Synchronization of extended chaotic systems

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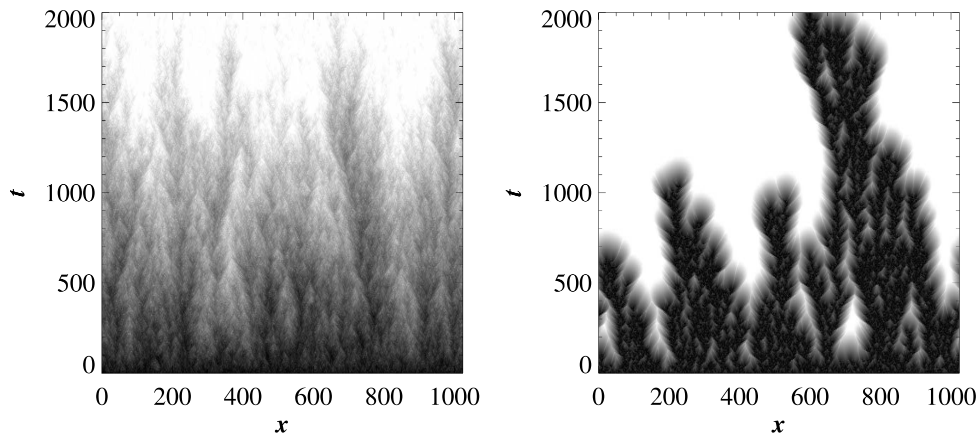

Le Monde.fr

- Du rôle de la biophysique dans l'applaudissement synchrone et (ou) chaotique

  Article publié le 26 Février 2000 – Par HERVE MORIN

  QUI AURAIT l'idée de s'intéresser à la synchronisation des applaudissements, et de comparer ce phénomène aux oscillations observées dans certaines réactions chimiques ou à la rythmique des " flashes " émis par les lucioles d'Asie ? Des scientifiques, bien sûr. " Avec un collègue, Zoltan Neda, de l'université de Cluj en Roumanie, je travaillais sur des problèmes de physique statistique, se souvient Yves Bréchet, du laboratoire de thermodynamique et physicochimie métallurgique de Grenoble. Nous sommes allés au Théâtre hongrois de Cluj."

- L'horloge biologique est indépendante des mécanismes de la vision.
  Une synchronisation par la lumière (16.04.99)

- Francisco Varela, le chercheur par qui la pensée se fait chair:
  Une question de synchronisation (18.02.99)

  " L'épilepsie est une synchronie pathologique tellement forte et généralisée que le cerveau perd toute fonctionnalité. En situation normale, les tâches cognitives s'y traduisent par des synchronies successives, courtes et localisées, portées par des oscillations gamma de fréquence de 30 à 80 hertz. "
  (Parkinson disease & essential tremor).

Plan of the Talk

Summary of Old Results

- Chaotic Synchronization in Low Dimensional Systems $\lambda_T = 0$
- Chaotic Synchronization in Spatially Extended Systems $V_F = 0$ (Diffusive Coupling)
  - The transition is analyzed as a a nonequilibrium phase transition
  - The transition is continuous and its critical properties correspond to
    - Multiplicative Noise (MN) $V_F = \lambda_T = 0$
    - Directed Percolation (DP) $V_F = 0 \quad ; \quad \lambda_T < 0$

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New Results

- Spatially Extended Chaotic Systems with Power-Law Coupling
- The synchronization transitions (STs) are continuous
- The critical indexes vary continuously with the interaction range
- The family of STs correspond to Anomalous Directed Percolation (ADP)
  - ADP has been found for Lévy-fligth spreading of epidemic processes
  - ADP critical exponents have been measured for stochastic lattice models

Low Dimensional Chaotic Systems

Chaotic Dynamics

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

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Systems Coupled via Stochastic forcing

Two replicas **u** and **w** of the same dynamical system:

$$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) + \gamma \cdot \eta(t) \qquad \mathbf{u}(0) \neq \mathbf{w}(0)$$

$$\dot{w}_k(t) = \varphi_k(\mathbf{w}(t)) + \gamma \cdot \eta(t)$$

η is a δ-correlated random variable $< \eta(t')\eta(t) >= \delta(t' - t)$.

For a sufficiently large noise amplitude $\gamma > \gamma_c$ the replicas can eventually synchronize.

Low Dimensional Chaotic Systems

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Mutually Coupled Systems

Two replicas **u** and **w** of the same dynamical system:

$$\dot{u}_k(t) = (1 - \gamma) \cdot \varphi_k(\mathbf{u}(t)) + \gamma \cdot \varphi_k(\mathbf{w}(t)) \quad \mathbf{u}(0) \neq \mathbf{w}(0)$$

$$\dot{w}_k(t) = (1 - \gamma) \cdot \varphi_k(\mathbf{w}(t)) + \gamma \cdot \varphi_k(\mathbf{u}(t))$$

For a sufficiently strong coupling $\gamma > \gamma_c$ the replicas can eventually synchronize

Low Dimensional Chaotic Systems

Chaotic Dynamics

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

Def: Synchronization is observed when the distance between replicas asymptotically vanishes

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} |u(t) - w(t)| = 0$$

Condition to observe synchronization in low dimensional systems :

the transverse Lyapunov exponent should be negative

$$\lambda_\perp = \lim_{t\to\infty} \lim_{z(0)\to 0} \ln \frac{z(t)}{z(0)} < 0$$

[Maritan & Banavar, PRL 72, 1451 (1994); Pikovsky, PLA 165, 33 (1992), PRL 73, 2931 (1994); Herzel & Freund, PRE 52, 3238 (1995); Lai & Zhou, EPL 43, 376 (1998)]

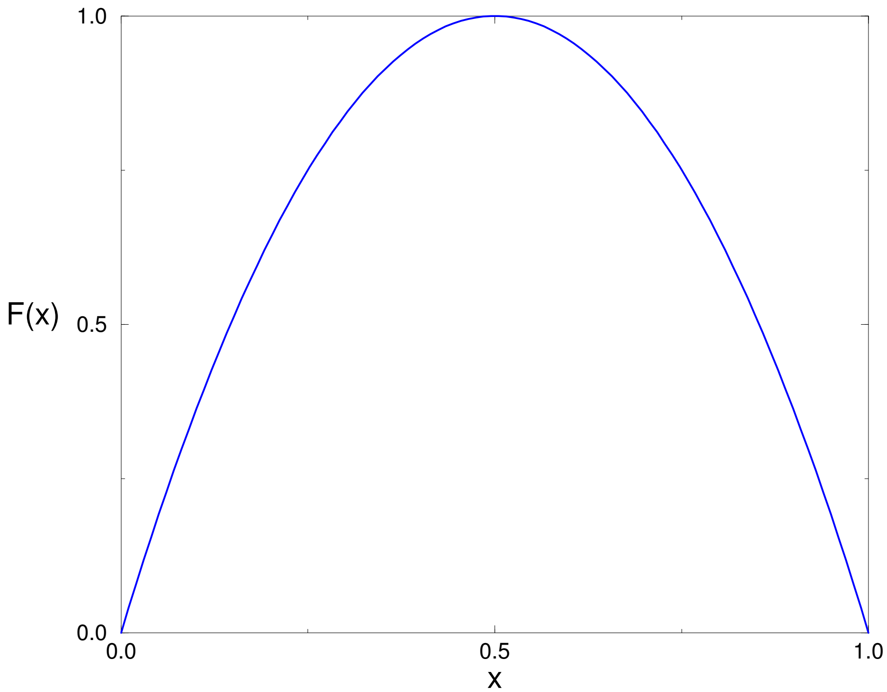
Spatially Extended Systems

Coupled Map Lattices

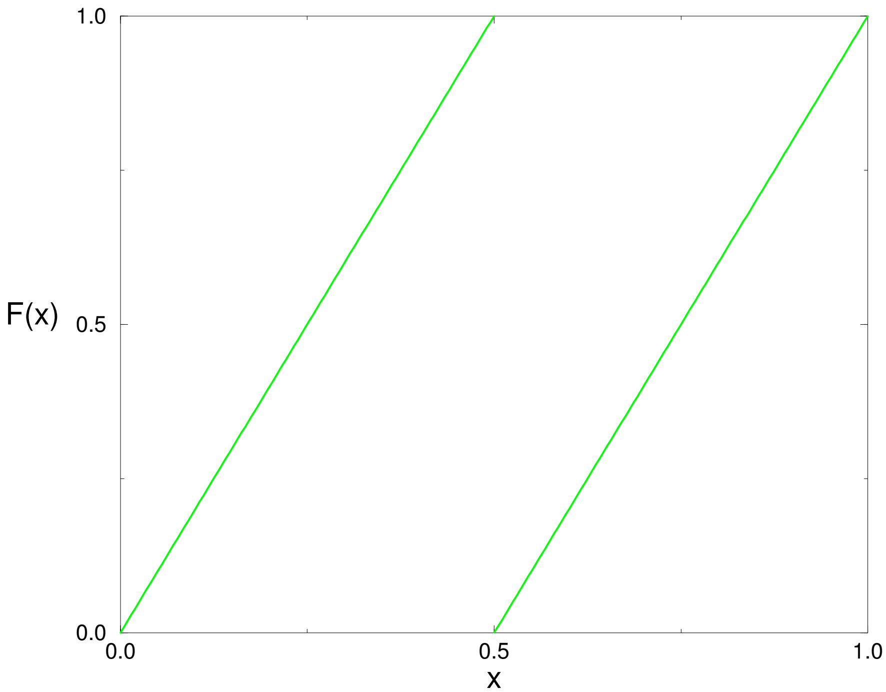
$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)u_x^t\right] \qquad \nabla_\varepsilon^2 u_x = \varepsilon\{[u_{x+1}+u_{x-1}]/2 - u_x\}$$

where x and t are discrete, F is a chaotic map, typically one dimensional.

Logistic Map F(x)=4x(1–x)



Bernoulli Shift   F(x)=Mod(2x,1)



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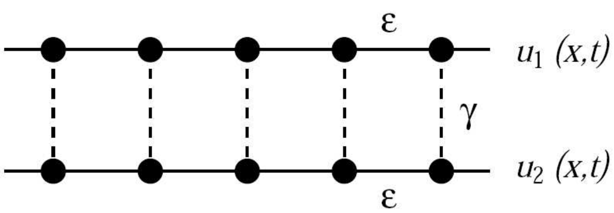

$$u_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^2)u_x^t\right] + \gamma\cdot F\left[(1+\nabla_\varepsilon^2)w_x^t\right]$$
$$w_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^2)w_x^t\right] + \gamma\cdot F\left[(1+\nabla_\varepsilon^2)u_x^t\right]$$

Stochastic Forcing

$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)u_x^t\right] + \gamma\cdot\zeta_x^t$$
$$w_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)w_x^t\right] + \gamma\cdot\zeta_x^t$$

where the noise is δ-correlated in space and time $<\zeta_x^t\zeta_y^s>\propto\delta_{x,y}\delta_{t,s}$.

The local difference field is defined as $z_x^t = |u_x^t - w_x^t|$.

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where x and t are discrete, F is a chaotic map, typically one dimensional.

Synchronization

For sufficiently strong coupling γ the spatially averaged difference field

$$\rho(t) = < z(t) > = \frac{1}{L} \sum_{x=1}^{L} z_x^t$$

could eventually vanish in the long time limit.

The synchronization transition is no longer fully described in terms of the transverse Lyapunov exponent (TLE).

An extreme nonlinearity in the local map F can induce transport of Finite Size Disturbances even for linearly stable states (i.e. Negative TLE).

A new indicator is required to fully characterize the transition for spatially extended systems.

Spatially Extended Systems

Coupled Map Lattices

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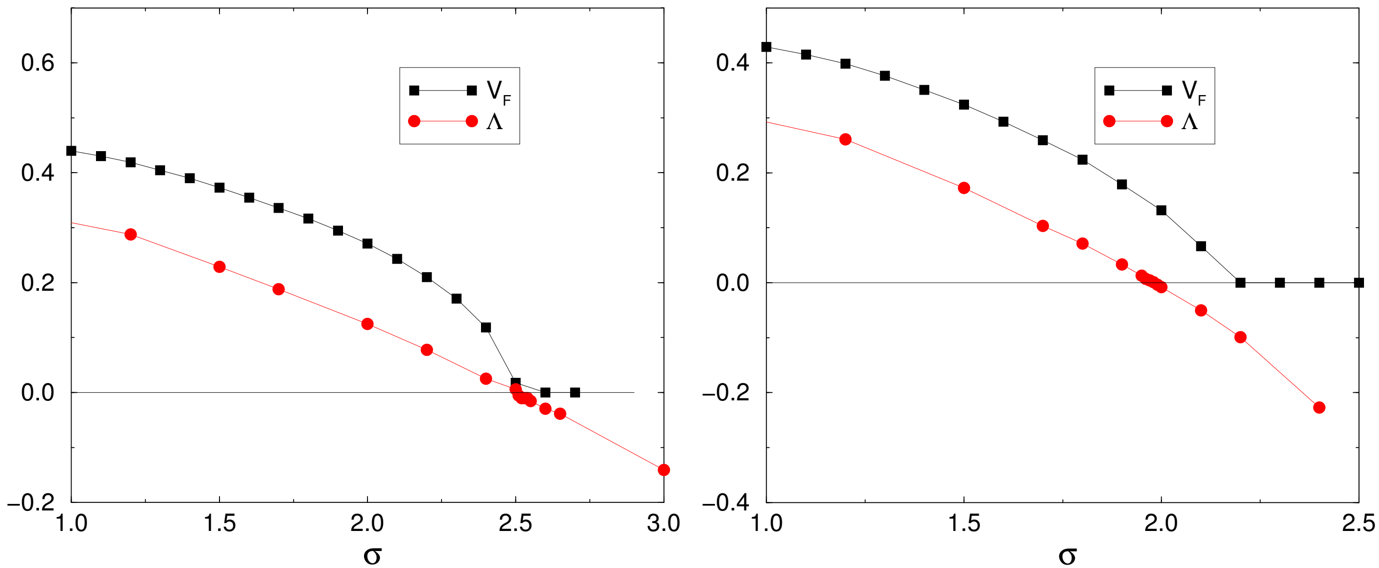
where x and t are discrete, F is a chaotic map, typically one dimensional.

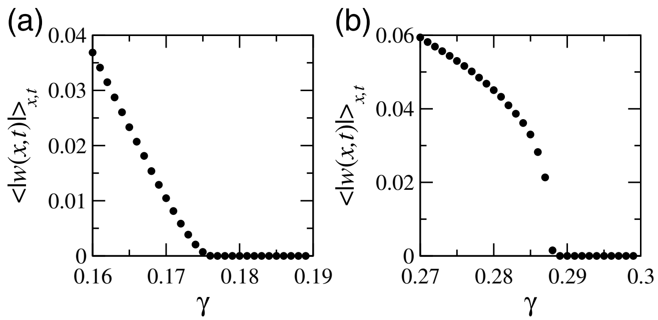
Propagation Velocity of Finite Size Perturbations

A droplet of unsynchronized sites ($N(0)$) is inserted in a completely synchronized state:

$$v_F = \lim_{t\to\infty} \frac{N(t)-N(0)}{2t}$$



Universality Classes



The Synchronization Transition is a Non-Equilibrium Phase Transition leading from an "active phase" ($\rho > 0$) to an "absorbing phase" ($\rho \equiv 0$).

The transition point a_c is located in the thermodynamic limit ($L \to \infty$) by the vanishing of the order parameter $\rho(t) \equiv < z(t) > \to 0$.

A continuous transition is typically characterized by a critical behavior :

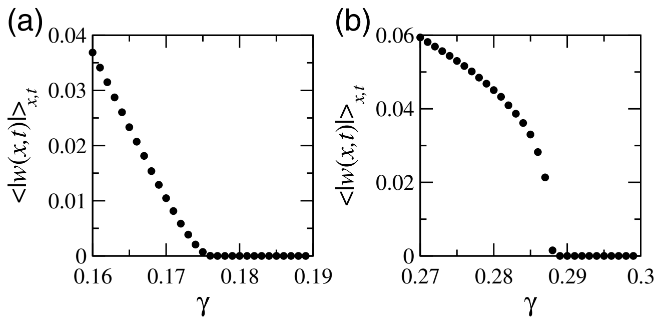
$$\rho(t) \propto t^{-\delta} \qquad \rho(t) = L^{-z\delta} g(t/L^z) \qquad \text{at} \qquad a \equiv a_c$$

$$< \rho >_t \propto |a - a_c|^{\beta}$$

$$L_c \propto |a - a_c|^{-\nu_\parallel} \qquad T_c \propto |a - a_c|^{-\nu_\perp}$$

only 3 exponents are independent (e.g. δ β and z)

Universality Classes



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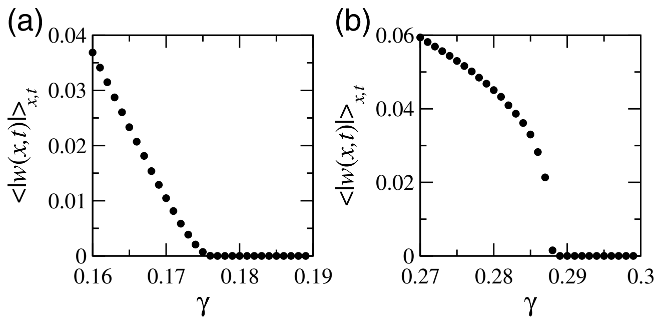
Two different types of transitions have been observed:

- Multiplicative Noise
  - $v_F = \lambda_\perp = 0$
  - Linear Effects rule the Transition
- Directed Percolation
  - $v_F = 0 \quad \lambda_\perp < 0$
  - Strong Nonlinear Effects ($|F'| >> 1$)

[Baroni, Livi & AT , PRE 63, 036226 (2001); Ahlers & Pikovsky, PRL, 88, 254101 (2002)]

Universality Classes



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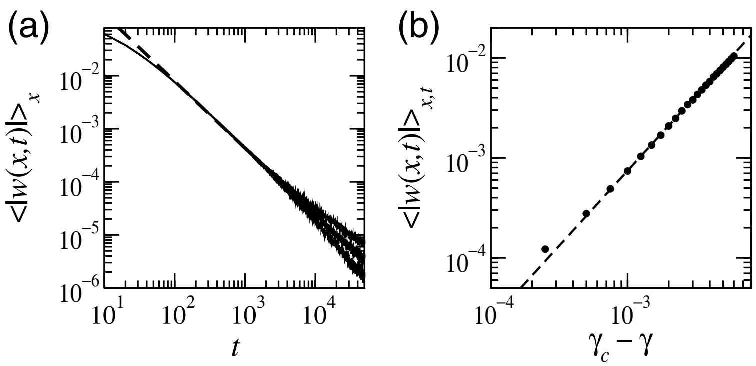
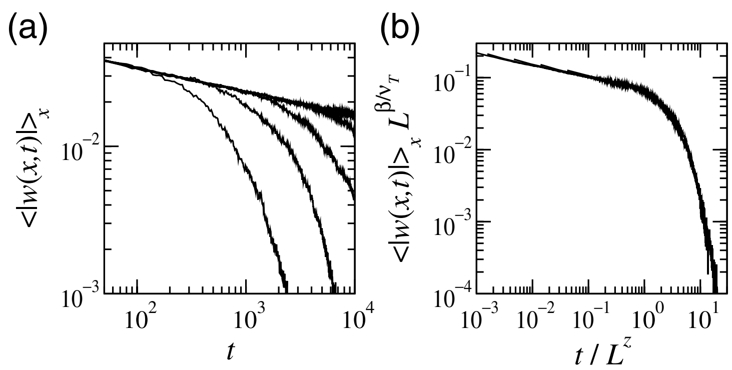




	MN	Tent	DP	Bernoulli
δ	1.10(5)	1.26(3)	0.159464(6)	0.16
β	1.70(5)	1.50(5)	0.276486(6)	0.28
z	1.53(7)	1.5	1.580745(6)	1.581

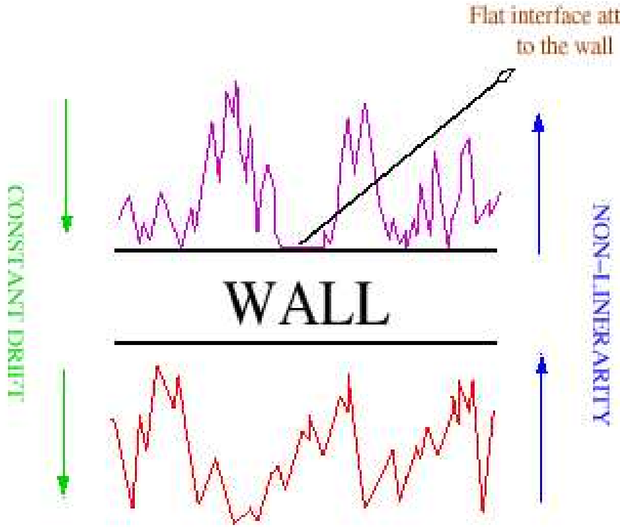
Ahlers & Pikovsky, PRL, 88, 254101 (2002); V. Ahlers , PhD Thesis (Berlin, 2001)

Multiplicative Noise

The corresponding field equation for the coarse-grained variable $w(x,t) = \bar{z}$ is:

$$\dot{w}(x,t) = \nabla^2 w(x,t) + aw(x,t) - bw^p(x,t) + w(x,t)\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and time and $p \geq 2$. Pikovsky & Kurths (94) have shown that this model describes the dynamics of CMLs within a linear framework.



This problem can be mapped on that of a depinning of a KPZ interface from a hard substrate through a Hopf-Cole Transformation $h(x,t) = -\ln w(x,t)$. This leads to a KPZ-like equation

$$\dot{h}(x,t) = \nabla^2 h(x,t) - (\nabla h(x,t))^2 - a' - be^{-(p-1)h(x,t)} + \eta(x,t)$$

The adsorbing state $w = 0$ is now mapped into $h = \infty$

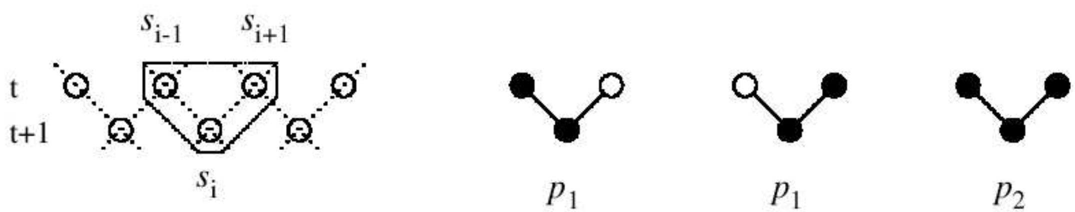
[M.A. Muñoz, cond-mat/0303650 (2003)]

Directed Percolation

The corresponding field equation is:

$$\dot{w}(x,t) = \nabla^2 w(x,t) + a w(x,t) - b w^2(x,t) + \sqrt{w(x,t)}\,\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and time.
This equation is usually associated to Infection Spreading Models: the Domany-Kinzel cellular automaton:



black sites are infected (active phase), white sites are healthy (absorbing phase).

- The infection spreads only by contact
- No revival of infection within healthy region: the absorbing state is stable

[H. Hinrichsen Adv. Phys. 49, 815-958 (2000)]

Some of the DP exponents have been for the first time measured in an experiment on a ring of oscillating ferrofluidic spikes at the transition to spatiotemporal intermittency [Rupp, Richter, & Rehberg, PRE 67, 036209 (2003)]

DP or not DP ?

- Microscopic models exhibiting DP critical behaviour are typically defined in terms of discrete and finite state variables (e.g. cellular automata).

[Ginelli,Livi,Politi, & AT PRE 67, 046217 (2003)]

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  [Peter Grassberger (1997)]

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  However it can be shown that :

  - An effective threshold W_c can be identified, below which the synchronization is ruled by linear (contracting) mechanisms: once $\rho(t) < W_c$ the system can no more escape from the absorbing (synchronized) state;

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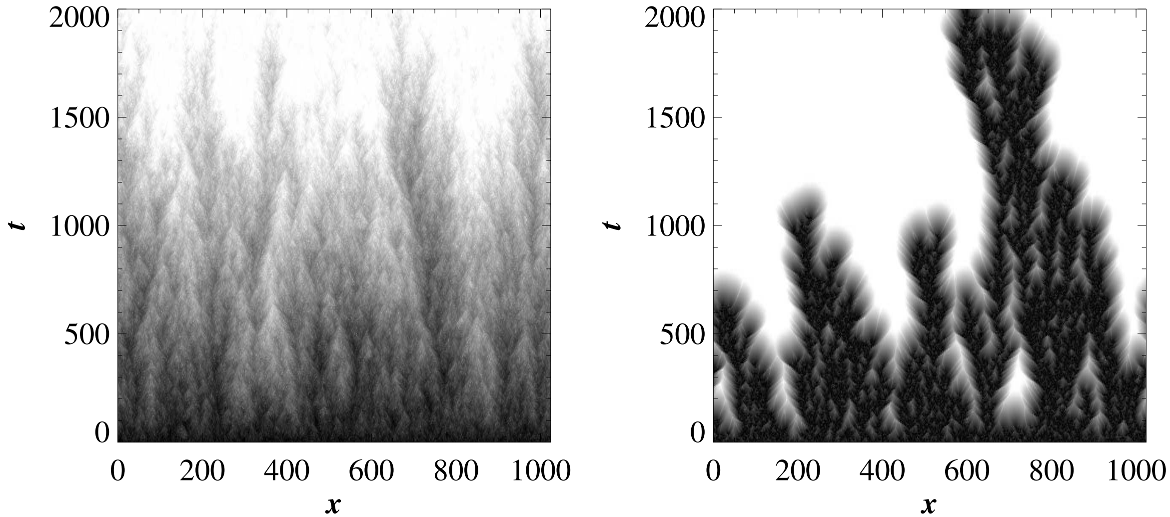
  - It is possible to derive heuristically the DP field equation starting from the difference field z_x^t evolution equation for replica of coupled lattices.

[Ginelli,Livi,Politi, & AT PRE 67, 046217 (2003)]

Summary of the first part

- In spatially extended systems (CMLs) with diffusive coupling two different synchronization transitions are observed :
  - if the linear behaviour prevails on nonlinear effects the transition belongs to the MN universality class;
  - if nonlinear effects dominate the dynamics DP scaling laws are observed.



In collaboration with:

Francesco Ginelli (Saclay - Paris)

- V. Ahlers* (Germany)
- R. Livi (Firenze)
- A. Pikovsky (Potsdam)
- L. Baroni* (Italy)
- D. Mukamel (Rehovot)
- A. Politi (Firenze)

* Working in private companies

Long-Range Interactions

Coupled Map Lattices with Power-Law Coupling

$$u_x^{t+1} = F\left[(1 + \nabla_\varepsilon^\sigma) u_x^t\right] \qquad \nabla_\varepsilon^\sigma u_x = -\epsilon u_x + \frac{\epsilon}{\eta(\sigma)} \sum_{m=1}^{M} \frac{u_{x-j_m(q)} + u_{x+j_m(q)}}{(j_m(q))^\sigma}$$

where $x \in [1, L]$ and t are discrete, $F = 2x$ (mod 1) is the Bernoulli map and periodic boundary conditions are assumed.

$$\eta(\sigma) = 2 \sum_{m=1}^{M} \frac{1}{(j_m(q))^\sigma} \qquad \text{normalization factor}$$

$\sigma \to 0$ Globally Coupled Maps $\qquad$ $\sigma \to \infty$ Usual CMLs

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$\sigma \to 0$ Globally Coupled Maps $\qquad \sigma \to \infty$ Usual CMLs

Coupling Schemes

- Fully Coupled: $j_m(q) = m$, $M = (L-1)/2$
- Reduced Coupling: $j_m(q) = q^m - 1$, $M = \log_q(L/2)$ with $q = 2, 4$ and 8

The coupling scheme does not alter the critical properties of the transition, but the reduced scheme is much faster ($\mathcal{O}(L \log_q L)$ versus $\mathcal{O}(L^2)$),

Chaotic Synchronization

The synchronization transition of two coupled replicas is studied

$$u_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^\sigma)u_x^t\right] + \gamma \cdot F\left[(1+\nabla_\varepsilon^\sigma)w_x^t\right]$$
$$w_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^\sigma)w_x^t\right] + \gamma \cdot F\left[(1+\nabla_\varepsilon^\sigma)u_x^t\right]$$

by examining the synchronization error $z_x^t = |u_x^t - w_x^t|$ for different coupling exponents σ.

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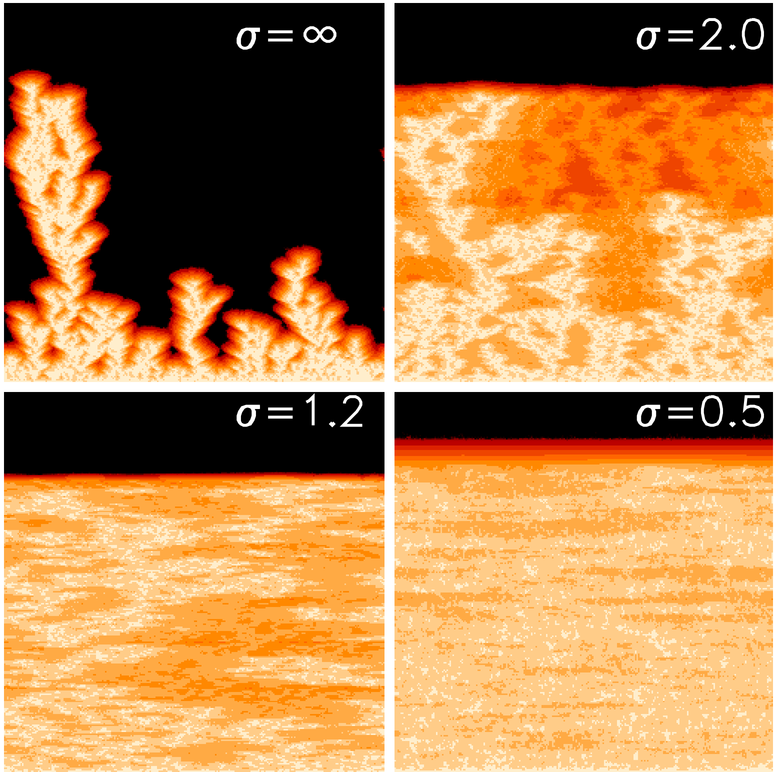
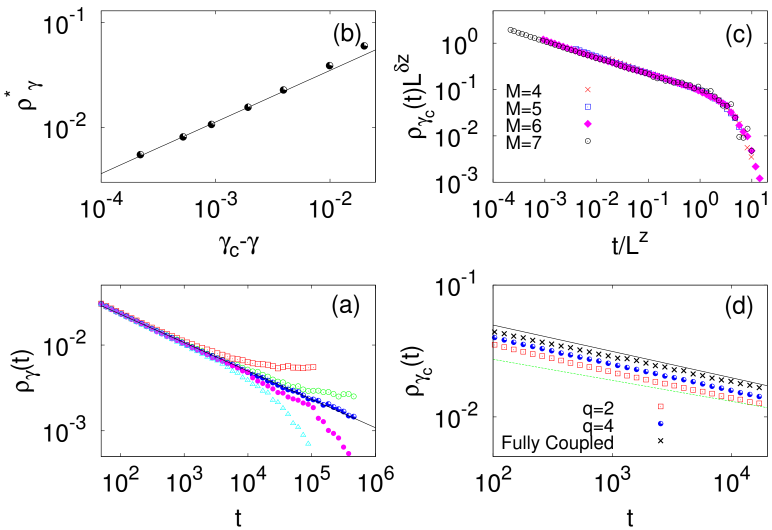


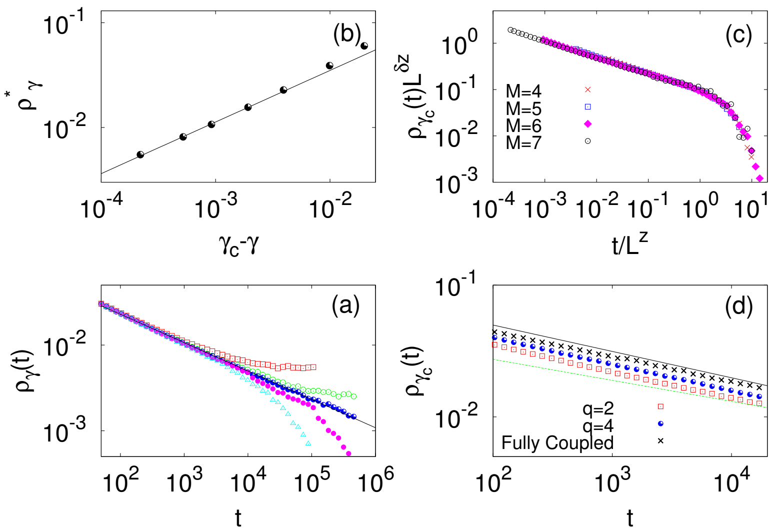
The critical exponents



System size $6 \times 10^4 \leq L \leq 4 \times 10^6$

Averages over $100 - 1000$ different realizations

The critical exponents



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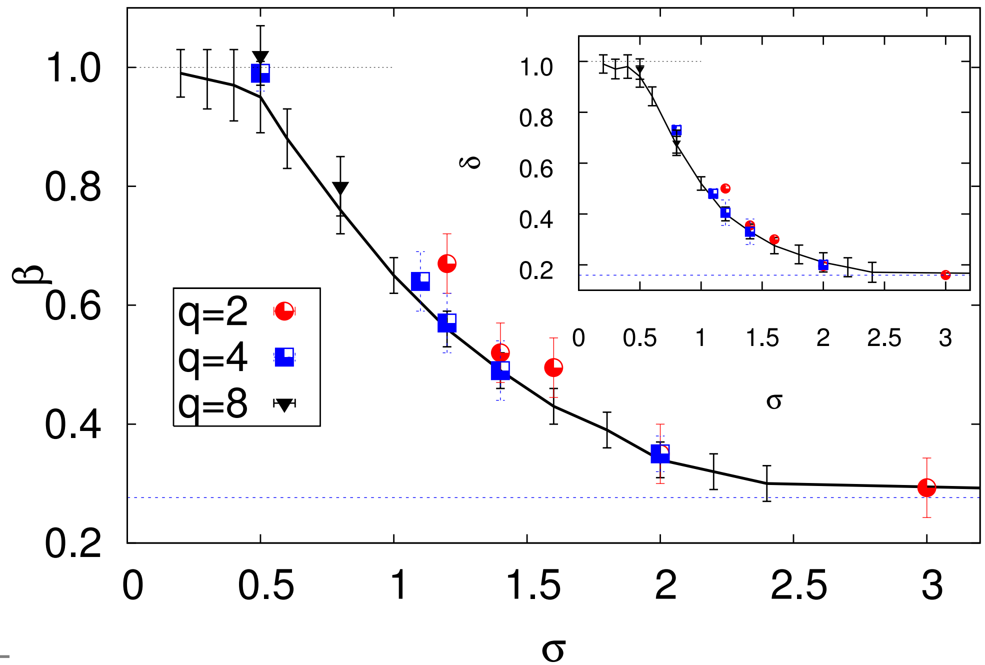


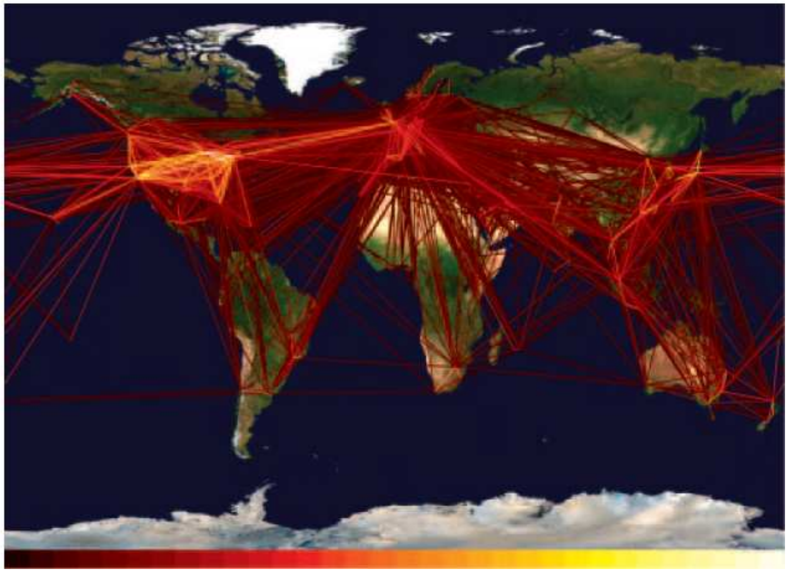
The critical exponents vary continuously, we have a family of universality classes labelled by the coupling exponent σ.

$\delta_{DP} \sim 0.16$ $\beta_{DP} \sim 0.27$

Anomalous Directed Percolation

Distribution of human travels

Brockmann et al.Nature (2006)



In many realistic spreadig processes short-range interactions do not appropriately describe the transport mechanism of the infection

- infectious disease transported by insects;
- disease spread triggered by aviation traffic;
- spreading agent subjected to a turbulent flow.

The motion of the agent can be super-diffusive.

Mollison in 1977 proposed a generalization of the usual DP in which the agent can perform Lévy flights, where the distribution of the spreading distances r is given by

$$P(r) \propto 1/r^{d+\sigma} \qquad \sigma > 0$$

d being the spatial dimension of the system.

Mollison, J R Stat Soc B 39 (1977) 283; Grassberger, Fractals in physics, (1986)

Field Theoretic Prediction

The generalization of the usual field equation to anomalous DP reads as:

$$\dot{w}(x,t) = (\nabla^2 + \nabla^\alpha)w(x,t) + aw(x,t) - bw^2(x,t) + \sqrt{w(x,t)}\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and the anomalous diffusion operator is defined as

$$\nabla^\sigma \mathrm{e}^{ikx} = -k^\sigma \mathrm{e}^{ikx}$$

The renormalization group calculations indicate that

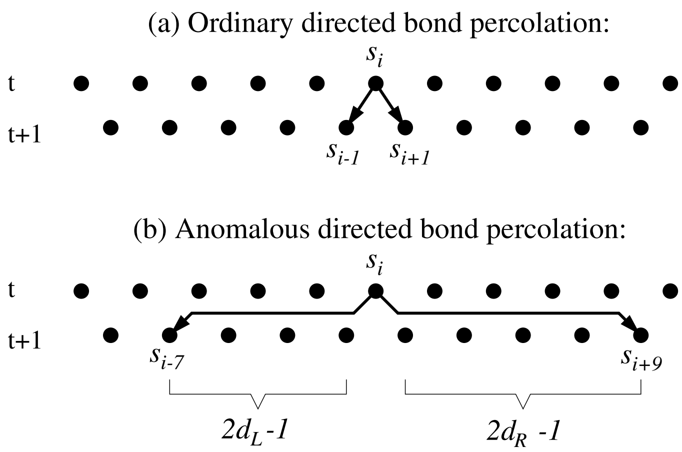
- for $\sigma < 0.5$ the mean-field description should become exact;
- for $\sigma > 2.0677(2)$ the usual DP results should be recovered

Mean-field exponents obtained by neglecting correlations are:

$$\beta_{MF} = \delta_{MF} = 1.0 \qquad z_{MF} = \sigma$$

Jannsen et al. EPJB 7 (1999) 137; Hinrichsen & Howard EPJB 7 (1999) 635.

Stochastic Lattice Model



$s_i = 1$ (infected) - $s_i(t) = 0$ (healthy)
Only infected sites can propagate the disease.
The control parameter is
the bond probability $0 \leq p \leq 1$

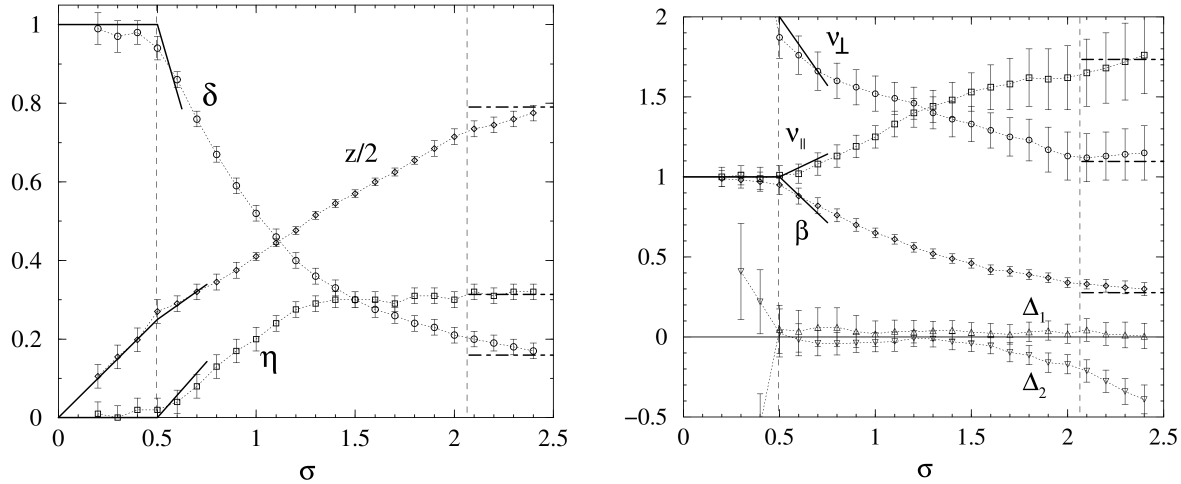
- At the next time $t + 1$ all the sites are initially healthy;
- two distances (d_L, d_R) are randomly generated from the distribution $P(r) \propto 1/r^{1+\sigma}$;
- the sites located at those distances from a site i (infected at time t) become infected if by choosing two random numbers (y_L, y_R) between 0 and 1
  - $s_{i+1-2d_L}(t = 1) = 1$ if $y_L < p$
  - $s_{i-1+2d_R}(t = 1) = 1$ if $y_R < p$

The length of the examined system was $L = 4 \times 10^{19}$, no finite size effects.

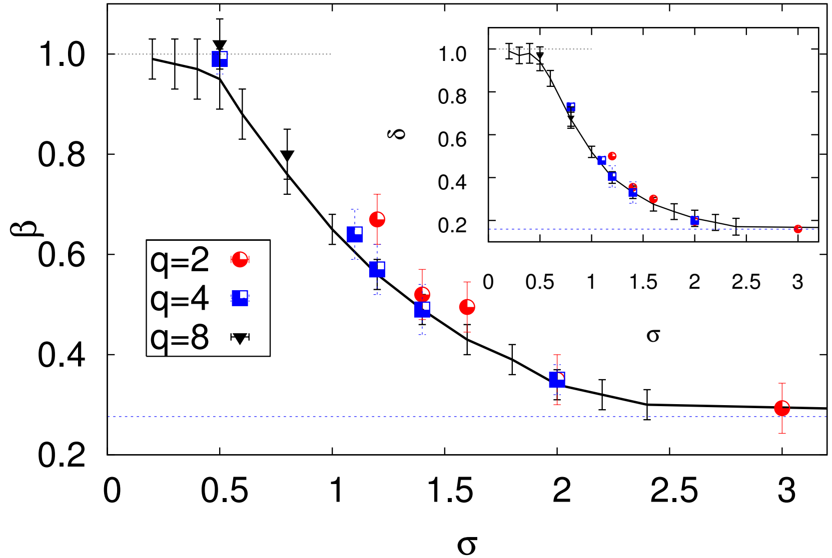
Hinrichsen & Howard EPJB 7 (1999) 635.

Critical Indeces

Extremely accurate estimation of the critical exponents in the whole range $0 < \sigma < 2.4$.



Our results for the chaotic synchronization transitions are in very good agreement

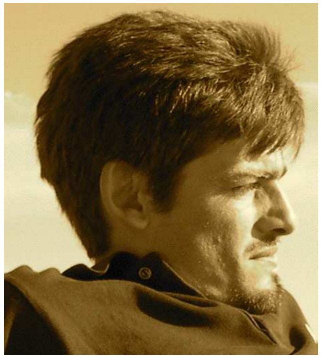


Conclusions & Perspectives

- The synchronization transition (ST) of two replicas of of chaotic discontinuous coupled maps with long range interactions is characterized by a continuum of universality classes labeled by the exponent σ.
- The critical properties of these STs correspond to Anomalous Directed Percolation, previously examined in the context of epidemic spreading.
- Preliminary results indicate that also for continuous maps the exponents depend on σ, but they do not belong to the anomalous DP class.
- Anomalous Multiplicative Noise has been not yet studied, therefore a completely open problem is to find to which universality class ST for continuous maps correspond.

Credits

- Claudio Juan Tessone - PhD in Physics (2006)
- Institut Mediterrani d'Estudis Avancats - Palma de Mallorca



- Massimo Cencini - Researcher (2005-)
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