The synchronization transition in spatially extended systems

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Plan of the Talk

Summary of Old Results

- Synchronization in Low Dimensional Systems $\lambda_T = 0$
- Synchronization in Spatially Extended Systems $V_F = 0$
  - Multiplicative Noise (MN) $V_F = \lambda_T = 0$
  - Directed Percolation (DP) $V_F = 0 \quad ; \quad \lambda_T < 0$

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New Results

- Simple Stochastic Model (a & Δ)
- Phase Diagram exhibiting both Transitions
- Strong Arguments supporting DP Scenario:
  - Discontinuous Limit ($\Delta = 0$)
  - Continuous Models ($\Delta > 0$)

Low Dimensional Chaotic Systems

Chaotic Dynamics

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

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Systems Coupled via Stochastic forcing

Two replicas $\mathbf{u}$ and $\mathbf{w}$ of the same dynamical system:

$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) + \sigma \cdot \eta(t)$ $\mathbf{u}(0) \neq \mathbf{w}(0)$

$\dot{w}_k(t) = \varphi_k(\mathbf{w}(t)) + \sigma \cdot \eta(t)$

η is a δ-correlated random variable $< \eta(t')\eta(t) >= \delta(t' - t)$.

For a sufficiently large noise amplitude $\sigma > \sigma_c$ the replicas can eventually synchronize.

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Mutually Coupled Systems

Two replicas **u** and **w** of the same dynamical system:

$$\dot{u}_k(t) = (1 - \sigma) \cdot \varphi_k(\mathbf{u}(t)) + \sigma \cdot \varphi_k(\mathbf{w}(t)) \quad \mathbf{u}(0) \neq \mathbf{w}(0)$$

$$\dot{w}_k(t) = (1 - \sigma) \cdot \varphi_k(\mathbf{w}(t)) + \sigma \cdot \varphi_k(\mathbf{u}(t))$$

For a sufficiently strong coupling $\sigma > \sigma_c$ the replicas can eventually synchronize

Low Dimensional Chaotic Systems

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$\dot{u}_k(t) = \varphi_k(\mathbf{u}(t)) \;\; k = 1, 2, 3, \ldots$ maximum Lyapunov exponent $\lambda > 0$

Def: Synchronization is observed when the distance between replicas asymptotically vanishes

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} |u(t) - w(t)| = 0$$

Condition to observe synchronization in low dimensional systems :

the transverse Lyapunov exponent should be negative

$$\lambda_\perp = \lim_{t\to\infty} \lim_{z(0)\to 0} \ln \frac{z(t)}{z(0)} < 0$$

[Maritan & Banavar, PRL 72, 1451 (1994); Pikovsky, PLA 165, 33 (1992), PRL 73, 2931 (1994); Herzel & Freund, PRE 52, 3238 (1995); Lai & Zhou, EPL 43, 376 (1998)]

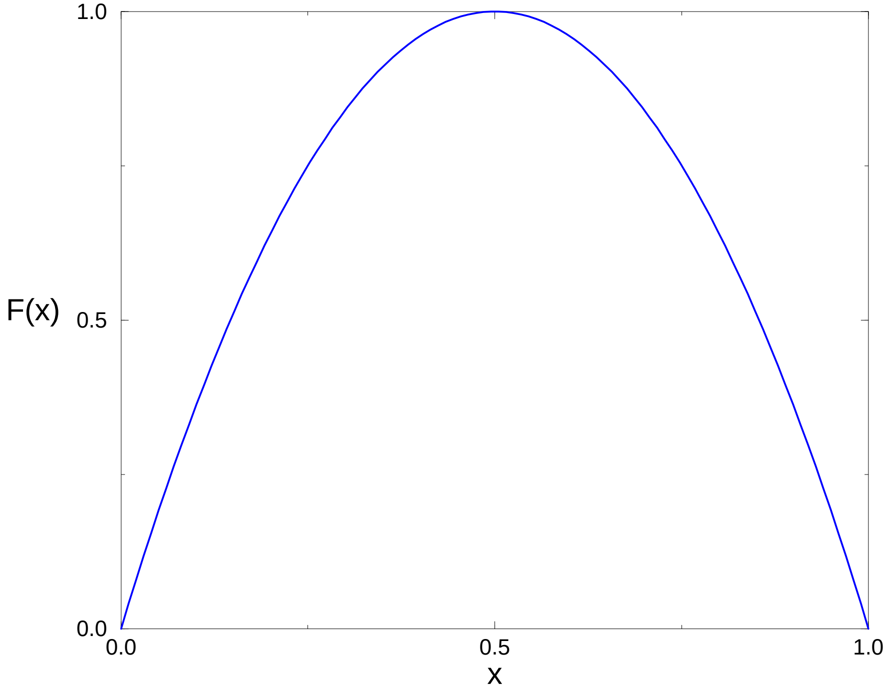
Spatially Extended Systems

Coupled Map Lattices

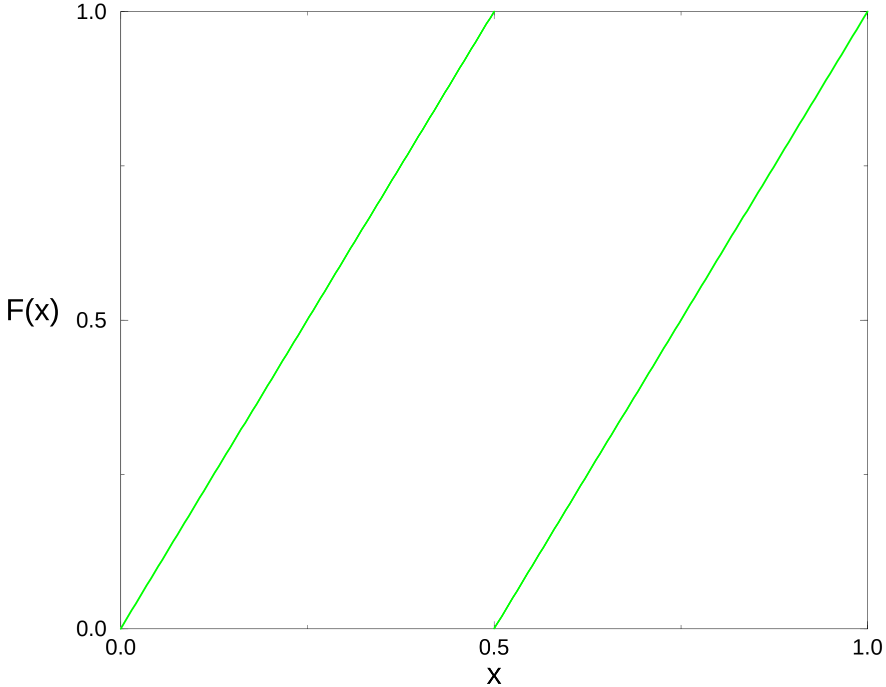
$$u_x^{t+1} = F\left[(1 + \nabla_\varepsilon^2) u_x^t\right] \qquad \nabla_\varepsilon^2 u_x = \varepsilon\{[u_{x+1} + u_{x-1}]/2 - u_x\}$$

where x and t are discrete, F is a chaotic map, typically one dimensional.

Logistic Map F(x)=4x(1−x)



Bernoulli Shift F(x)=Mod(2x,1)



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Mutually Coupled

$$u_x^{t+1} = (1-\sigma)F\left[(1+\nabla_\varepsilon^2)u_x^t\right] + \sigma\cdot F\left[(1+\nabla_\varepsilon^2)w_x^t\right]$$
$$w_x^{t+1} = (1-\sigma)F\left[(1+\nabla_\varepsilon^2)w_x^t\right] + \sigma\cdot F\left[(1+\nabla_\varepsilon^2)u_x^t\right]$$

Stochastic Forcing

$$u_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)u_x^t\right] + \sigma\cdot\zeta_x^t$$
$$w_x^{t+1} = F\left[(1+\nabla_\varepsilon^2)w_x^t\right] + \sigma\cdot\zeta_x^t$$

where the noise is δ-correlated in space and time $<\zeta_x^t\zeta_y^s>\propto\delta_{x,y}\delta_{t,s}$.

The local difference field is defined as $z_x^t = |u_x^t - w_x^t|$.

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Synchronization

For sufficiently strong coupling σ the spatially averaged difference field

$$\rho(t) =< z(t) >= \frac{1}{L}\sum_{x=1}^{L} z_x^t$$

could eventually vanish in the long time limit.

The synchronization transition is no longer fully described in terms of the transverse Lyapunov exponent (TLE).

An extreme nonlinearity in the local map F can induce transport of Finite Size Disturbances even for linearly stable states (i.e. Negative TLE).

A new indicator is required to fully characterize the transition for spatially extended systems.

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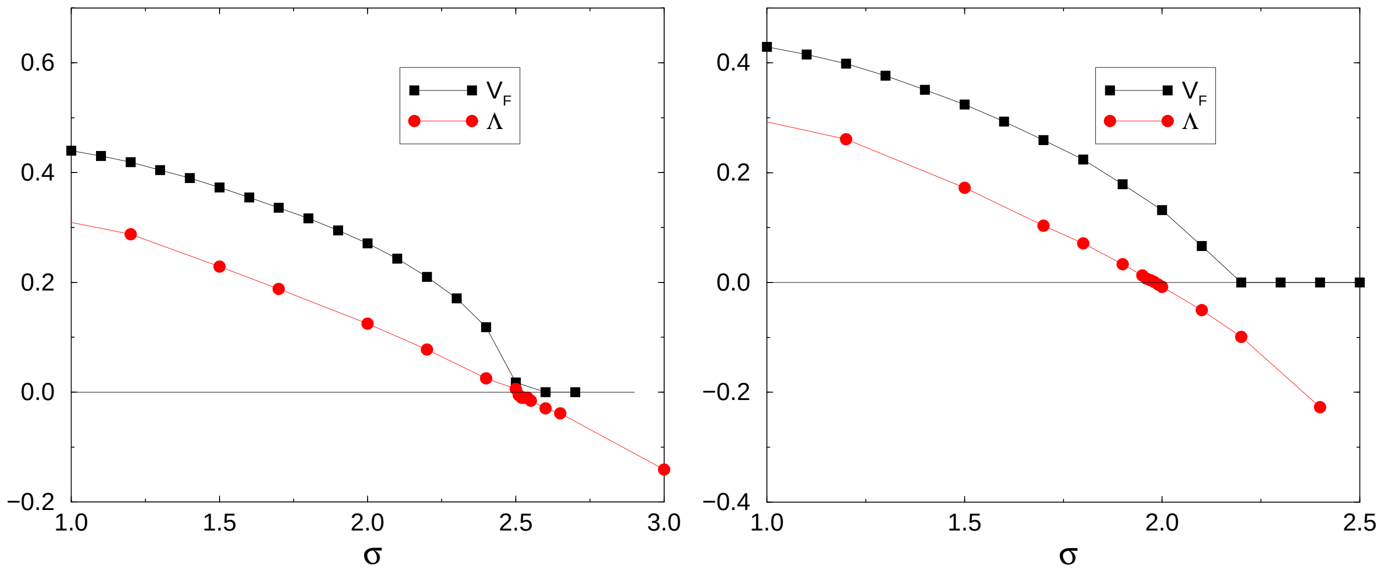
where x and t are discrete, F is a chaotic map, typically one dimensional.

Propagation Velocity of Finite Size Perturbations

A droplet of unsynchronized sites ($N(0)$) is inserted in a completely synchronized state:

$$v_F = \lim_{t\to\infty} \frac{N(t)-N(0)}{2t}$$



Universality Classes

The Synchronization Transition is a Non-Equilibrium Phase Transition leading from an "active phase" ($\rho > 0$) to an "absorbing phase" ($\rho \equiv 0$).

The transition point a_c is located in the thermodynamic limit ($L \to \infty$) by the vanishing of the order parameter $\rho(t) \equiv < z(t) > \to 0$.

A continuous transition is typically characterized by a critical behavior :

$$\rho(t) \propto t^{-\delta} \qquad < \rho >_t \propto |a - a_c|^{\beta}$$

$$\rho(t) = L^{-z\delta} g(t/L^z)$$

$$L_c \propto |a - a_c|^{-\nu_{\|}} \qquad T_c \propto |a - a_c|^{-\nu_{\perp}}$$

only 3 exponents are independent.

Universality Classes

The Synchronization Transition is a Non-Equilibrium Phase Transition leading from an "active phase" ($\rho > 0$) to an "absorbing phase" ($\rho \equiv 0$).

Two different types of transitions have been observed:

- Multiplicative Noise
  - $v_F = \lambda_\perp = 0$
  - Linear Effects rule the Transition
- Directed Percolation
  - $v_F = 0 \quad \lambda_\perp < 0$
  - Strong Nonlinear Effects ($|F'| >> 1$)

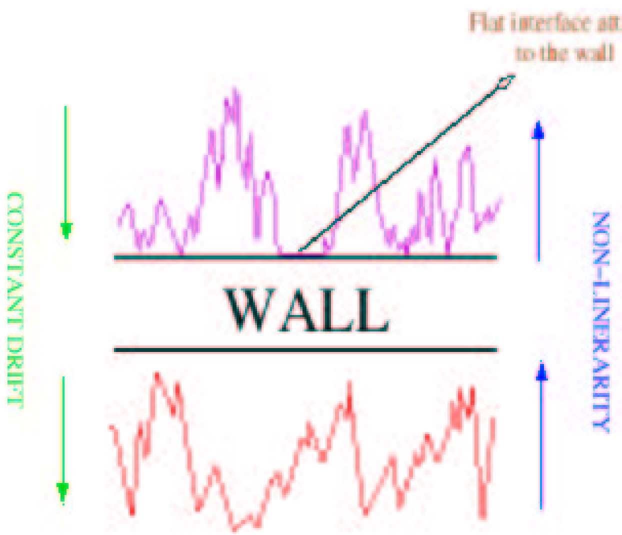
[Baroni, Livi & AT , PRE 63, 036226 (2001); Ahlers & Pikovsky, PRL, 88, 254101 (2002)]

Multiplicative Noise

The corresponding mean-field equation is:

$$\dot{w}(x,t) = \nabla^2 w(x,t) + a w(x,t) - b w^p(x,t) + w(x,t)\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and time and $p \geq 2$. Pikovsky & Kurths (PRE, 49, 898 (1994)) have shown that this model describes the synchronization transition within a linear framework.



This problem can be mapped on that of a depinning of a KPZ interface from a hard substrate through a Hopf-Cole Transformation $h(x,t) = -\ln w(x,t)$. This leads to a KPZ-like equation

$$\dot{h}(x,t) = \nabla^2 h(x,t) - (\nabla h(x,t))^2 - a' - b e^{-(p-1)h(x,t)} + \eta(x,t)$$

The adsorbing state $w = 0$ is now mapped into $h = \infty$

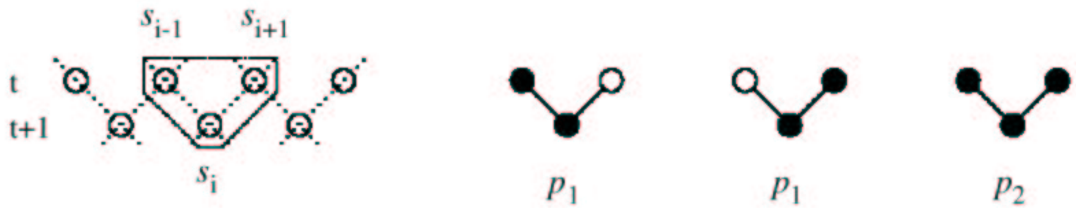
[M.A. Muñoz, cond-mat/0303650 (2003)]

Directed Percolation

The corresponding field equation is:

$$\dot{w}(x,t) = \nabla^2 w(x,t) + aw(x,t) - bw^2(x,t) + \sqrt{w(x,t)}\eta(x,t)$$

where η is a Gaussian noise δ-correlated in space and time.
This equation is usually associated to Infection Spreading Models: the Domany-Kinzel cellular automaton:



black sites are infected (active phase), white sites are healthy (absorbing phase).

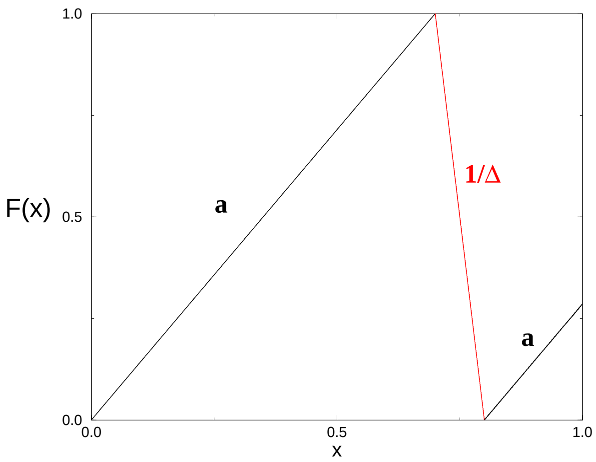
- The infection spreads only by contact
- No revival of infection within healthy region: the absorbing state is stable

[H. Hinrichsen Adv. Phys. 49, 815-958 (2000)]

An experiment on ferrofluids has measured for the first time some of the DP critical exponents [Rupp, Richter, & Rehberg, PRE 67, 036209 (2003)]

Random Multipliers Model

A stochastic model is introduced to mimic the dynamics of the difference field z_x^t for 2 chains of mutually coupled CMLs.



$$z_x^{t+1} = \begin{cases} 1, & \text{w.p.} \quad p = a v_x^t \\ a v_x^t, & \text{w.p.} \quad 1-p \end{cases} \quad , \quad \text{if } v_x^t > \Delta$$

$$z_x^{t+1} = \begin{cases} v_x^t/\Delta, & \text{w.p.} \quad p = a\Delta \\ a v_x^t, & \text{w.p.} \quad 1-p \end{cases} \quad , \quad \text{if } v_x^t \leq \Delta$$

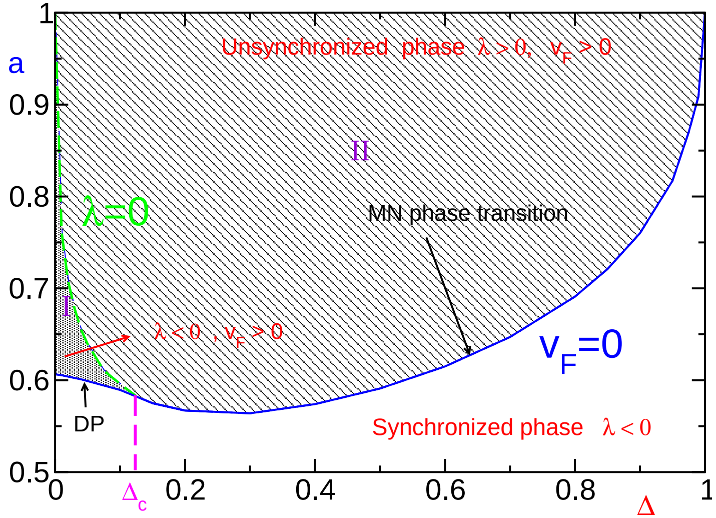
where $v_x^t = (1 + \nabla_\epsilon^2) z_x^t$ and PBC are assumed.

The model is controlled by two parameters a and Δ, for fixed coupling $\varepsilon = 2/3$.
The stochastic nature of the model avoids the emergence of possible long time correlations as in the original deterministic CMLs.

- For small $\Delta < \Delta_c$, the nonlinear mechanisms prevail over the linear ones
- For $\Delta > \Delta_c$, the linear analysis is sufficient to describe the dynamics

Ginelli, Livi, & Politi JPA 35, 499 (2002)

Phase Diagram



- Region II: due to the linear instability any perturbation of the synchronized state will persist forever independently of L
- Region I: a finite perturbation can eventually die in a finite chain, but its life time increases exponentially with L

The critical properties of the model have been studied mainly by analizing $\rho(t)$ (the averaged density of unsynchronized sites). But the definition of ρ requires to fix a threshold W in order to distinguish a synchronized site ($z_x^t < W$) from an unsynchronized one ($z_x^t > W$).

Ginelli,Livi,Politi, & AT PRE 67, 046217 (2003)

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- Therefore, one is obliged to fix an arbitrary threshold W below which trajectories are assumed to be synchronized.
- A priori, one cannot exclude that due to large fluctuations the system will be driven out of the absorbing state, sooner or later.
- The existence of an effective absorbing state will be shown by analyzing the first passage times.

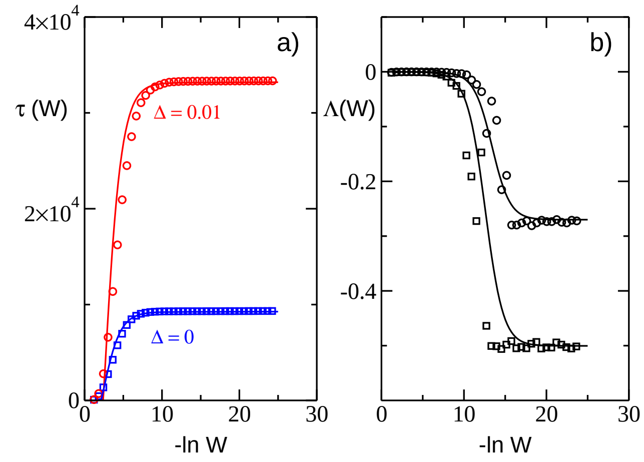
First Passage Times

$\tau(W)$ is the (ensemble) average time needed for $\rho(t)$ to become smaller than a certain threshold for the first time.

By analytical and scaling arguments it can be shown that:

$$\tau(W) = \frac{\ln W}{\lambda_\perp} - L^z g(W L^{\delta z}, W_c)$$

the first term accounts for linear stable behaviour, while the second term for nonlinear effects.



The linear stable behaviour holds below a certain threshold $W_c \propto L^{-z(1+\delta)}$, that vanishes in the thermodynamic limit.

In a finite cellular automaton the minimal meaningful density is $\rho_m = 1/L$, W_c plays the role of ρ_m in continuous systems.

Conclusions

- In spatially extended systems (CMLs) two different synchronization transitions are observed : if the linear behaviour prevails on nonlinear effects the transition belongs to the MN universality class, if nonlinear effects dominate DP scaling laws are observed.
- A stochastic model able to reproduce both the transitions is introduced and studied;
- The analysis of the first passage times has shown that the synchronization transition may indeed belong to the DP universality class for discontinous ($\Delta = 0$) and continuous ($\Delta > 0$) cases.

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Last Developments

- A microscopic model, introduced to study the depinning transition of an interface from an attractive wall, exhibits the same transition scenario; [Ginelli,Ahlers,Livi,Mukamel,Pikovsky,Politi, & AT, cond-mat/0302588];
- Complete synchronization has been analytically studied for CMLs with power law coupling, the results found are only partially correct since nonlinear effects are neglected. [C. Anteneodo et al., nlin.CD/0308014] - [L. Biven & AT, work in progress]