been studied in Ref. 10 and are a two-parameter family which is conveniently parameterized by the mean wavenumber  $q$  and the temporal period  $T$  of the modulation. The wave source selects the MAW wavenumber  $q$  and the meandering period selects the MAW period  $T$ . Two bifurcations (Hopf where MAWs emerge and saddle-node where they vanish) delimit the three distinct regimes seen in simulations and experiments (Fig. 1(d-f) and Ref. 2).

So far we examined both selection criteria independently in a 1D system. In a 2D system both are uniquely determined by the control parameters. Now we can draw tentative bifurcation diagrams of spirals that meander in the Eckhaus unstable regime as shown in Fig. 2. The spiral becomes unstable by meandering near an Eckhaus instability of the selected wave train via one of the following two scenarios. Either the Eckhaus instability appears just before (panel (a) where the modulation amplitude jumps upon small changes in control parameter  $\mu$ ) or shortly after the meandering (panel (b) where damped modulations continuously increase). Near the meander instability, the motion of the core has a small amplitude and saturation may not be reached within the finite system. However, for larger systems the onset of superspirals tends to a first order transition in (a) and a second

order transition in (b). Careful observation of the onset can thus reveal the locations of the two contributing instabilities.

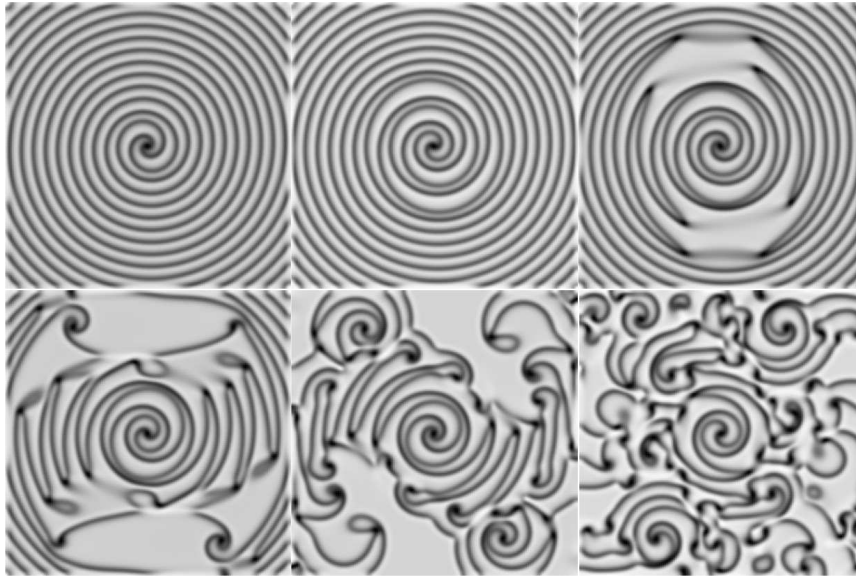


Figure 3. Spiral breakup at  $\gamma = 0.21$  and  $n = 2$ . The snapshots show  $|A|^2$ . From left to right and top to bottom,  $t=25, 50, 75, 100, 125$  and  $200$ . White(black) corresponds to the larger (smaller)  $|A|^2$ . Other parameters are  $\epsilon = 1.0$ ,  $\nu = 0.5$ ,  $c_1 = -1.4$  and  $c_3 = -0.3$ .

#### 4. Resonant Forcing

Here, we briefly illustrate how a resonant 2:1-forcing ( $n = 2$ ) destabilizes rotating spirals in the CGLE by assuming that  $\gamma \neq 0$ . Upon external forcing, frequency-locked standing wave patterns are obtained for  $\nu > -c_3\epsilon$ , while defect turbulence arises for  $\nu < -c_3\epsilon$  as in <sup>5</sup> for strong enough forcing. If the forcing crosses the threshold of instability, spirals in 2D become unstable and form new defects at a certain distance from their center, see Fig. 3. These defects evolve into new spirals, which interact with the original spiral and among each other. As the external forcing strength increases, the size of the surviving core region shrinks and finally disappears. Far from the spiral core, defects result from the merging of fronts. In the unforced CGLE ( $\gamma = 0$ ), a spiral becomes more unstable and breakup (BU) occurs as  $c_3$  is decreased. A shift of  $c_3$  towards more negative values induces

absolute instability and BU occurs without forcing due to the absolute Eckhaus instability. The breakup process upon external forcing is essentially as follows: as  $\gamma$  increases, BU occurs via the merging of fronts. A recent stability analysis shows that indeed the absolute instability of the periodic waves far from the core is related to modes that represent front interactions<sup>6</sup>.

## 5. Conclusions

We have analysed spiral instabilities in periodically forced oscillatory media. First, superspirals have been found if the convective Eckhaus instability is excited by the intrinsic oscillatory meandering instability due to a nonlinear problem<sup>4</sup>. Our tentative bifurcation diagram of superspirals links their far-field breakup to a saddle-node bifurcation of spirals. Second, spirals in resonantly forced oscillatory media can be destabilized by suitable resonant 2:1 forcing that causes a front merging instability in the emitted wavetrain<sup>6</sup>.

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