

Analysis and Simulation of Waves in Reaction-Diffusion Systems

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Abstract

The paper considers reaction-diffusion systems in excitable media for studying the dynamics of related travelling waves. In particular, the one-dimensional Fitzhugh-Nagumo model is considered to apply two classical approaches of feedback systems and derive results of structural kind. The comparison with simulations obtained by a new integration procedure indicates the possible usefulness of such methods in this specific context.

1 Introduction

A broad attention has been devoted from many years to the study of pattern formation in distributed systems, due to the importance of this problem in the fields of biology, chemistry, physics and ecology (see, for example, [1][2] and references therein). A particular interest is related to the case of reaction-diffusion equations in one-dimensional and two-dimensional excitable systems, which can be used to model the electrical activity of biological tissues -nerve fibers, cardiac muscle, brain tissues- and to reproduce a lot of main phenomena experimentally observed. For example, the onset of solitary pulses, periodic wave trains, circular waves and also spiral waves, i.e. spatial waves of this shape, periodic in time and rotating in the plane. These spatiotemporal patterns model real phenomena such as neural action potential along the axonal membrane, organized contractions of atria and ventricles, cortical waves in the brain, etc. . It is therefore quite important, particularly in cardiology and neurobiology, but also in many problems concerning chemical activities, the knowledge of the laws of such behaviors.

The study of these systems generally presents significant difficulties and its results are far from having detected many essential characteristics of the above phenomena. A common approach is to use simplified models in order to reproduce in a qualitative way the actual

system behavior and to grasp the mechanisms which give rise to the various situations. It is evident that the understanding of such fundamental processes is a key point in view of a policy of their control, as can be required in many situations.

The purpose of this paper is to give a contribution to the analysis of reaction-diffusion systems in excitable media, by introducing approaches coming from the area of control engineering and using frequency methods for feedback systems. As a paradigm of interesting systems is considered the well-known Fitzhugh-Nagumo model [1][2], presented in Section 2 and described in its main features concerning the possibility of sustaining travelling waves. The describing function method of analysis [3][4] is used and its results are discussed in Section 3, while a technique for relay systems (the Tsytkin method [5] [6]) is applied in Section 4, with comparisons and comments of related results. Due to the difficulty of simulating these kind of systems, a particular attention has been devoted to such problem. Then, a new integration scheme [7] is proposed in Section 5, putting in evidence some specific advantages of this algorithm, here used to obtain the true behavior of Fitzhugh-Nagumo model. The brief conclusion of Section 6 ends the paper.

2 The Fitzhugh-Nagumo model

As a paradigm of reaction-diffusion systems in excitable media, consider the well-known Fitzhugh-Nagumo (FHN) model of equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u + f(u) - v \\ \frac{\partial v}{\partial t} &= \epsilon(u - \gamma v) . \end{aligned} \quad (1)$$

The variables $u = u(x, t)$ and $v = v(x, t)$ depend on one or two-dimensional space x , in addition to time t , and represent the activator and the inhibitor state, respectively. The symbol ∇^2 denotes the Laplacian,

ϵ (a small term) and γ are system parameters and f is a S-shaped nonlinear function such that only one equilibrium point exists.

Equations (1) are a valuable qualitative model of biological media (nerve fibers and muscle tissues such as heart tissue) and present a wide variety of solutions known as travelling waves [1][2]. Corresponding to real phenomena also in chemical and physical activities, these waves are primarily plane waves which have a fixed profile and propagate along the medium in a fixed direction with constant speed. The term plane waves, means that they are independent of any space variation perpendicular to the direction of propagation and the system nonlinearity only allows certain profiles with its own velocity. Two-dimensional space models of eq.s (1) can also exhibit waves with curvilinear propagation front. The most important case is that of spiral waves having the form of a single spiral in the plane, rotating about a point [1][2].

In order to study the dynamics of these waves, consider the case of one spatial dimension. The usual approach is looking for solutions of (1) $u(x, t) = U(z)$ and $v(x, t) = V(z)$ with $z = x - ct$, representing waves moving to the right with speed c . In this moving coordinate system eq.s (1) become

$$\begin{aligned} -cU' &= U'' + f(U) - V \\ -cV' &= \epsilon(U - \gamma V), \end{aligned} \tag{2}$$

where $U' \doteq dU/dz$, etc.

The study of this ordinary differential system can now be directed to seek homoclinic orbits and limit cycles. In fact the first ones, starting from the rest point and finishing to it, can be thought as solitary travelling pulses in the original system (1), while the second ones result in periodic wave trains. The determination of the above dynamics can allow one to derive the essential features of such analysis. In fact, although the considered waves are nonlinear, one can define an important characteristic of the medium, namely its dispersion curve, giving the wave speed c as a function of the wavelength λ , i.e. the distance on the x scale between successive wave fronts. The typical form of dispersion law for model of eq.s (1) is shown in Figure 1, where the upper branch indicates stable solutions and the lower branch refers to unstable solutions. In practice, there exists a lower bound for speed propagation, corresponding to the minimum wavelength of periodic trains and an upper bound of speed propagation which asymptotically refers, for $\lambda \rightarrow \infty$, to solitary pulses.

Unfortunately, the study of homoclinic orbits and limit cycles in a nonlinear system as (2) is not straightforward, particularly as the wave speed c is to be deter-

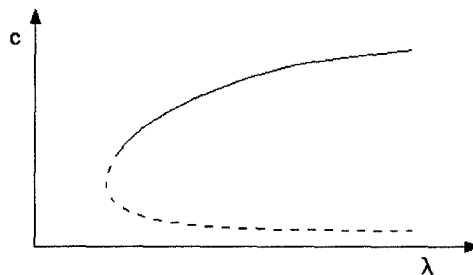


Figure 1: Dispersion curve of FHN model

mined at the same time of the wave profiles U and V .

The usual method followed in this problem exploits the fact that the parameter ϵ is small and therefore (2) represent a singular perturbation system, with a slow and a fast variable, V and U respectively. This allows one to utilize an asymptotic transition layer approach [1][2][8] deriving simplified dynamics conditions. In particular, the case where f is assumed to be cubic

$$f(u) = u(u - a)(1 - u), \tag{3}$$

where $0 < a < 1$, has been mainly considered, leading to a dispersion curve which is a good approximation of the upper branch (stable solutions) of the true one, apart its spurious extension up to the origin. The results necessarily follow from a numerical approach and it is not easy a link of the main characteristics of such diagram with the original system parameters.

Another approach consists in introducing a piecewise linear form for the nonlinearity f , such as

$$f(u) = -u + \frac{1}{2}[1 + \text{sgn}(u - a)], \tag{4}$$

and letting $\gamma = 0$ [1][2]. This appears a crude approximation of the function (3) but the crucial point is that its general shape is preserved, so leading to similar behaviors of the corresponding systems. On the other hand, it must be recalled that the cubic FHN model has been originally proposed as a reasonable caricature of the classical complete model of Hodgkin-Huxley.

By solving linear problems for eq.s (2) in z domain and applying continuity conditions where switch occurs it is possible to obtain conditions for the desired dynamics. A dispersion curve similar to that of Figure 1 can be derived, and this result is given as a function of ϵ , which is not simply viewed as a small parameter as in the previous approach. Some reduced numerical computations are needed also in this case, but the intuition on the mechanisms of travelling waves result to be enhanced.

The next Sections are devoted to present two different approaches for seeking the dynamics of interest of (2), corresponding to travelling waves of (1). These approaches come from control engineering and generally utilize frequency methods for feedback systems.

3 Describing Function Method

The elimination of variable V in (2) leads to the third order differential equation

$$\frac{d^3U}{dz^3} + (c - \frac{\epsilon\gamma}{c})\frac{d^2U}{dz^2} - \epsilon\gamma\frac{dU}{dz} + \frac{\epsilon}{c}U + \frac{df(U)}{dz} - \frac{\epsilon\gamma}{c}f(U) = 0 \tag{5}$$

where $c \neq 0$.

By considering the Laplace transform from z to the complex variable s , (5) can be clearly separated in a linear part and in a nonlinear one. The former is dynamic and can be represented by its transfer function $L(s)$ given by

$$L(s) = \frac{s + \epsilon\gamma/c}{s^3 + (c - \epsilon\gamma/c)s^2 - \epsilon\gamma s + \epsilon/c}, \tag{6}$$

while the latter is simply described by the static non-linearity f . The two parts are connected in feedback as shown in Figure 2.

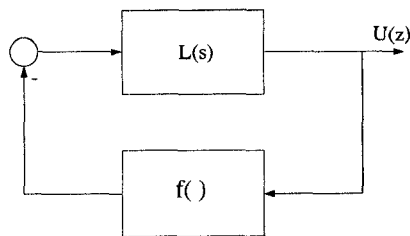


Figure 2: Lur'e system

This system structure, sometimes called of Lur'e, is well-known and widely studied in control engineering for different aspects of nonlinear dynamics and stability [3][4][5]. In particular, the structure of Figure 2 is suitable for applying harmonic balance techniques to seek periodic solutions U developed in Fourier series. In order to obtain qualitative results we can limit this approach to the first harmonic by using the classical describing function method.

Assume that any $2\pi/\omega$ solution $U_0(z)$ of (5) can be represented as

$$U_0(z) \doteq A + B \cos \omega z, \quad B > 0, \quad \omega > 0. \tag{7}$$

The separate balance along the loop of Figure 2 of 0 and ω frequency components, neglecting the higher order harmonics at the output of the nonlinearity f , leads to the equations

$$\begin{aligned} A[1 + L(0)N_0(A, B)] &= 0 \\ 1 + L(j\omega)N_1(A, B) &= 0. \end{aligned} \tag{8}$$

Here N_0 and N_1 represent the bias gain and the first harmonic gain of the nonlinearity f , that is

$$\begin{aligned} N_0(A, B) &= \frac{1}{2\pi A} \int_{-\pi}^{\pi} f(A + B \cos \omega z) dz \\ N_1(A, B) &= \frac{1}{\pi B} \int_{-\pi}^{\pi} f(A + B \cos \omega z) e^{-j\omega z} dz, \end{aligned} \tag{9}$$

while $L(0)$ and $L(j\omega)$ are the same steady-state gains of the linear system described by $L(s)$.

Conditions (8) are two algebraic equations, the first one real and the second complex, and have to be solved for A , B , and ω . Then, the characteristics of approximate limit cycles can be frequently derived in a structural (non numerical) form showing their dependence on system parameters. Of course, due to the assumptions of method, the accuracy of such predictions strongly depends on the loop attenuation of higher harmonics [3][4][5]. A possible measure of error can be a distortion index, viewed as the ratio between the amounts of higher harmonics and considered harmonics along the loop, in correspondence to the obtained solution.

Coming back to (5) and then to system of Figure 2, where $L(s)$ is given by (6) and f by the cubic form (3), we apply conditions (8). In particular, the second one requires the intersection of the polar plot of $L(j\omega)$ with the locus of $-1/N_1$. Since N_1 is real (f is a single valued nonlinearity) this intersection must correspond to $Im[L(j\omega)] = 0$. According to the indicative Figure 3 this gives

$$\omega = \frac{1}{c} \sqrt{\epsilon - \epsilon^2 \gamma^2} \tag{10}$$

while the remaining conditions (8) result in

$$\begin{aligned} -5A^3 + 5(a + 1)A^2 - [2(a + \frac{1}{\gamma}) + \frac{4}{3}(a + 1)^2 + a + \frac{1}{\gamma}]A \\ + \frac{2}{3}(a + 1)(a + \frac{1}{\gamma}) = 0, \end{aligned} \tag{11}$$

and, by the corresponding roots A , in

$$B = \sqrt{-4A^2 + \frac{8}{3}(a + 1)A + a + \epsilon + \frac{\epsilon - \epsilon^2 \gamma^2}{c^2}}. \tag{12}$$

Since eqs. (8) are solved only when ω and B are real, the following conclusions can be drawn:

- for parameter values of interest ω is real ($\epsilon < 1/\gamma^2$) and a necessary condition to have a solution is

$$c^2 > 3 \frac{\epsilon - \epsilon^2 \gamma^2}{1 + a^2 - a - 3\epsilon \gamma} \quad (13)$$

- according to (13), eq. (11) has generally three real roots A . Two of them can give real values for B leading to the indicative bifurcation diagram of Figure 4. In a certain range of ϵ two limit cycles are predicted vanishing at a tangent bifurcation. This has the graphical interpretation shown in Figure 3 as double intersection of loci of $L(j\omega)$ and $-1/N_1$.

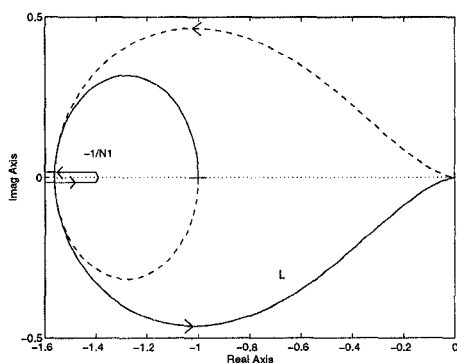


Figure 3: Nyquist diagram

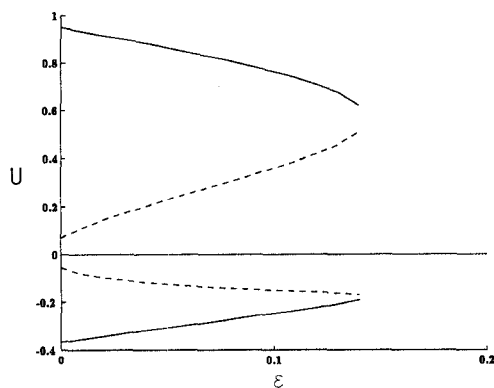


Figure 4: Limit cycle amplitude

In terms of the original problem of travelling waves in FHN model it is to consider that $z = x - ct$ in the periodic solution (7) and then the related wavelength is $\lambda = 2\pi/\omega$. Therefore, eq. 10 indicates the dispersion curve as a straight line to which correspond, with different profiles, two solutions which are presumably one stable (the larger) and one unstable (the smaller).

The results concerning two different cases ($\epsilon = 0.0004$ and $\epsilon = 0.02$, with $a = 0.01$ and $\gamma = 2.5$) are shown

in Figure 5 and compared with the true ones obtained by numerical simulations (see Section 5). It appears evident that the describing function method generally fails in reproducing the dispersion curve. In particular, the saturation of such curve is not predicted, so that the existence of solitary waves in the system is not indicated. Moreover, it can be observed that the obtained results tend to have some accuracy only for low values of the wavelength λ , when the distortion in the scheme of Figure 2 decreases, in particular for larger ϵ . On the other hand, when a successful application of singular perturbation is possible it seems reasonable that first order harmonic balance is hard to employ and vice-versa.

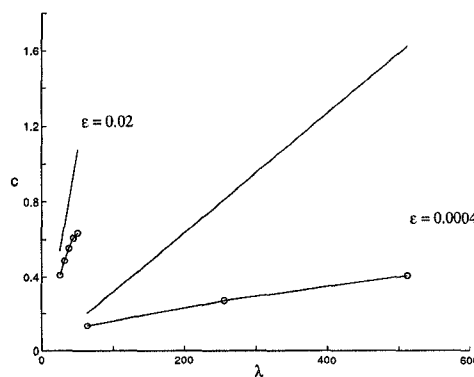


Figure 5: Computed (describing function method) and simulated (o) dispersion curves

4 Tsyarkin Method

Coming back to feedback system of Figure 2 as a representation of eq. (5), assume now that the function f is modeled by the piecewise linear form (4), with $\gamma = 0$. Simple computations lead to the corresponding block scheme of Figure 6 where a bias is indicated, the linear

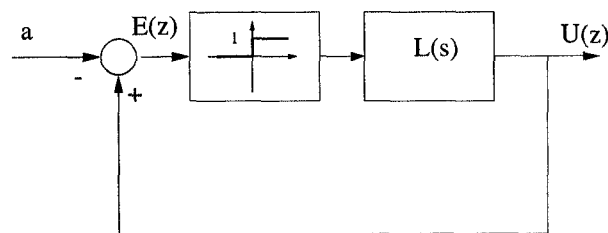


Figure 6: Relay feedback system

transfer function is

$$L(s) = \frac{s}{s^3 + cs^2 + \epsilon/c} \quad (14)$$

and the nonlinearity is reduced to a relay with input $E(z)$ and output 0 or 1.

This class of systems has a certain interest in control engineering for its importance in practical applications and presents specific opportunity of analysis [5] [6]. In particular, when a limit cycle occurs the relay output signal necessarily becomes a periodic square wave of fixed amplitude, whose parameters (the switching times, and then the frequency) are unknown. An indicative behavior is shown in Figure 7 where only one pulse for period is assumed. For determining the pa-

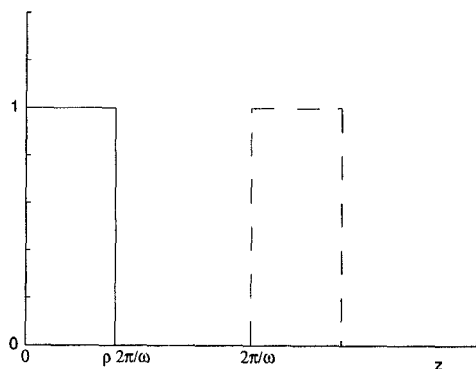


Figure 7: Periodic pulse train

rameters ω and ρ (the relative pulse duration) we can derive the steady-state output of the linear system $L(s)$ to this periodic square wave and then match the corresponding switching times at the input $E(z)$ of the relay.

Observe that also this idea exploits the feedback structure of the system. In Section 3 an approximate harmonic input to the nonlinearity has been assumed and the consequent balance along the loop has been imposed, with some simplified assumptions. Now, we give the exact shape of the nonlinearity output and again impose the balance along the loop through the linear subsystem. Apart from this analogy, the describing function method of Section 3 is an approximate technique, while this relay system approach is oriented to give exact results.

The application of the above idea can follow a frequency domain procedure, originally proposed by Tsypkin [5][6]. The waveform of Figure 7 is written in Fourier series and, as input of the linear system, allows one to derive the corresponding Fourier series of output $U(z)$, through suitable functions depending on $L(s)$ at frequencies $s = j\omega k$, for $k = 0, 1, \dots, \infty$. In a similar way is obtained the series of $U'(z) \doteq dU(z)/dz$.

Then, assuming the positive relay switching at $z = 0$ as in Figure 7, it must be

$$\begin{aligned}
 U(0) = 0 \quad , \quad U'(0) > 0 \\
 U(\rho 2\pi/\omega) = 0 \quad , \quad U'(\rho 2\pi/\omega) < 0 .
 \end{aligned}
 \tag{15}$$

Applying these relations to the above series of $U(z)$ and $U'(z)$ for the system of Figure 6 with $L(s)$ of eq. (14), we obtain

$$\begin{aligned}
 \text{Im}[\Lambda(0, \omega) - \Lambda(2\pi\rho, \omega)] &= a\pi \\
 \text{Re}[\Lambda(0, \omega) - \Lambda(2\pi\rho, \omega)] &> 0 \\
 \text{Im}[\Lambda(0, \omega) - \Lambda(-2\pi\rho, \omega)] &= -a\pi \\
 \text{Re}[\Lambda(0, \omega) - \Lambda(-2\pi\rho, \omega)] &> 0 ,
 \end{aligned}
 \tag{16}$$

where the complex function $\Lambda(\theta, \omega)$ is defined as

$$\begin{aligned}
 \text{Re}[\Lambda(\theta, \omega)] &\doteq \sum_{k=1}^{\infty} \{ \text{Re}[L(jk\omega)] \cos k\theta + \text{Im}[L(jk\omega)] \sin k\theta \} \\
 \text{Im}[\Lambda(\theta, \omega)] &\doteq \sum_{k=1}^{\infty} \frac{1}{k} \{ \text{Im}[L(jk\omega)] \cos k\theta - \text{Re}[L(jk\omega)] \sin k\theta \} .
 \end{aligned}
 \tag{17}$$

The equality conditions in (16) are two nonlinear equations to be solved in ω and ρ for determining a limit cycle of the system. The solution can be numerical or graphical. The latter, even if appears much more laborious than in describing function method, can better give an idea of the exact role of system parameters in the studied dynamics. The inequality conditions in (16) have only to be checked.

The application of the procedure has led to the results reported in Figure 8, where the obtained dispersion curves are drawn for two different cases ($\epsilon = 0.001$ and $\epsilon = 0.05$, with $a = 0.01$) concerning stable solutions. The comparison with simulations shows a quite good accuracy of the method, which can result an efficient tool to study in some detail the mechanism of travelling waves in reaction-diffusion systems. In particular, it appears a promising technique to derive some structural information about the basic elements of such phenomena.

5 Integration Scheme

The integration of the FHN eq.s (1) has been performed adopting a new time-splitting scheme, in particular a so called Leap Frog algorithm [7]. The time-splitting technique is based on separating the evolution operator in a linear part (reduced to the Laplacian operator) and a nonlinear part, whose effects are suitably combined via the Trotter formula [7]. Then, usually the linear part is solved in the spectral domain by two FFTs, while the nonlinear part is integrated using a simple finite difference (Euler) scheme.

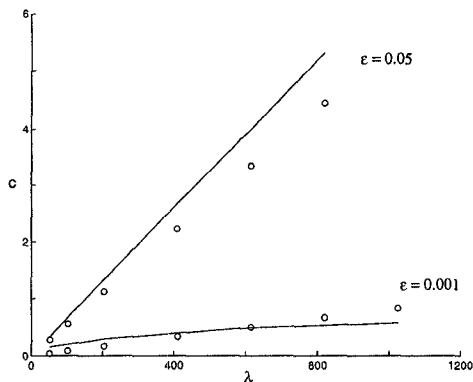


Figure 8: Computed (Tsypkin method) and simulated (o) dispersion curves

The novelty of the algorithm here used consists in performing also the integration of the linear part in space domain. This corresponds in the particular case of FHN model, to evaluate the related value u at $t + dt$ as the convolution integral

$$\int K(y, dt)u(x - y, t)dy , \tag{18}$$

where the expression of the kernel K is

$$K(y, t) = \frac{1}{\sqrt{4\pi t}}e^{-|y|^2/4t} . \tag{19}$$

The main idea behind this scheme deals with the evaluation of the integral (18) on a discrete spatial grid of N points. A very limited number N_c of sites around the central value of the convolution is used. We employ a "modified" kernel instead of discretizing directly the expression (19) and such a kernel is constructed to reproduce, with a given accuracy, the first terms of the Fourier expansion of true kernel. Technical details about the implementation of such algorithm are in [7].

This technique is local in space and can be quite fast with respect to the usual pseudo-spectral algorithms, maintaining almost the same integration precision, as shown in the considered applications [7]. In particular, the computational burden of the present scheme will scale as $N_c/\ln(N)$ with respect to that requested by the spectral algorithms. Therefore, for grids with a sufficiently high number of points the scheme will be faster than the algorithms employing the FFTs.

Another point to stress is that, due to the locality of this algorithm, now different boundary conditions can be treated employing the same "modified" kernel in the greater part of the grid, while only at the boundaries the convolution should be handled differently. This suggest that such integration technique could be fruit-

fully employed in problems with complicate boundaries where spectral algorithms are no more suitable.

6 Conclusion

The paper has considered a well-known model of reaction-diffusion systems in excitable media, namely the Fitzhugh-Nagumo equation. To study the dynamics of travelling waves for systems of this kind, two classical frequency methods of feedback systems have been employed, for deriving more qualitative information than usually is obtained on the mechanism of such behaviours. The comparison of preliminary results with numerical simulations, obtained by a new integration scheme, has put in evidence positive and negative aspects of proposed approaches.

References

- [1] J.D. Murray, *Mathematical Biology*, Springer, Berlin, 1989.
- [2] P. Grindrod, *Patterns and Waves. The Theory and Application of Reaction-Diffusion Equations*, Clarendon Press, Oxford, 1991.
- [3] H.K. Khalil, *Nonlinear Systems*, Mc Millan Pub.Company, New York, 1992.
- [4] M. Vidyasagar, *Nonlinear System Analysis*, 2nd ed., Prentice Hall, Englewood Cliffs, 1993.
- [5] D.P. Atherton, *Stability of Nonlinear Systems*, Research Study Press, Chichester, 1981.
- [6] Ya.Z. Tsypkin, *Relay Control Systems*, Cambridge Univ. Press, Cambridge, 1984.
- [7] A. Torcini, H. Frauenkron and P. Grassberger, Studies of phase turbulence in the one-dimensional complex Ginzburg-Landau equation, *Phys. Rev. E*, 55 p.5073, 1997.
- [8] J. Tyson and J.P. Keener, Singular perturbation theory of travelling waves in excitable media: a review, *Physica D*, 32, pp.327-361, 1988.