Chaotic Synchronization of spatially extended systems

Alessandro Torcini

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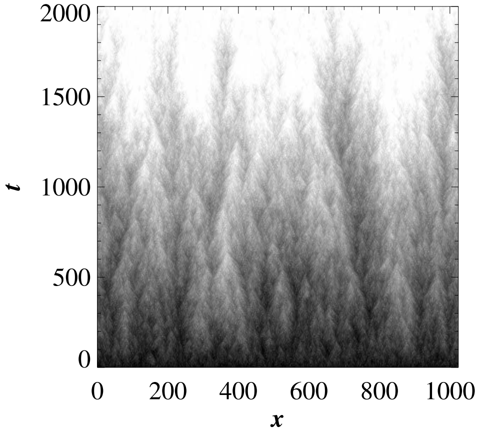

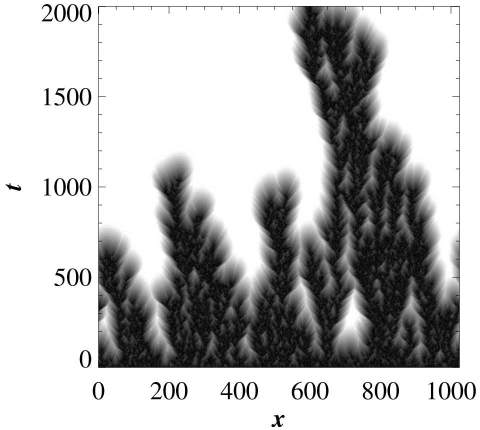

Plan of the Talk

Summary of Old Results

- Chaotic Synchronization in Low Dimensional Systems $\lambda_T = 0$
- Chaotic Synchronization in Spatially Extended Systems $V_F = 0$ (Diffusive Coupling)
  - The transition is analyzed as a a non-equilibrium phase transition
  - The transition is continuous and its critical properties correspond to
    - Multiplicative Noise (MN) $V_F = \lambda_T = 0$
    - Directed Percolation (DP) $V_F = 0 \quad ; \quad \lambda_T < 0$

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New Results

- Spatially Extended Chaotic Systems with Power-Law Coupling
- The synchronization transitions (STs) are continuous
- The critical indexes vary continuously with the interaction range
- The family of STs correspond to Anomalous Directed Percolation (ADP)
  - ADP has been found for Lévy-flight spreading of epidemic processes
  - ADP critical exponents have been measured for stochastic lattice models

Low Dimensional Chaotic Systems

Chaotic Dynamics

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

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Systems Coupled via Stochastic forcing

Two replicas **u** and **w** of the same dynamical system:

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) + \gamma \cdot \eta(t)$ $\mathbf{u}(0) \neq \mathbf{w}(0)$

$\dot{w}_k(t) = \varphi_k(\mathbf{w}(t)) + \gamma \cdot \eta(t)$

η is a δ-correlated random variable $< \eta(t')\eta(t) >= \delta(t' - t)$.

For a sufficiently large noise amplitude $\gamma > \gamma_c$ the replicas can eventually synchronize.

Low Dimensional Chaotic Systems

Chaotic Dynamics

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

Mutually Coupled Systems

Two replicas **u** and **w** of the same dynamical system:

$$\dot{u}_k(t) = (1-\gamma)\cdot\varphi_k(\mathbf{u}(t)) + \gamma\cdot\varphi_k(\mathbf{w}(t)) \quad \mathbf{u}(0) \neq \mathbf{w}(0)$$

$$\dot{w}_k(t) = (1-\gamma)\cdot\varphi_k(\mathbf{w}(t)) + \gamma\cdot\varphi_k(\mathbf{u}(t))$$

For a sufficiently strong coupling $\gamma > \gamma_c$ the replicas can eventually synchronize

Low Dimensional Chaotic Systems

Chaotic Dynamics

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

Def: Synchronization is achieved if the distance between replicas asymptotically vanishes

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} |u(t) - w(t)| = 0$$

In order to observe synchronization in low dimensional systems :

$$\lambda_\perp = \lim_{t\to\infty} \lim_{z(0)\to 0} \ln \frac{z(t)}{z(0)} < 0$$

the transverse Lyapunov exponent (TLE) should be negative .

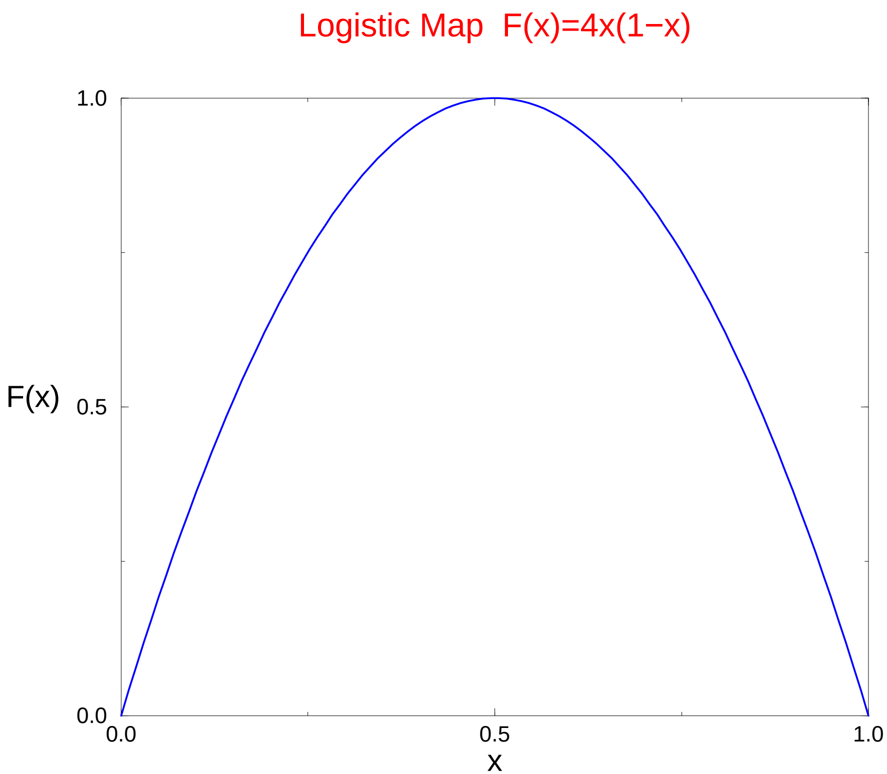
Maritan & Banavar, PRL (1994); Pikovsky, PLA (1992), PRL (1994);
Herzel & Freund, PRE (1995); Lai & Zhou, EPL (1998)

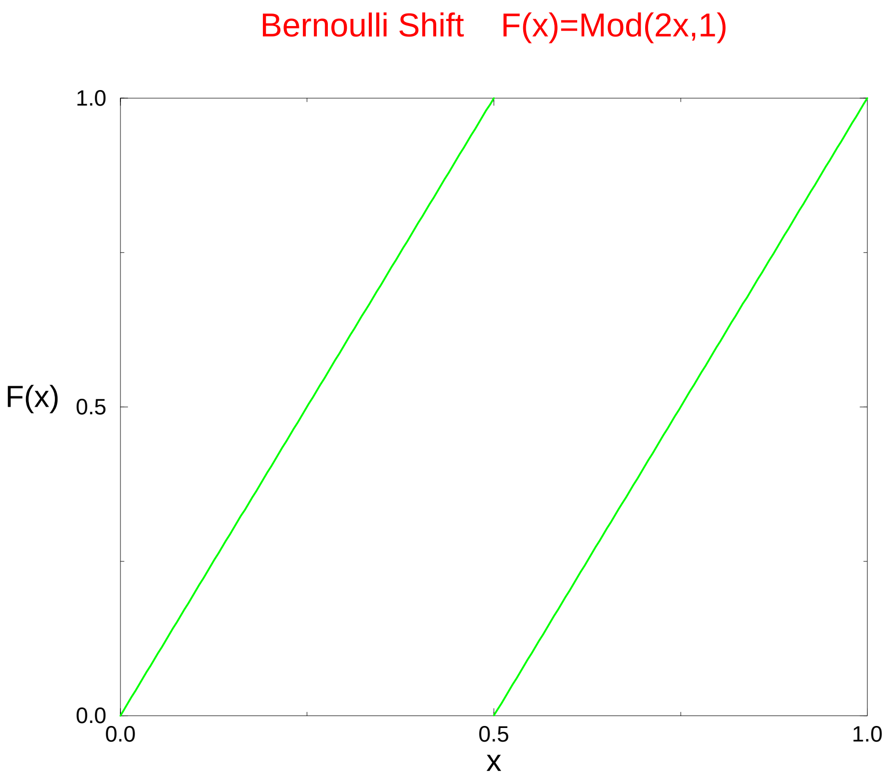
Spatially Extended Systems

Coupled Map Lattices

$$u_x^{t+1} = F\left[(1 + \nabla_\varepsilon^2) u_x^t\right] \qquad \nabla_\varepsilon^2 u_x = \varepsilon\{[u_{x+1} + u_{x-1}]/2 - u_x\}$$

where x and t are discrete, F is a chaotic map, typically one dimensional.





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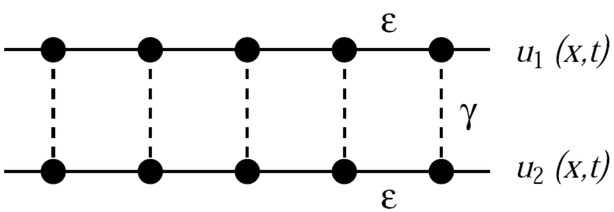


$$u_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^2)u_x^t\right] + \gamma \cdot F\left[(1+\nabla_\varepsilon^2)w_x^t\right]$$
$$w_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^2)w_x^t\right] + \gamma \cdot F\left[(1+\nabla_\varepsilon^2)u_x^t\right]$$

Stochastic Forcing

$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)u_x^t\right] + \gamma \cdot \zeta_x^t$$
$$w_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)w_x^t\right] + \gamma \cdot \zeta_x^t$$

where the noise is δ-correlated in space and time $< \zeta_x^t \zeta_y^s > \propto \delta_{x,y}\delta_{t,s}$.

The local difference field is defined as $z_x^t = |u_x^t - w_x^t|$.

Spatially Extended Systems

Coupled Map Lattices

$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)u_x^t\right] \qquad \nabla_\varepsilon^2 u_x = \varepsilon\{[u_{x+1}+u_{x-1}]/2 - u_x\}$$

where x and t are discrete, F is a chaotic map, typically one dimensional.

Synchronization

For sufficiently strong coupling γ the spatially averaged difference field

$$\rho(t) =< z(t) >= \frac{1}{L}\sum_{x=1}^{L} z_x^t$$

could eventually vanish in the long time limit.

The synchronization transition is no longer fully described in terms of the transverse Lyapunov exponent (TLE).

An extreme nonlinearity in the local map F can induce transport of Finite Size Disturbances even for linearly stable states (i.e. Negative TLE).

A new indicator is required to fully characterize the transition for spatially extended systems.

Spatially Extended Systems

Coupled Map Lattices

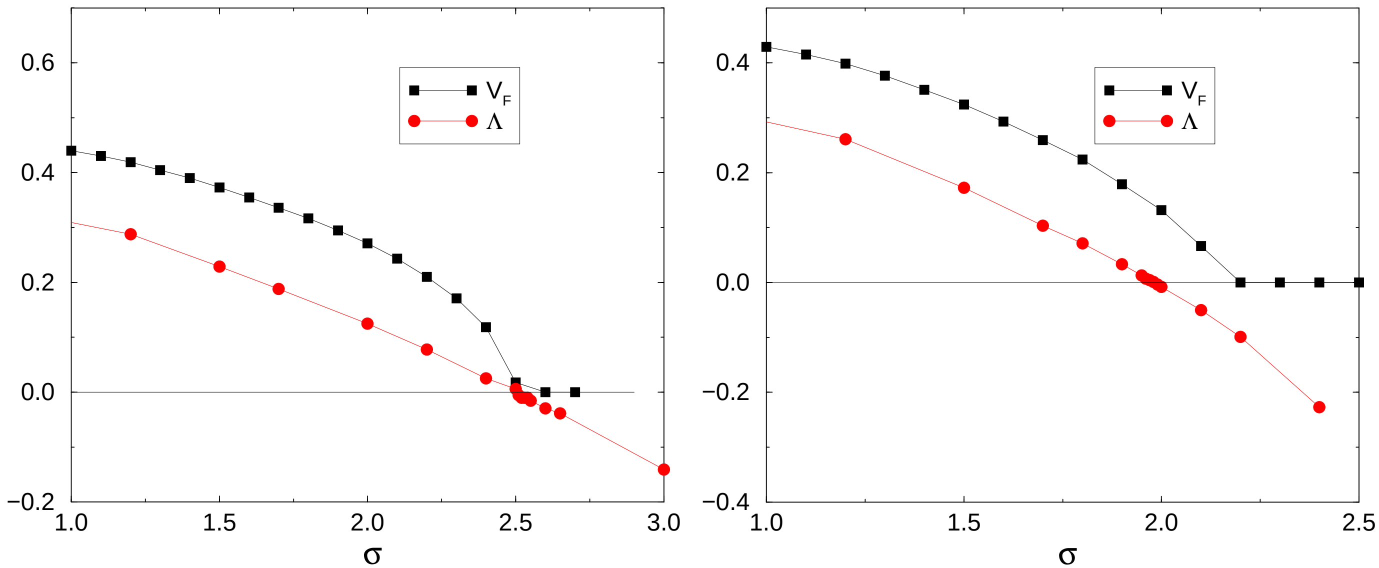
$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)u_x^t\right] \qquad \nabla_\varepsilon^2 u_x = \varepsilon\{[u_{x+1}+u_{x-1}]/2 - u_x\}$$

where x and t are discrete, F is a chaotic map, typically one dimensional.

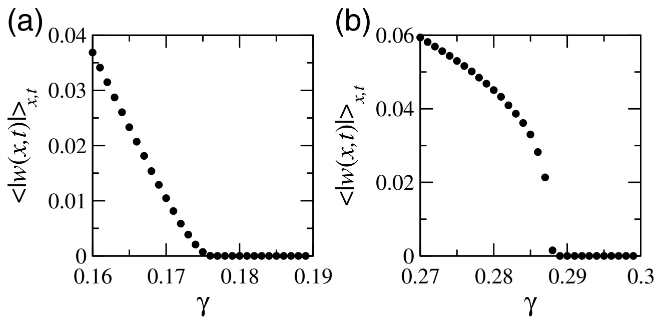
Propagation Velocity of Finite Size Perturbations

A droplet of unsynchronized sites ($N(0)$) is inserted in a completely synchronized state:

$$v_F = \lim_{t\to\infty} \frac{N(t)-N(0)}{2t}$$



Universality Classes



The Synchronization Transition is a Non-Equilibrium Phase Transition leading from an "active phase" ($\rho > 0$) to an "absorbing phase" ($\rho \equiv 0$).

The transition point a_c is located in the thermodynamic limit ($L \to \infty$) by the vanishing of the order parameter $\rho(t) \equiv < z(t) > \to 0$.

A continuous transition is typically characterized by a critical behavior :

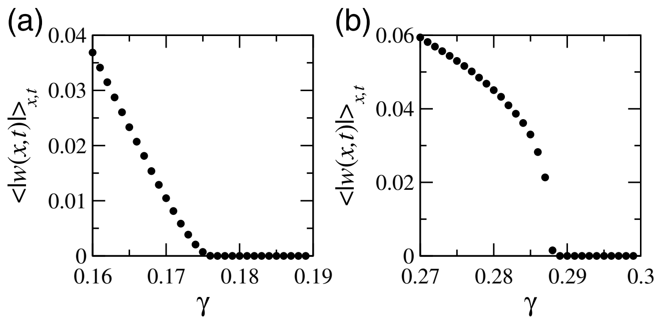
$$\rho(t) \propto t^{-\delta} \qquad \rho(t) = L^{-z\delta} g(t/L^z) \qquad \text{at} \qquad a \equiv a_c$$

$$< \rho >_t \propto |a - a_c|^{\beta}$$

$$L_c \propto |a - a_c|^{-\nu_\parallel} \qquad T_c \propto |a - a_c|^{-\nu_\perp}$$

only 3 exponents are independent (e.g. δ β and z)

Universality Classes



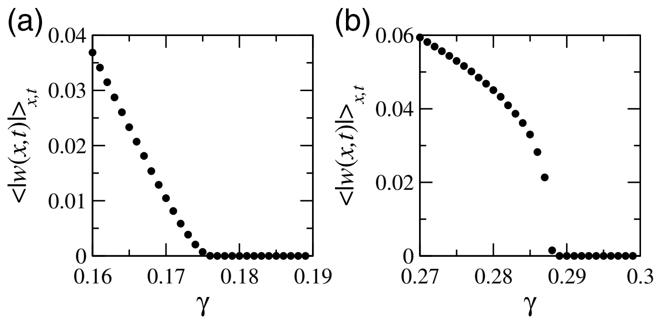
The Synchronization Transition is a Non-Equilibrium Phase Transition leading from an "active phase" ($\rho > 0$) to an "absorbing phase" ($\rho \equiv 0$).

Two different types of transitions have been observed:

- Multiplicative Noise
  - $v_F = \lambda_\perp = 0$
  - Linear Effects rule the Transition
- Directed Percolation
  - $v_F = 0 \quad \lambda_\perp < 0$
  - Strong Nonlinear Effects ($|F'| >> 1$)

Baroni, Livi & AT , PRE 63, 036226 (2001); Ahlers & Pikovsky, PRL, 88, 254101 (2002)

Universality Classes



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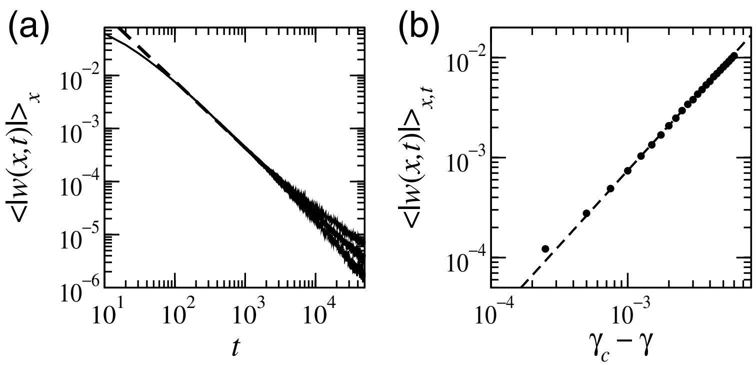
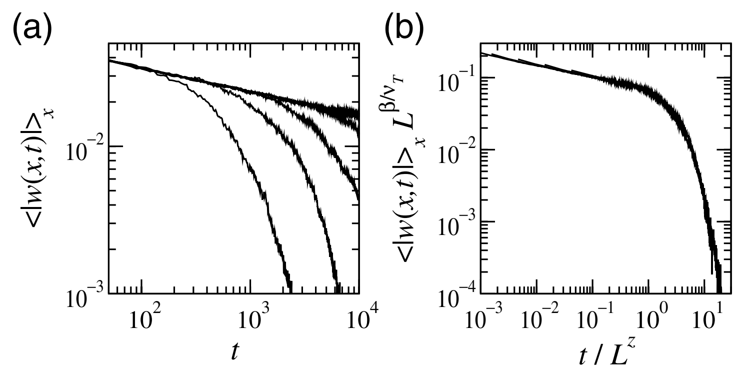




	MN	Tent	DP	Bernoulli
δ	1.10(5)	1.26(3)	0.159464(6)	0.16
β	1.70(5)	1.50(5)	0.276486(6)	0.28
z	1.53(7)	1.5	1.580745(6)	1.581

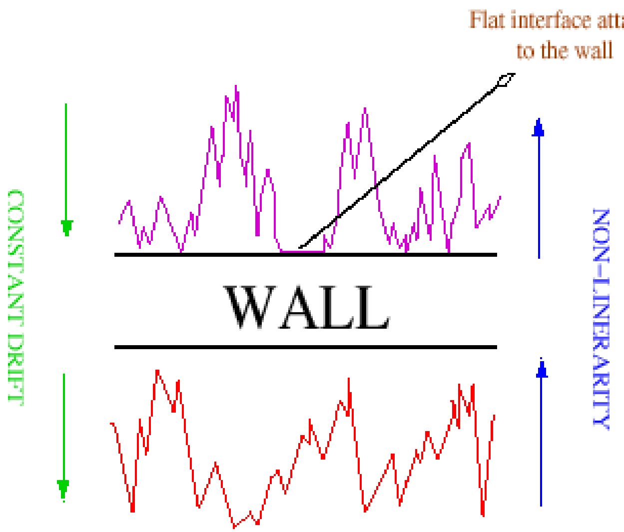
Ahlers & Pikovsky, PRL, 88, 254101 (2002); V. Ahlers , PhD Thesis (Berlin, 2001)

Multiplicative Noise

The corresponding field equation for the coarse-grained variable $w(x,t) = \bar{z}$ is:

$$\dot{w}(x,t) = \nabla^2 w(x,t) + aw(x,t) - bw^p(x,t) + w(x,t)\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and time and $p \geq 2$. Pikovsky & Kurths (94) have shown that this model describes the dynamics of CMLs within a linear framework.



This problem can be mapped on that of a depinning of a KPZ interface from a hard substrate through a Hopf-Cole Transformation $h(x,t) = -\ln w(x,t)$. This leads to a KPZ-like equation

$$\dot{h}(x,t) = \nabla^2 h(x,t) - (\nabla h(x,t))^2 - a' - b e^{-(p-1)h(x,t)} + \eta(x,t)$$

The adsorbing state $w = 0$ is now mapped into $h = \infty$

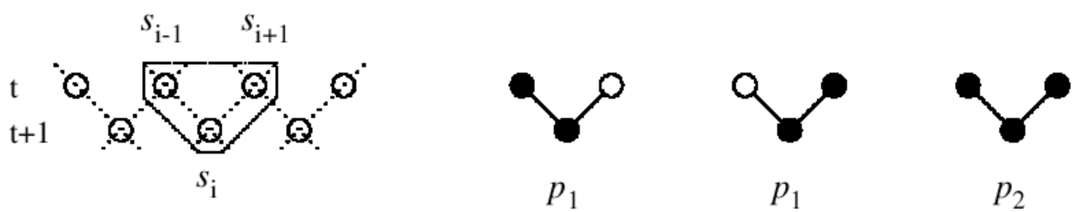
[M.A. Muñoz, cond-mat/0303650 (2003)]

Directed Percolation

The corresponding field equation is:

$$\dot{w}(x,t) = \nabla^2 w(x,t) + aw(x,t) - bw^2(x,t) + \sqrt{w(x,t)}\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and time.
This equation is usually associated to Infection Spreading Models: the Domany-Kinzel cellular automaton:



black sites are infected (active phase), white sites are healthy (absorbing phase).

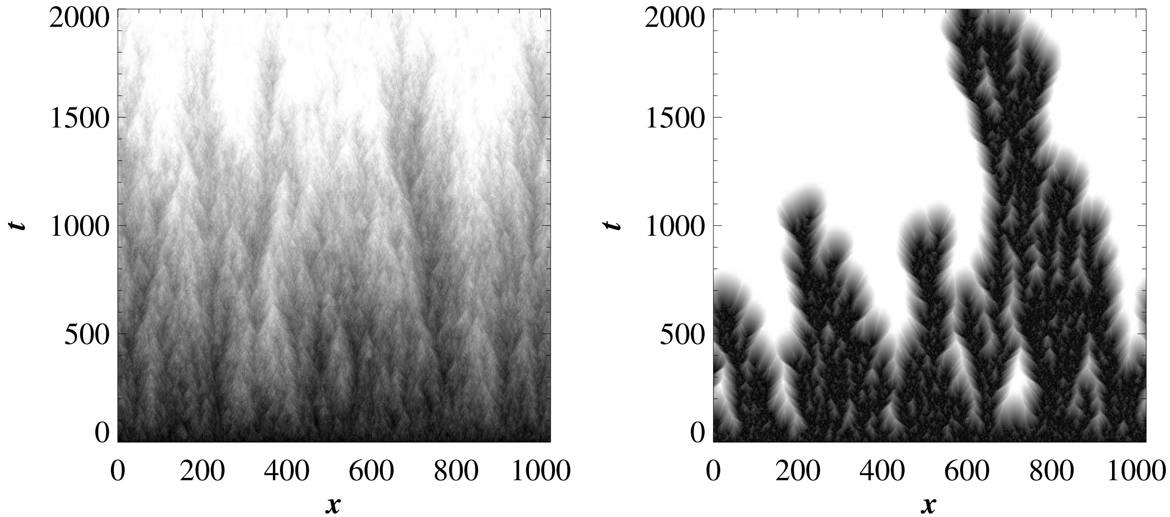
- The infection spreads only by contact
- No revival of infection within healthy region: the absorbing state is stable

[H. Hinrichsen Adv. Phys. 49, 815-958 (2000)]

Experimental measure of DP exponents for a ring of oscillating ferrofluidic spikes at the transition to spatiotemporal intermittency Rupp, Richter, & Rehberg, PRE (2003)

Summary of the first part

- In spatially extended systems (CMLs) with diffusive coupling two different synchronization transitions are observed :
  - if the linear behaviour prevails on nonlinear effects the transition belongs to the MN universality class;
  - if nonlinear effects dominate the dynamics DP scaling laws are observed.



In collaboration with:

Francesco Ginelli (Saclay - Paris)

- V. Ahlers* (Germany)
- R. Livi (Firenze)
- A. Pikovsky (Potsdam)
- L. Baroni* (Italy)
- D. Mukamel (Rehovot)
- A. Politi (Firenze)

* Working in private companies

Long-Range Interactions

Coupled Map Lattices with Power-Law Coupling

$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^\sigma)u_x^t\right] \qquad \nabla_\varepsilon^\sigma u_x = -\epsilon u_x + \frac{\epsilon}{\eta(\sigma)}\sum_{m=1}^{M}\frac{u_{x-j_m(q)}+u_{x+j_m(q)}}{(j_m(q))^\sigma}$$

where $x \in [1, L]$ and t are discrete, $F = 2x$ (mod 1) is the Bernoulli map and periodic boundary conditions are assumed.

$$\eta(\sigma) = 2\sum_{m=1}^{M}\frac{1}{(j_m(q))^\sigma} \qquad \text{normalization} \quad \text{factor}$$

$\sigma \to 0$ Globally Coupled Maps $\qquad \sigma \to \infty$ Usual CMLs

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Coupling Schemes

- Fully Coupled: $j_m(q) = m$, $M = (L-1)/2$
- Reduced Coupling: $j_m(q) = q^m - 1$, $M = \log_q(L/2)$ with $q = 2, 4$ and 8

The coupling scheme does not alter the critical properties of the transition, but the reduced scheme is much faster ($\mathcal{O}(L \log_q L)$ versus $\mathcal{O}(L^2)$),

Chaotic Synchronization

The synchronization transition of two coupled replicas is studied

$$u_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^\sigma)u_x^t\right] + \gamma \cdot F\left[(1+\nabla_\varepsilon^\sigma)w_x^t\right]$$
$$w_x^{t+1} = (1-\gamma)F\left[(1+\nabla_\varepsilon^\sigma)w_x^t\right] + \gamma \cdot F\left[(1+\nabla_\varepsilon^\sigma)u_x^t\right]$$

by examining the synchronization error $z_x^t = |u_x^t - w_x^t|$ for different coupling exponents σ.

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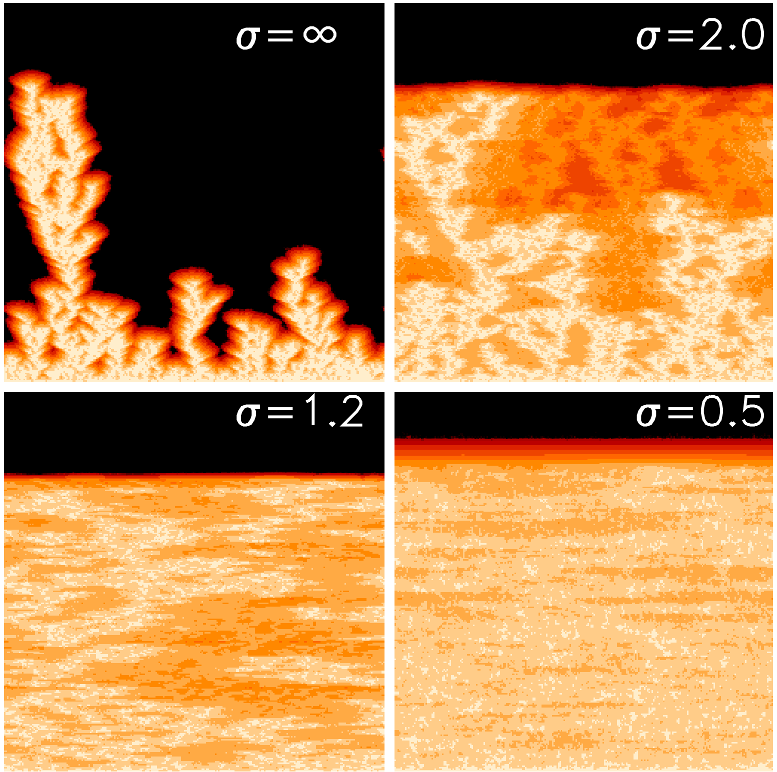
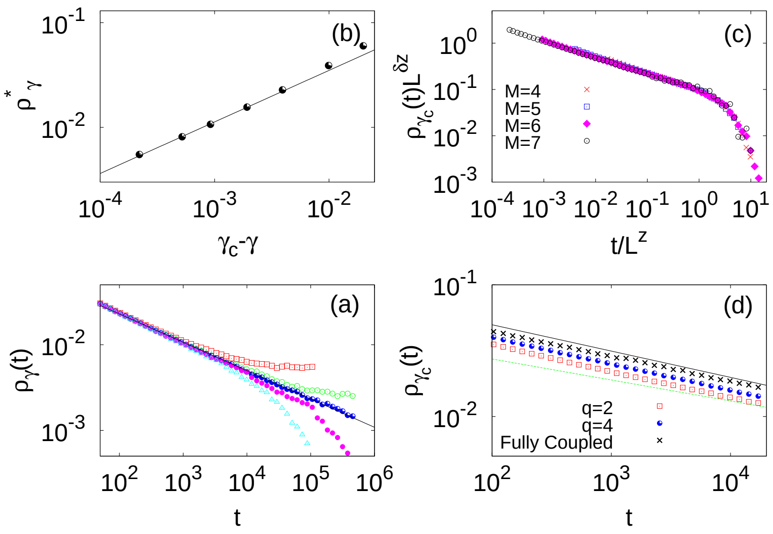


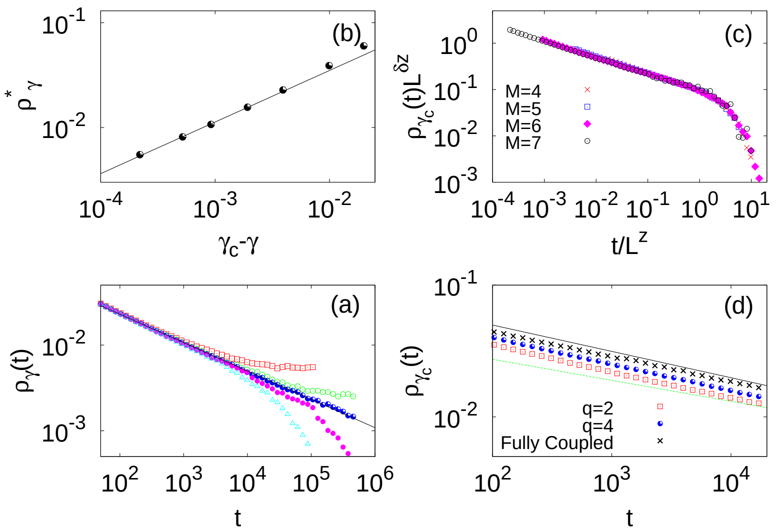
The critical exponents



System size $6 \times 10^4 \leq L \leq 4 \times 10^6$

Averages over $100 - 1000$ different realizations

The critical exponents



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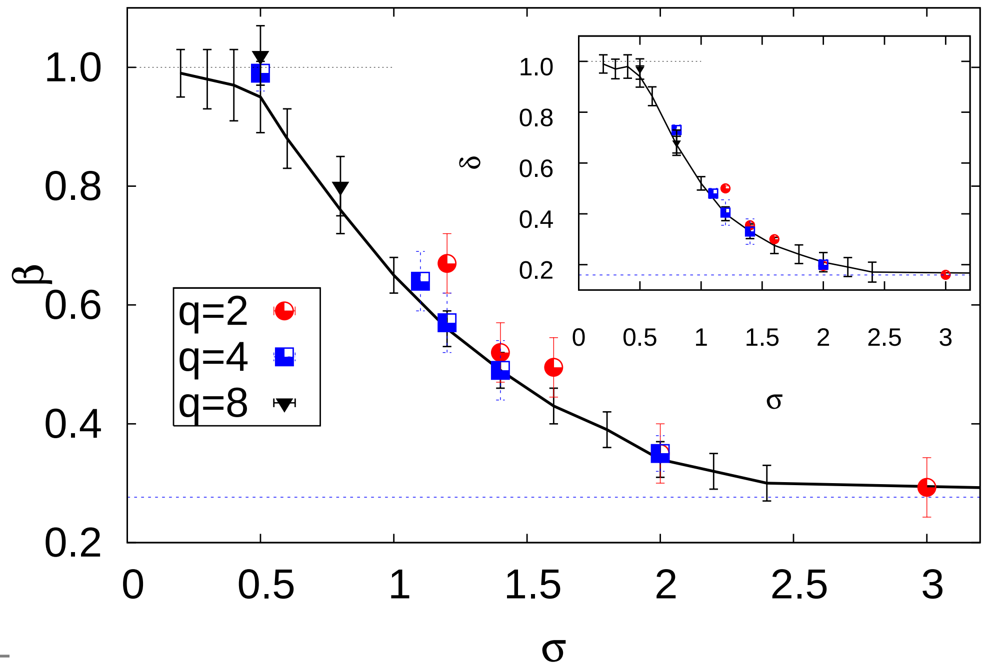


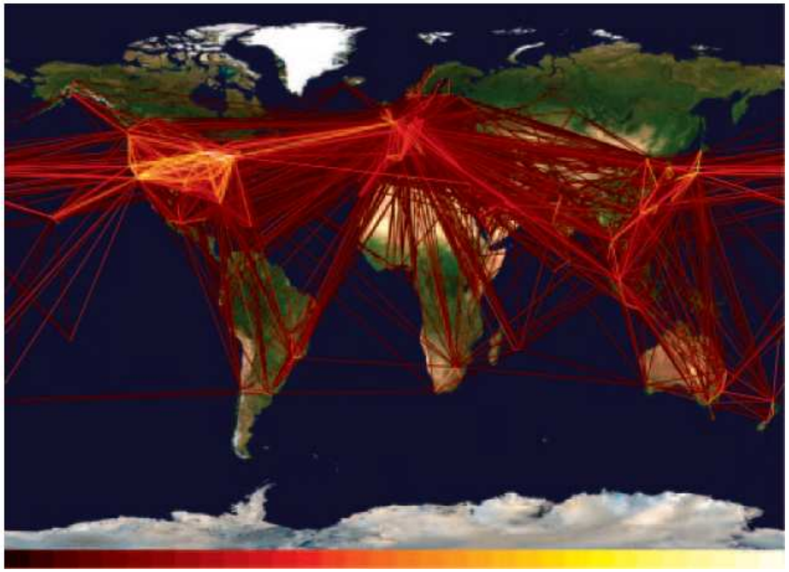
The critical exponents vary continuously, we have a family of universality classes labelled by the coupling exponent σ.

$\delta_{DP} \sim 0.16$ $\beta_{DP} \sim 0.27$

Anomalous Directed Percolation

Distribution of human travels

Brockmann et al.Nature (2006)



In many realistic spreading processes short-range interactions do not appropriately describe the transport mechanism of the infection

- infectious disease transported by insects;
- disease spread triggered by aviation traffic;
- spreading agent subjected to a turbulent flow.

The motion of the agent can be super-diffusive.

Mollison in 1977 proposed a generalization of the usual DP in which the agent can perform Lévy flights, where the distribution of the spreading distances r is given by

$$P(r) \propto 1/r^{d+\sigma} \qquad \sigma > 0$$

d being the spatial dimension of the system.

Mollison, J R Stat Soc B 39 (1977) 283; Grassberger, Fractals in physics, (1986)

Field Theoretic Prediction

The generalization of the usual field equation to anomalous DP reads as:

$$\dot{w}(x,t) = (\nabla^2 + \nabla^\alpha) w(x,t) + a w(x,t) - b w^2(x,t) + \sqrt{w(x,t)}\,\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and the anomalous diffusion operator is defined as

$$\nabla^\sigma \mathrm{e}^{ikx} = -k^\sigma \mathrm{e}^{ikx}$$

The renormalization group calculations indicate that

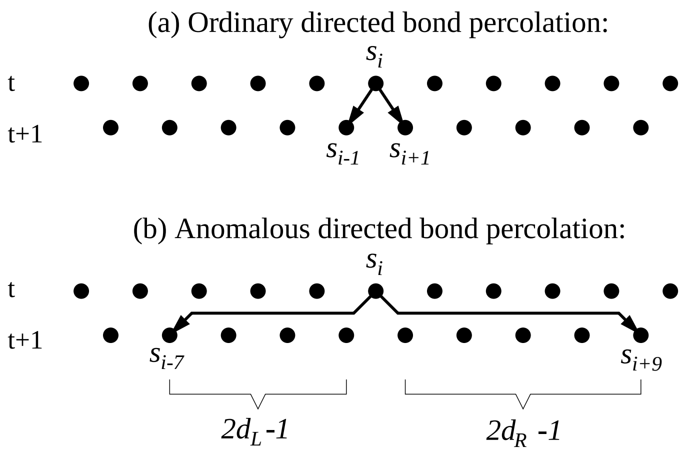
- for $\sigma < 0.5$ the mean-field description should become exact;
- for $\sigma > 2.0677(2)$ the usual DP results should be recovered

Mean-field exponents obtained by neglecting correlations are:

$$\beta_{MF} = \delta_{MF} = 1.0 \qquad z_{MF} = \sigma$$

Jannsen et al. EPJB 7 (1999) 137; Hinrichsen & Howard EPJB 7 (1999) 635.

Stochastic Lattice Model



$s_i = 1$ (infected) - $s_i(t) = 0$ (healthy)
Only infected sites can propagate the disease.
The control parameter is
the bond probability $0 \leq p \leq 1$

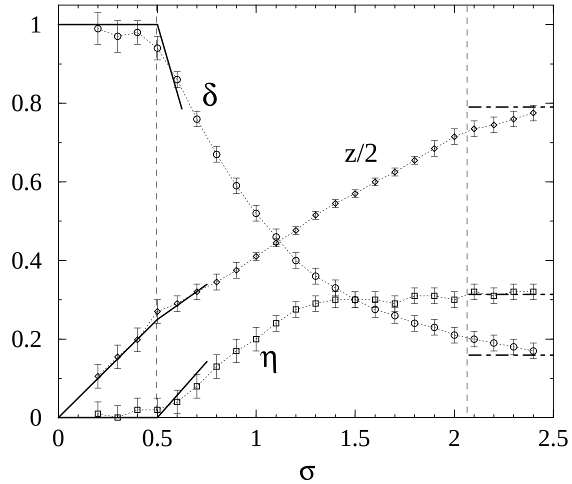
- At the next time $t + 1$ all the sites are initially healthy;
- two distances (d_L, d_R) are randomly generated from the distribution $P(r) \propto 1/r^{1+\sigma}$;
- the sites located at those distances from a site i (infected at time t) become infected if by choosing two random numbers (y_L, y_R) between 0 and 1
  - $s_{i+1-2d_L}(t = 1) = 1$ if $y_L < p$
  - $s_{i-1+2d_R}(t = 1) = 1$ if $y_R < p$

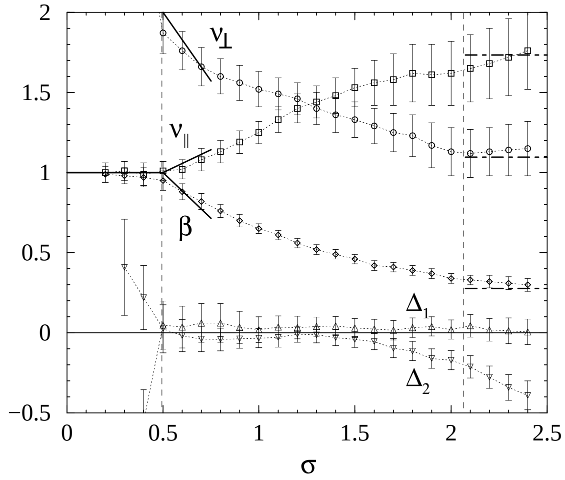
The length of the examined system was $L = 4 \times 10^{19}$, no finite size effects.

Hinrichsen & Howard EPJB 7 (1999) 635.

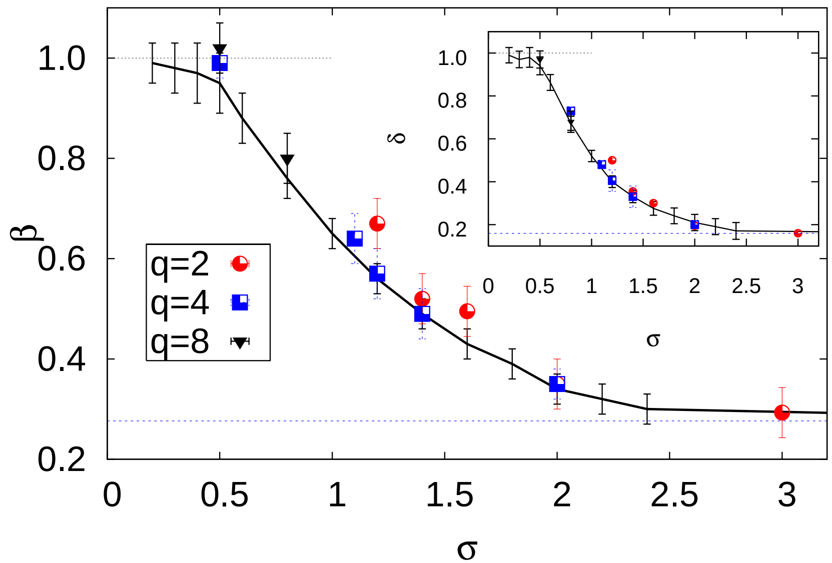
Critical Indexes

Extremely accurate estimation of the critical exponents in the whole range $0 < \sigma < 2.4$.





Our results for the chaotic synchronization transitions are in very good agreement

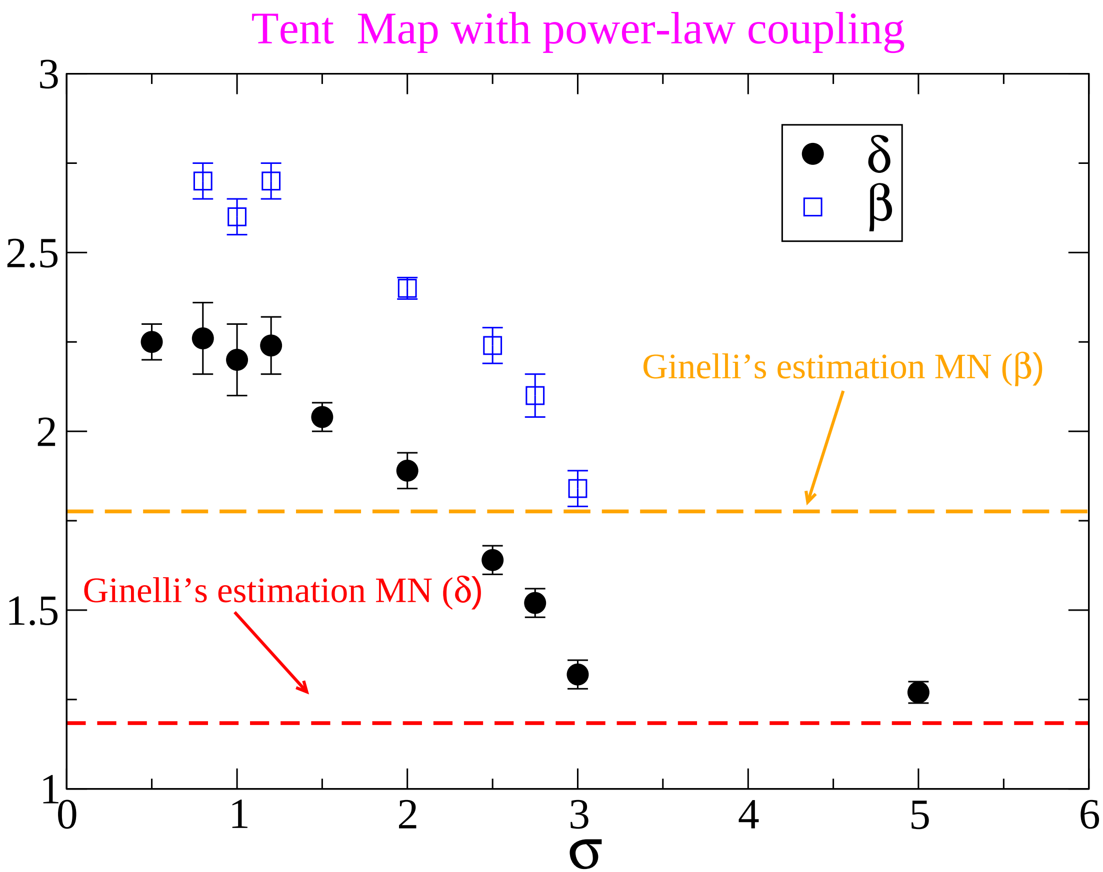


Conclusions & Perspectives

- The synchronization transition (ST) of two replicas of of chaotic discontinuous coupled maps with long range interactions is characterized by a continuum of universality classes labeled by the exponent σ.
- The critical properties of these STs correspond to Anomalous Directed Percolation, previously examined in the context of epidemic spreading.
- Preliminary results indicate that also for continuous maps the exponents depend on σ, but they do not belong to the anomalous DP class.
- Anomalous Multiplicative Noise has been not yet studied, therefore a completely open problem is to find to which universality class ST for continuous maps correspond.

C.J. Tessone, M. Cencini & AT , PRL (2006)

Continuous Maps



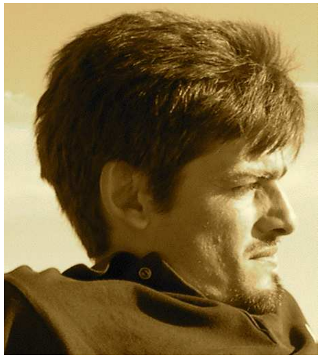
Kissinger et al. (2005) - Multiplicative noise - numerical estimations

$\delta_{MN} = 1.184(10)$ and $\beta_{MN} = 1.776(15)$

Munoz (2000) – Mean-field results β depends on noise amplitude, it is not universal.

Credits

- Claudio Juan Tessone - Post-Doc (2007-)
- ETH - Zurich - Switzerland

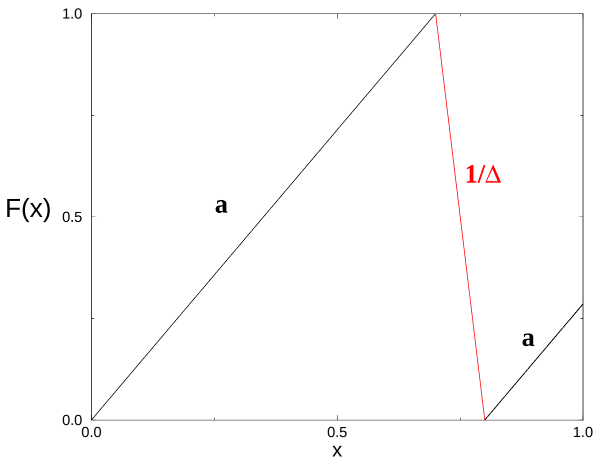


- Massimo Cencini - Researcher (2005-)
- INFM – CNR - Rome

http://www.fi.isc.cnr.it/users/alessandro.torcini/

Random Multipliers Model

A stochastic model is introduced to mimic the dynamics of the difference field z_x^t for 2 chains of mutually coupled CMLs.



$$z_x^{t+1} = \begin{cases} 1, & \text{w.p.} \quad p = a v_x^t \\ a v_x^t, & \text{w.p.} \quad 1-p \end{cases}, \quad \text{if } v_x^t > \Delta$$

$$z_x^{t+1} = \begin{cases} v_x^t/\Delta, & \text{w.p.} \quad p = a\Delta \\ a v_x^t, & \text{w.p.} \quad 1-p \end{cases}, \quad \text{if } v_x^t \leq \Delta$$

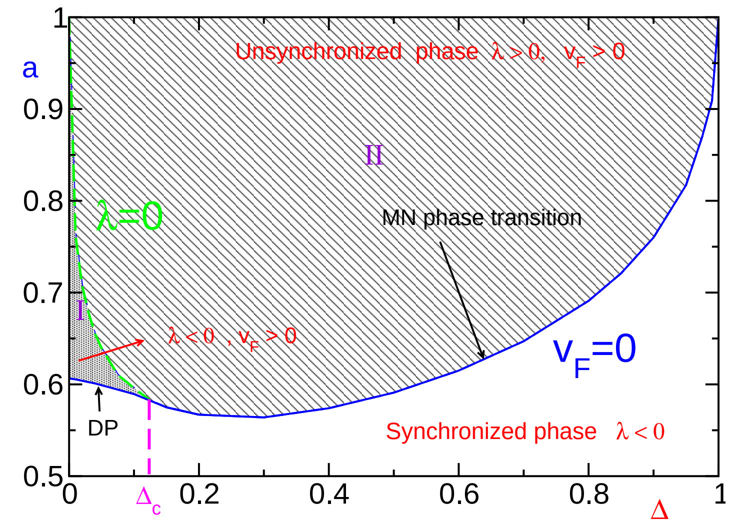
where $v_x^t = (1 + \nabla_\epsilon^2) z_x^t$ and PBC are assumed.

The model is controlled by two parameters a and Δ, for fixed coupling $\varepsilon = 2/3$.
The stochastic nature of the model avoids the emergence of possible long time correlations as in the original deterministic CMLs.

- For small $\Delta < \Delta_c$, the nonlinear mechanisms prevail over the linear ones
- For $\Delta > \Delta_c$, the linear analysis is sufficient to describe the dynamics

Ginelli, Livi, & Politi JPA 35, 499 (2002)

Phase Diagram



- Region II: due to the linear instability any perturbation of the synchronized state will persist forever independently of L
- Region I: a finite perturbation can eventually die in a finite chain, but its life time increases exponentially with L

The critical properties of the model have been studied mainly by analizing $\rho(t)$ (the averaged density of unsynchronized sites). But the definition of ρ requires to fix a threshold W in order to distinguish a synchronized site ($z_x^t < W$) from an unsynchronized one ($z_x^t > W$).

Ginelli,Livi,Politi, & AT PRE 67, 046217 (2003)

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- Therefore, one is obliged to fix an arbitrary threshold W below which trajectories are assumed to be synchronized.
- A priori, one cannot exclude that due to large fluctuations the system will be driven out of the absorbing state, sooner or later.
- The existence of an effective absorbing state will be shown by analyzing the first passage times.

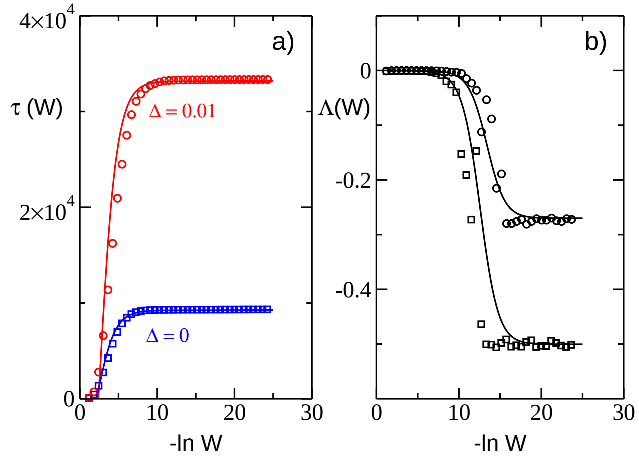
First Passage Times

$\tau(W)$ is the (ensemble) average time needed for $\rho(t)$ to become smaller than a certain threshold for the first time.

By analytical and scaling arguments it can be shown that:

$$\tau(W) = \frac{\ln W}{\lambda_\perp} - L^z g(W L^{\delta z}, W_c)$$

the first term accounts for linear stable behaviour, while the second term for nonlinear effects.



The linear stable behaviour holds below a certain threshold $W_c \propto L^{-z(1+\delta)}$, that vanishes in the thermodynamic limit.

In a finite cellular automaton the minimal meaningful density is $\rho_m = 1/L$, W_c plays the role of ρ_m in continuous systems.