LYAPUNOV SPECTRA OF COUPLED MAP LATTICES

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The relationship between the Lyapunov spectrum of diffusively coupled one-dimensional maps and the spectrum of the discrete Schrödinger operator is stressed. As a result, an analytic expression is derived for the Lyapunov spectrum for uniformly expanding maps. It is also shown that for a coupling strength larger than a critical value, the spectrum extends to $-\infty$.

One-dimensional lattices of coupled maps represent a relatively simple class of models, useful to investigate spatio-temporal behavior [1]. One of the most complete indicators of chaotic dynamics is represented by the thermodynamic limit (i.e. for the chain length L tending to ∞) of the spectrum of Lyapunov exponents (LS). In fact, knowledge of the LS allows one to determine the Kolmogorov–Sinai entropy from the Pesin formula, as well as the dimension density from the Kaplan–Yorke conjecture [2]. The application of standard techniques to extract the characteristic exponents allowed one to test the existence of the thermodynamic limit in various cases ranging from simplectic maps to continuous-time, and continuous-space models [3]. Recently, some theoretical results have also been obtained, regarding the qualitative shape of the spectrum [4]. However, a general qualitative theory of LS is still lacking, and there are very few rigorous results even in simple cases.

The aim of the present Letter is to stress the strict analogy existing between the computation of the LS for diffusively coupled one-dimensional maps, and the more familiar and widely studied estimation of the spectrum of the discrete Schrödinger operator. The relationship between the two classes of problems allows for a partial transfer of well-known techniques to the field of nonlinear systems.

Let us start with the general equation

$$x^i_{n+1} = F(y^i_n) , \qquad (1)$$

where the subscript n and superscript i denote time and space coordinates, $F(y)$ is a map of the unit interval and y^i_n is given by

$$y^i_n = \tfrac{1}{2}\epsilon x^{i-1}_n + (1-\epsilon) x^i_n + \tfrac{1}{2}\epsilon x^{i+1}_n , \qquad (2)$$

ϵ measuring the strength of the diffusive coupling. We recall that ϵ is bounded between 0 (uncoupled maps) and 1 (odd sublattice decoupled from the even one), to guarantee that y, and hence $F(y)$, be still confined to the unit interval. The evolution in the tangent space is obtained by linearizing eq. (1),

$$\delta x^i_{n+1} = F'(y^i_n) [\tfrac{1}{2}\epsilon \delta x^{i-1}_n + (1-\epsilon)\delta x^i_n + \tfrac{1}{2}\epsilon \delta x^{i+1}_n] . \qquad (3)$$

In the case of stationary solutions, the explicit time dependence contained in $F'(y^i_n)$ cancels out and the eigenvalue problem associated with eq. (3) becomes

$$m\delta x^i = F'(y^i) [\tfrac{1}{2}\epsilon \delta x^{i-1} + (1-\epsilon)\delta x^i + \tfrac{1}{2}\epsilon \delta x^{i+1}] , \qquad (4)$$

where the logarithm of the absolute value of m gives the Lyapunov exponent. Eq. (4) is formally equivalent to the stationary Schrödinger equation in the tight-binding approximation,

$$\psi^{i+1}+\psi^{i-1}+(\omega-V^i)\psi^i=0\,, \tag{5}$$

where ω is the frequency and V^i is the potential. A comparison between eqs. (4) and (5) shows the analogy between the two models: the identification

$$\tfrac{1}{2}\epsilon(\omega-V^i)\equiv 1-\epsilon-m/F'(y^i) \tag{6}$$

suggests that the multiplier m plays the role of the frequency ω in the Schrödinger operator, whereas the spatial dependence of $F'(y^i)$ mimicks the variation of the potential V^i. Such an equivalence leads to an exact correspondence between Anderson localization and the properties of a frozen random state. This confirms a suggestion put forward in ref. [5]. The more general case of spatio-temporal chaos, instead, carries a novel difficulty associated with the temporal dependence of the pseudo potential $F'(y^i)$. An analogy with random walks in a random environment has been used in ref. [4] for simplectic maps. However, in so far as we consider the simple piecewise linear map

$$F(y)=ry\,\mathrm{Mod}(1)\,, \tag{7}$$

we can get rid both of the space and time dependence of $F'(y^i)$. In this special case the local multiplier is everywhere equal to r. This model is equivalent to the free particle case in the language of the Schrödinger operator, where an analytical solution can be derived. The integrated density of states $N'(\omega)$ of the Schrödinger equation is the well-known expression $(V=0)$

$$N'(\omega)=1-\pi^{-1}\arccos(\omega/2)\,. \tag{8}$$

The spectrum is confined between $\omega_{\min}=-2$ and $\omega_{\max}=2$, where it exhibits square-root Van Hove singularities. Despite the simple structure of the model, some interesting results are already obtained in this case. In fact, the Lyapunov exponent λ is, by definition, the logarithm of the absolute value of m. From eqs. (6) and (7) we find

$$m=r\,[1-\epsilon(1+\tfrac{1}{2}\omega)]\,. \tag{9}$$

The maximum value of the multiplier is reached for the minimum energy which, independently of the coupling strength, is $m_{\max}=r$, a value coinciding with the multiplier of the single isolated map. Therefore, the maximum Lyapunov exponent coincides with that of the isolated map $\ln r$. The minimum multiplier is reached at the maximum energy of the associated Schrödinger equation,

$$m_{\min}=r(1-2\epsilon)\,. \tag{10}$$

For $\epsilon<1/2$, $m_{\min}$ is positive, and its logarithm yields the minimum Lyapunov exponent. In this range of ϵ-values, the LS is given by

$$N(\lambda)=1-\pi^{-1}\arccos(e^{\lambda}/\epsilon r+1-1/\epsilon)\,. \tag{11}$$

A typical behavior is reported in fig. 1 (curve a) for $\epsilon=1/3$ and $r=2$. For $\epsilon>1/2$, $m_{\min}$ is negative, meaning that the spectrum of m-values extends around 0. As a consequence, the Lyapunov spectrum which does not discriminate between negative and positive m's extends to $-\infty$ (for $m=0$) and is, more in general, given by

$$N(\lambda)=N'(e^{\lambda})-N'(-e^{\lambda})\,. \tag{12}$$

Therefore, we can distinguish two different ranges of λ-values: $I_1=[\ln r+\ln(2\epsilon-1),\ln r]$, not affected by negative multipliers, where eq. (11) still holds, and $I_2=(-\infty,\ln r+\ln(2\epsilon-1)]$ where two separate contributions are added. The two regimes are separated by a square-root type of singularity, deriving from the above mentioned Van Hove singularity. A typical plot is shown in fig. 2 for $\epsilon=2/3$, and $r=2$ (curve

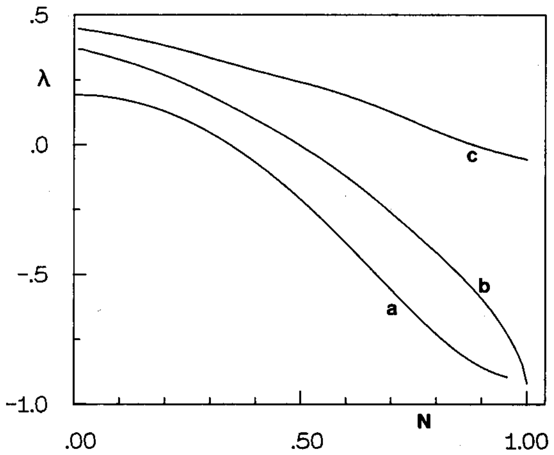

Fig. 1. Lyapunov spectra of model (7) with $r=2$ (a), of the logistic map at crisis (b), and of the tent map (c), with a coupling strength $\epsilon=1/3$. Curve a, resulting from analytic expression (11), has been shifted down by 0.5 units for clarity reasons. Curves b and c follow from numerical simulations performed with chains of length $L=100$, and 2^{15} iterations.

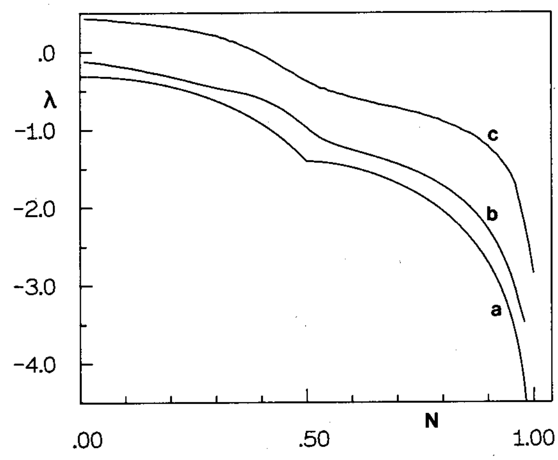

Fig. 2. Same as in fig. 1, for $\epsilon=2/3$. Curves a and b are shifted down by 1 and 0.5 units, respectively. The last two exponents of curve b (logistic map) have been discarded because of the inaccuracy deriving from the strong contraction rate.

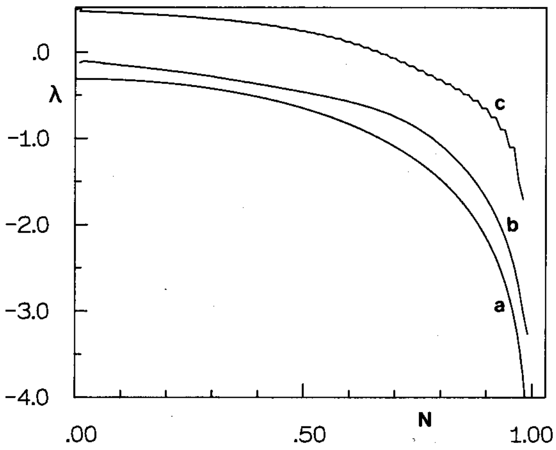

Fig. 3. Same as in fig. 1, for $\epsilon=1$. Curves a and b are shifted down by 1 and 0.5 units, respectively. The last two exponents of curve c and the last one of curve b have been discarded for the same reasons as in fig. 2. The steps appearing in curve c are essentially due to the existence of two uncoupled sub-chains for $\epsilon=1$.

a). Finally, in the limit case $\epsilon=1$, the interval I_1 vanishes and the intermediate singularity disappears overlapping with the first one (see curve a in fig. 3).

This problem has been already investigated by Kaneko in ref. [5] and Kaspar and Schuster in ref. [6]. They limited the analysis to the onset of chaos (i.e. $r=1+\delta$, $\delta\approx 10^{-3}$), where only the first Lyapunov exponents are needed to estimate the fractal dimension of the chain. Such exponents have been computed under the hypothesis of a continuous spatial structure: an approximation valid only around the maximum exponent.

The origin of the singular behavior in the LS can be better understood by observing a general property of the tangent space evolution matrix (3). This can be written in the following form for a lattice of L sites with periodic boundary conditions,

$$J_n^L=\Phi_n^L\cdot J^L(\epsilon)\,, \tag{13}$$

where the diagonal matrix $(\Phi_n^L)_{i,j}=F'(y_n^i)\delta_{i,j}$ carries all the time (n) dependence, and $J^L(\epsilon)$ is the Jacobi matrix,

$$J^L(\epsilon)=\begin{pmatrix} 1-\epsilon & \epsilon/2 & 0 & \dots & 0 & \epsilon/2 \\ \epsilon/2 & 1-\epsilon & \epsilon/2 & \dots & 0 & 0 \\ 0 & \epsilon/2 & 1-\epsilon & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1-\epsilon & \epsilon/2 \\ \epsilon/2 & 0 & 0 & \dots & \epsilon/2 & 1-\epsilon \end{pmatrix}. \tag{14}$$

It is a simple but instructive task to compute the eigenvalues of $J^L(\epsilon)$. First of all one should notice that the form (14) corresponds to the free particle Schrödinger matrix (see eq. (5)). Accordingly, $J^L(\epsilon)$ commutes with the translation matrix T defined on a generic vector as

$$T\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{L-1} \\ v_L \end{pmatrix}=\begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_L \\ v_1 \end{pmatrix}. \tag{15}$$

The eigenvalues t_k of T are the roots of unity,

$$t_k=\exp(2i\pi k/L),\quad 0\leqslant k\leqslant L-1\,. \tag{16}$$

The eigenvectors are of the form $(1,\ t_k,\ t_k^2,\ \dots,\ t_k^{L-1})$, i.e. plane waves. By applying $J^L(\epsilon)$ to vectors belonging to this base, one obtains all the eigenvalues [#1]

$$J_k^L=1-2\epsilon\sin^2(\pi k/L),\quad k=0,\dots,L-1\,. \tag{17}$$

It is straightforward to observe that for any fixed L

[#1] One referee pointed out that the same expression was obtained in ref. [7].

there exists a set of ϵ-values such that the determinant of J^L vanishes,

$$\epsilon_k = [2\sin^2(\pi k/L)]^{-1} . \quad (18)$$

This result is valid independent of the local dynamics, due to the factorization property (13). The range of ϵ-values is limited from below by 1/2. Therefore, the map (1) has no singularities in the range [0, 1/2], for any choice of the local dynamics $F(y)$. This result confirms and generalizes what was previously found for map (7): the LS extends to $-\infty$ for $\epsilon > 1/2$. It is worth noticing that whenever the determinant of J^L vanishes, the dynamics of the system happens to be constrained on a suitable sub-manifold of the phase space. We shall report elsewhere on how this fact affects the statistical properties of macroscopic observables.

We conclude by showing some numerical simulations we have performed for the logistic map at crisis ($F(y)=4y(1-y)$), and the tent map ($F(y)=2y$, $0<y<1/2, f(y)=2-2y$, $1/2<y<1$). The results are reported in figs. 1–3 and correspond to curves b and c, respectively. In fig. 1 all the three spectra are bounded from below (being $\epsilon=1/3$), while in figs. 2, 3, clear-cut evidence of a divergence to $-\infty$ is reported. Notice that the intermediate singularity, occurring for map (7), is smoothed out for the other two maps. We conjecture this to be a consequence of the random dependence of $F'(y^i)$ on the spatial variable. In the Schrödinger problem we know that the square-root singularities occurring at the band edge of the spectrum for a periodic potential, are replaced by the "flat" Lifshitz tails, for a random potential. Something similar may reasonably occur in the case of spatio-temporal chaos, where the dynamics provides a source of randomness.

We end the Letter by noting that the analysis developed in eqs. (13)–(18) does not only allow one to determine the properties of the determinant, and to recover the results for map (7), but it also allows one to perform a sort of mean field theory for generic maps. The first step in this direction is performed by observing that a homogeneous solution (constant in space) remains such under time evolution for any choice of ϵ, and its dynamics is obviously equivalent to that of the single map. In such a case the diagonal matrix Φ_n^L reduces to the product of the identity matrix times the scalar quantity $F'(y_n)$. The evolution in the tangent space then reduces to the product of matrices which differ only for a multiplicative factor. The final effect on the LS is that the Lyapunov exponents are nothing but the logarithm of the eigenvalues J_k^L plus the Lyapunov exponent of the single uncoupled map. As a result, it turns out that eqs. (8) and (12) take into account the λ dependence for any homogeneous solution of a generic map, the dependence on the model being confined to a trivial shift. In particular, it is possible to show that in the case of short chains ($L=2$–5), in suitable ranges of ϵ-values, the homogeneous state is stable. This corresponds to the case when only one Lyapunov exponent is strictly positive. Further steps in the derivation of a mean field theory would require replacing the invariant measure of the single map with the actual measure of the system.

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