Introduction to Extended Conformal Symmetries and Applications

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Lectures given 10.-18.03.98 at
Universidad Nacional de la Plata, La Plata, Argentina

Abstract

We first illustrate the goal of conformal invariance in two dimensions by reviewing classical systems such as the $o(4)$ symmetry of the hydrogen atom. Then we introduce the Virasoro algebra and discuss its minimal models. By further extending the Virasoro algebra we proceed to $\mathcal{W}$-algebras. The example $\mathcal{W}(2,3)$ is discussed in some detail before we comment on construction methods and some general features. We conclude by describing how quasifinite representations of $\mathcal{W}_{1+\infty}$ arise in fractional quantum Hall systems.

1. Introduction: What are symmetries good for?

These notes intend to provide a pedagogical introduction to the concepts behind extended conformal symmetries and their applications. Issues that would usually be considered trivial are therefore going to be discussed in some detail, while more advanced ones will only be sketched.

To introduce some concepts, we address the following question first: Why do we understand atomic physics? There is certainly a long list of answers, but the fact that the hydrogen atom is a quantum mechanical integrable system is certainly part of it. The integrability of the hydrogen atom can be considered as the reason why it is possible to compute its spectrum and wavefunctions exactly. Integrability can be traced to an enhancement of symmetry: Rotational symmetry $o(3)$ is not sufficient to solve a $3+1$ dimensional system (such as the hydrogen atom), but here symmetry is enhanced to $o(4)$.

Let us briefly sketch how the $o(4)$ symmetry can be used to find the spectrum of the hydrogen atom. Here we essentially follow section 30 of [1] – other accounts include chapter

VI in [2]. The Hamilton operator of the hydrogen atom is given by

$$H = \frac{\vec{p}^{\,2}}{2\mu} - \frac{\kappa}{r}\,, \tag{1.1}$$

where μ is the reduced mass. The angular momentum $\vec{L}$ and Lenz' vector $\vec{M} = \frac{1}{2\mu}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{\kappa}{r}\vec{r}$ are conserved quantities, *i.e.*

$$[\vec{L}, H] = [\vec{M}, H] = 0\,. \tag{1.2}$$

The conserved quantities $\vec{L}$ and $\vec{M}$ form an algebra

$$su(2)\times su(2) \cong so(4) \tag{1.3}$$

(to see this, one has to change the basis) subject to the relations

$$\begin{aligned} \vec{L}\cdot\vec{M} &= \vec{M}\cdot\vec{L} = 0\,,\\ \vec{M}^2 &= \frac{2H}{\mu}(\vec{L}^2 + \hbar^2) + \kappa^2\,. \end{aligned} \tag{1.4}$$

Using these relations, the second order Casimir operator of $so(4)$ reads

$$C_2 = \vec{M}^2 - \frac{2H}{\mu}\vec{L}^2\,. \tag{1.5}$$

Now one has to look for finite-dimensional irreducible representations of $so(4)$ that satisfy the relations (1.4). This leads to the well-known spectrum

$$E_k = -\frac{\mu\kappa^2}{2\hbar^2(2k+1)^2}\,, \tag{1.6}$$

where the main quantum number is given by $n = 2k+1$ and the degeneracy of the level E_k is $2k+1$.

Another example of the fundamental rôle of symmetries in physics is given by the relevance of the representation theory of the Lorentz group [3] for elementary particle physics. The study of the representation theory reinforces the labeling of particles by their mass, spin and (in the massless case) by their helicity. The mathematics involved is lengthy and we refrain from further comments.

Since interacting quantum field theories (QFTs) are notoriously difficult, it is highly desirable in this case to obtain answers of physical importance on the basis of symmetry grounds only. QFTs can be thought of as quantum mechanical systems with infinitely many degrees of freedom. It is therefore clear that symmetries must also be infinite to yield sufficiently strong restrictions. Such cases do indeed exist, though not unexpectedly one has to look for rather "exotic" situations. The class of examples we are aiming at is given by conformally invariant two-dimensional systems.

To illustrate conformal invariance let us stick for the time being to three spatial dimensions and take a look at the phase diagram of such a commonplace object as water (H_2O) – see Fig. 1. Our interest is in the phase transition between the liquid and gaseous state. Generically, this is a first order transition. However, this first order transition line ends in a second order critical point where we find conformal invariance. Experimentally one observes bubbles in droplets in bubbles and this on all scales. More precisely, there is no characteristic scale at this second order critical point, *i.e.* the system becomes scale invariant. Very often, scale invariant systems are automatically conformally invariant. In dimensions greater than two, this means they are also invariant under inversion at the unit sphere in addition to scale invariance and the usual symmetries (rotations and translations).

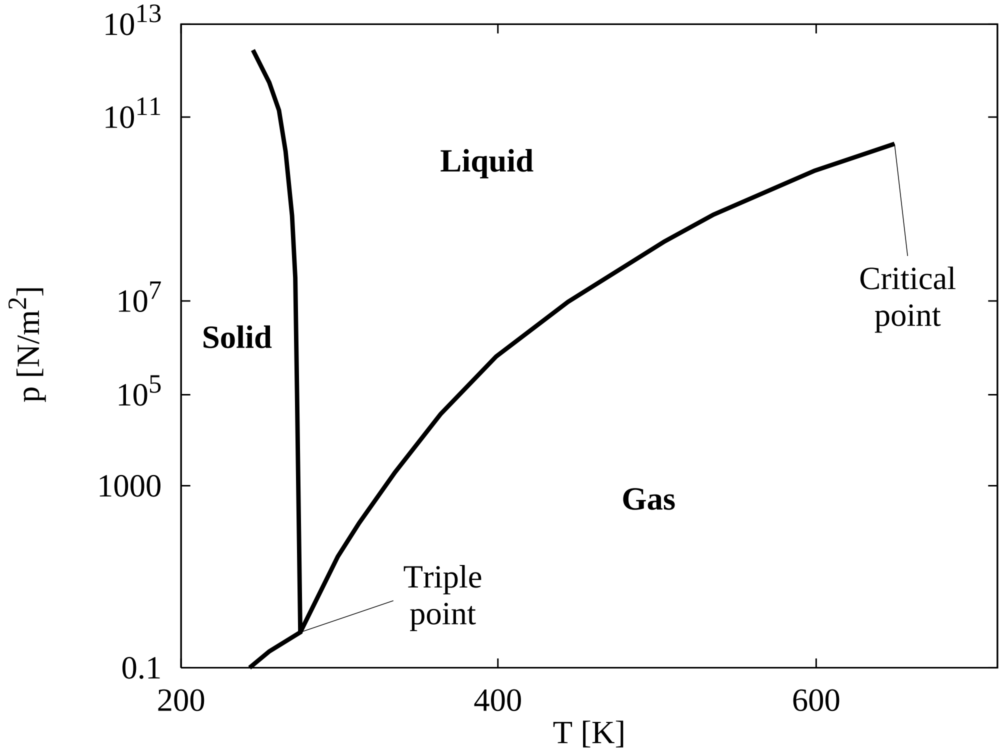

Fig. 1: Phase diagram of water. Adapted from [4].

2. The Virasoro algebra and its minimal models

The program of the previous section will be implemented in this one which will be devoted to the Virasoro algebra. We will briefly summarize some ideas - for more extended reviews with mathematical emphasis see e.g. [5, 6] and for the physical aspects [7].

As a first step, let us look at the conformal group in two dimensions. Two dimensions are special in sofar as one can introduce a single complex coordinate z. Conformal transformations act on $z \in \overline{\mathbb{C}}$ by

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}\,,\ ad-bc=1\,. \tag{2.1}$$

This is a well-defined group of transformations on the complex plane $\mathbb{C}$ compactified with one point at infinity. Its two-fold cover group is $SL(2,\mathbb{C})$. The transformation (2.1) include in particular the special transformations mentioned in the preceding section:

- Translations: $c=0$, $a=d=1$, $b \in \mathbb{C}$.
- Rotations: $b=c=0$, $|a|=|d|=1$.
- Scale transformations: $b=c=0$, $a=d^{-1} \in \mathbb{R}$.
- The inversion: $a=d=0$, $b=-c=1$.

This group is only six-dimensional and does not yet exhibit the infinitely many constraints that we are looking for. Actually, any (anti-)holomorphic transformation is (locally) a conformal transformation (defined in the sense that it preserves angles). However, there are obstructions to promoting them to a group of globally well-defined transformations. Therefore we postpone such an extension and pass to the Lie algebra first.

It is convenient to introduce real lightcone coordinates t on $(1+i)\mathbb{R} \subset \mathbb{C}$ and $\bar{t}$ on $(1-i)\mathbb{R} \subset \mathbb{C}$. Then one introduces generators

$$\ell_n = t^{-n+1}\partial_t \tag{2.2}$$

and analogously $\bar{\ell}_n$. The six generators $\ell_{\pm 1}$, ℓ_0, $\bar{\ell}_{\pm 1}$ and $\bar{\ell}_0$ are seen to generate the Lie algebra corresponding to (2.1). $\ell_1 + \bar{\ell}_1$ generates translations in the imaginary (time) direction and $\ell_1 - \bar{\ell}_1$ in the real (space) direction, while $\ell_0 - \bar{\ell}_0$ generates rotations and $\ell_0 + \bar{\ell}_0$ scale transformations.

This suggests the identification of $\ell_1 + \bar{\ell}_1$ with the Hamiltonian and of $\ell_1 - \bar{\ell}_1$ with the translation operator of a physical system. However, this is not quite correct, since mostly one does actually not work in the plane, but on a cylinder. The latter can be mapped onto the plane by the logarithm, under which (time) translations along the cylinder are mapped onto scale transformations and (spatial) translations along the circumference are mapped to rotations. Therefore, the Hamiltonian is identified with $\ell_0 + \bar{\ell}_0$ and the translation operator with $\ell_0 - \bar{\ell}_0$.

In the sequel we will concentrate on one chiral component (*i.e.* omit the objects with a bar), but it should be kept in mind that a second copy has to be added to describe the full physical system.

Now it is easy to relax the constraint on n in (2.2) and allow $n \in \mathbb{Z}$. Then one finds a closed infinite-dimensional Lie algebra, the so-called 'Witt algebra' with commutation relations

$$[\ell_n, \ell_m] = (n-m)\ell_{m+n}\,. \tag{2.3}$$

This algebra for one chiral component actually has a geometrical meaning, namely as the Lie algebra $\mathrm{diff}(S^1, \mathbb{R})$ of the diffeomorphisms of the circle S^1 into the real numbers.

A further subtlety has to be taken into account, which is that we have to deal with projective representations since we are interested in quantum systems. Recall that if a representation R of a group G leaves quantum mechanical expectation values invariant (*i.e.* $\langle n| R^\dagger(g)AR(g) |m\rangle = \langle n| A |m\rangle$ for all $g \in G$), then so does $\tilde{R}(g) = \exp(i\varphi_g)R(g)$ with a g-dependent phase factor. The representation $\tilde{R}$ is called a 'projective representation' which is a permitted symmetry for a quantum mechanical system. If we pass to the Lie algebra, the derivative of φ_g at the identity gives rise to an additional constant which obviously commutes with the entire Lie algebra. This is the reason why when dealing with Lie algebras one can trade projective representations of the Lie algebra for ordinary representations of a central extension of this Lie algebra.

The Witt algebra (2.3) has a unique central extension, the so-called 'Virasoro algebra'

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{C}{12}(n^3 - n)\delta_{m+n,0}\,, \tag{2.4}$$

where C is a central element ($[L_n, C] = 0$).

Historically, the search for the correct central extension of the Witt algebra was an important issue, and Virasoro himself only had a highest weight condition for string vacua – in particular he did not have the central extension. This central extension may have appeared first in [8] as a two-cocycle during the computation of the entire cohomology ring of the Witt algebra.

Let us now briefly sketch the proof that any central extension of (2.3) is isomorphic to (2.4). We will use a modified version [9] of a proof which can also be found in [5]. A generic central extension $\tilde{L}_n$ of the generators ℓ_n is given by the commutation relations

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{m+n} + a(n,m)\,, \tag{2.5}$$

and the problem is to fix the central elements $a(n,m)$. First we change the basis to $L_0 := \frac{1}{2}[\tilde{L}_1, \tilde{L}_{-1}]$ and $L_n := -\frac{1}{n}[\tilde{L}_0, \tilde{L}_n]$. The new central extension in this basis then satisfies $a(0,n) = a(1,-1) = 0$. The Jacobi-identity $[L_0, [L_n, L_m]] + \text{cycl.} = 0$ yields the condition $(n+m)a(n,m) = 0$ which implies $a(n,m) = b(n)\delta_{m+n,0}$. Now the Jacobi identity $[L_{-1}, [L_{-n+1}, L_n]] + \text{cycl.} = 0$ leads to the recurrence relation $(n-2)b(n) = (n+1)b(n-1)$. Its unique solution is $b(n) = \frac{C}{12}(n^3 - n)\delta_{m+n,0}$, where the prefactor $\frac{1}{12}$ is convention.

The above way to the Virasoro algebra (2.4) may seem a bit lengthy or artificial, but it actually turns out that many second order phase transitions in two-dimensional statistical systems have this symmetry. So, we proceed with the second important issue that was illustrated by the example of the hydrogen atom: The representations of the underlying symmetry.

We define a highest weight representation characterized by numbers c and h by a cyclic highest weight vector $|h\rangle$ that satisfies

$$L_n |h\rangle = 0 \qquad \forall n > 0\,, \tag{2.6a}$$

$$L_0 |h\rangle = h |h\rangle\,, \tag{2.6b}$$

$$C |h\rangle = c |h\rangle\,. \tag{2.6c}$$

By abuse of notation, we will sometimes not distinguish C and c below. It is then meant in the sense (2.6c).

Existence of highest weight representations is ensured since one can easily introduce a special class – so-called 'Verma representations' $M_{c,h}$. Denote the universal enveloping algebra of the Virasoro algebra by $U(\mathrm{Vir})$ and consider the left-ideal $\mathcal{I} \subset U(\mathrm{Vir})$ generated by

$$
\begin{aligned}
&L_n \qquad n > 0\,, && (2.7a)\\
&L_0 - h\mathbb{1}\,, && (2.7b)\\
&C - c\mathbb{1}\,. && (2.7c)
\end{aligned}
$$

Then $M_{c,h} = U(\mathrm{Vir})/\mathcal{I}$. Clearly, (2.7a)–(2.7c) imply (2.6a)–(2.6c), respectively, *i.e.* $M_{c,h}$ is indeed a highest weight representation.

The above construction may be a bit abstract. A more explicit description is given by the fact that the following states are a *basis* for $M_{c,h}$ [1])

$$L_{-n_r} \cdots L_{-n_1} \,|\, h\rangle \qquad 0 < n_1 \leq n_2 \ldots \leq n_r\,. \tag{2.8}$$

Cyclicity of $|\,h\rangle$, the highest weight condition (2.6) and the commutation relations (2.4) ensure that the states (2.8) span any highest weight representation. The special property of $M_{c,h}$ is their linear independence.

As a simple consequence one can easily compute the character of $M_{c,h}$

$$\mathrm{ch}_{c,h} = \mathrm{tr}_{M_{c,h}}\, q^{L_0 - \frac{c}{24}} = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}^{\infty} 1 - q^n}\,. \tag{2.9}$$

The denominator of (2.9) is the generating function for the partitions of an integer n. More precisely, from the (formal) identity $(1-q)^{-1} = \sum_{r=0}^{\infty} q^r$, one sees that the factor $(1-q^n)^{-1}$ in (2.9) counts those states in (2.8) with all $n_i = n$. The infinite product arises when taking the possibilities with different n_i into account.

The Verma representation $M_{c,h}$ is not always irreducible. This is for example the case for $h=0$. Consider $L_{-1}\,|\,0\rangle$ at $h=0$. It is easy to check that

$$
\begin{aligned}
L_n L_{-1}\,|\,0\rangle &= 0 \qquad\qquad \forall n > 0\,, && (2.10a)\\
L_0 L_{-1}\,|\,0\rangle &= L_{-1}\,|\,0\rangle\,, && (2.10b)
\end{aligned}
$$

i.e. $|\,1\rangle = L_{-1}\,|\,0\rangle$ is a highest weight vector spanning its own invariant submodule. For generic c, the quotient $M_{c,0}/M_{c,1}$ is an irreducible module. This quotient $M_{c,0}/M_{c,1}$ can be formally obtained from $M_{c,0}$ by setting $L_{-1}\,|\,0\rangle = 0$. $L_{-1}\,|\,0\rangle$ is also called a 'singular vector' in $M_{c,0}$.

[1]) This follows from the Poincaré-Birkhoff-Witt basis in $U(\mathrm{Vir})$.

Though generically $M_{c,0}/M_{c,1}$ is irreducible, we are actually interested in the exceptional cases where it is not. The Lie algebra representation theory of all subrepresentations of $M_{c,h}$ was worked out in [10, 11]. However, neither the requirement of irreducibility nor of unitarity is sufficient to reach the desired goal of restricting the permitted representations to a finite number for a given c.

The seminal paper [12] finally reached this goal by introducing further structures that arise naturally in a QFT framework. Thus, we now add some concepts that are familiar from QFT. First, we define local 'fields' as power series in a formal variable z

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-d(\phi)}\phi_n \tag{2.11}$$

where $d(\phi)$ denotes the 'conformal dimension' of ϕ. Next, one requires the existence of a so-called 'vertex map' Y which is an *isomorphism* between the irreducible vacuum module and the space of local fields:

$$Y(\phi_{-d(\phi)}\,|0\rangle\,, z) = \phi(z)\,. \tag{2.12}$$

Finally, one implements a non-linear structure via the normal-ordering procedure

$$(\phi\,\psi)_n := \sum_{j\geq d(\phi)} \phi_{-j}\psi_{n+j} + \sum_{j<d(\phi)} \psi_{n+j}\phi_{-j}\,. \tag{2.13}$$

A specific field is the one associated to the Virasoro algebra:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n\,. \tag{2.14}$$

It turns out to be (a component of) the energy-momentum tensor. The conformal dimension of $T(z)$ is two. The derivative ∂ increases the dimension by one, and conformal dimensions are additive under normal ordering (2.13).

The vertex map (2.12) combined with the non-linear normal-ordering structure (2.13) are the key ingredients leading to minimal models. As an example, let us look at the following composite field

$$\Lambda = (T\,T) - \frac{3}{10}\partial^2 T\,. \tag{2.15}$$

This field has conformal dimension four. It vanishes precisely iff $\Lambda_{-4}\,|0\rangle$ is orthogonal to $\Lambda_{-4}\,|0\rangle$ and $L_{-4}\,|0\rangle$ [2]). Here orthogonality is understood with respect to the contravariant form introduced by $L_n^\dagger = L_{-n}$. To see this and anticipating already the consequences, we thus compute

$$\begin{aligned}\langle h|L_4\Lambda_{-4}|h\rangle &= \frac{2(25c+40h+118)h}{5}\,,\\ \langle h|\Lambda_4\Lambda_{-4}|h\rangle &= \frac{16(100h^2+840h+439)h+25c^2+10(10h+11)(10h+1)c}{50}\,.\end{aligned} \tag{2.16}$$

[2]) Since $L_{-1}\,|0\rangle$ vanishes, there are at most two linearly independent states at level four in the vacuum module. These can be chosen to be $\Lambda_{-4}\,|0\rangle$ and $L_{-4}\,|0\rangle$.

Requiring both of them to vanish for $h = 0$, one finds that c must be given by $c = -\frac{22}{5}$ (except for the trivial case $c = 0$). Now we insert $c = -\frac{22}{5}$ into (2.16), equate both expressions to zero and solve for h. We find that these two conditions are satisfied precisely for two values of h: $h = 0$ and $h = -\frac{1}{5}$. This means that if we impose the QFT conditions, only 2 representations of the Virasoro algebra are permitted at $c = -\frac{22}{5}$ [3])!

This is an example of the series of Virasoro-minimal models. Each of them is characterized by a different field that vanishes in the vacuum representation and whose vanishing restricts the allowed representations to a finite set. The Virasoro-minimal models are given by the following discrete sets of central charges and associated highest weights:

$$\begin{aligned} c_{p,q} &= 1 - 6\frac{(p-q)^2}{pq}\,, && p,\ q \text{ coprime} \\ h_{r,s} &= \frac{(rp-sq)^2-(p-q)^2}{4pq}\,, && 1 \leq r \leq q-1,\ 1 \leq s \leq p-1\,. \end{aligned} \tag{2.17}$$

We have already pointed out above that if we require only irreducibility of the representations of the Virasoro algebra as a *Lie* algebra, we would not find a finite set of highest weight representations. Then, even at the $c_{p,q}$ given in (2.17) the condition on r and s in the second line would have to be relaxed in such a way to admit an infinite set. The crucial observation that extra (QFT inspired) conditions restrict the representations to a finite set goes back to the famous paper [12]. This notion is made mathematically precise using the language of vertex operator algebras. It was proven more recently [13] that the only irreducible representations of the Virasoro vertex operator algebra are characterized by (2.17).

The minimal model (2.17) at $c_{p,q}$ is characterized by a null field with conformal dimension $(p-1)(q-1)$ (or a corresponding lowest non-trivial singular vector in the vacuum module). Clearly, the vanishing of (2.15) at $c_{2,5} = -\frac{22}{5}$ is the simplest case. Then there are two cases with different null fields of dimension six ($c_{3,4} = \frac{1}{2}$ and $c_{2,7} = -\frac{68}{7}$), and then the dimension increases further, making a direct approach as above increasingly difficult and finally impossible.

It should be mentioned that the unitary cases among (2.17) are those with $q = p + 1$.

Apart from the Lee-Yang-edge singularity, well-known examples of Virasoro-minimal models include the Ising model with $c_{3,4} = \frac{1}{2}$ and $h \in \{0, \frac{1}{2}, \frac{1}{16}\}$ as well as the three-state Potts model with $c_{5,6} = \frac{4}{5}$ and $h \in \{0, \frac{2}{5}, \frac{7}{5}, 3, \frac{1}{8}, \frac{13}{8}, \frac{1}{40}, \frac{21}{40}, \frac{2}{3}, \frac{1}{15}\}$. These conformal dimensions characterize the long-distance behaviour of correlation functions at criticality – a two-point function at a distance x behaves asymptotically as $x^{-h-\bar{h}}$, where h and $\bar{h}$ correspond to the left and right copy of the Virasoro algebra (in most relevant cases one has $h = \bar{h}$).

The Ising model is abundant in two-dimensional second order phase transition. This can be attributed to its simplicity. In fact, the machinery of conformal field theory is not

[3]) This actually corresponds to a non-unitary statistical model known as the Lee-Yang-edge singularity.

needed in this case, or conversely everything can be realized in terms of a free fermion. The dimension $h = \frac{1}{2}$ just arises from the free fermion and $h = \frac{1}{16}$ is the vacuum energy of a free fermion subject to antiperiodic boundary conditions.

However, other minimal models can be realized as well. For example, if one deposits an N_2 monolayer on the hexagonal graphite surface as shown in Fig. 2, the nitrogen molecules can have three inequivalent orientations. We can thus realize the three-state Potts model. The asymptotic behaviour of the correlation functions at the second order phase transition (*i.e.* if one adjusts pressure and temperature suitably) is then predicted by conformal field theory.

In order to obtain off-critical quantities, one needs a description in terms of a perturbed conformal field theory. For example, a thermal perturbation corresponds to a perturbation by the energy operator ε. The exponent of the specific heat is then given by the scaling relation $\alpha = 2 - 4h_\varepsilon$, where h_ε is the chiral dimension of ε ($\frac{1}{2}$ for the Ising model). So, scaling relations can be used to relate the off-critical exponents to those at criticality.

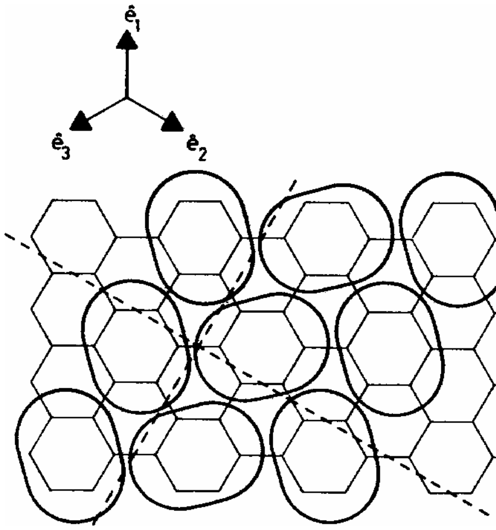

Fig. 2: An N_2 monolayer adsorbed on a graphite surface. Taken from [14].

Let us now return shortly to representation theoretical aspects. Further important quantities are the characters of the irreducible modules $V_{c,h}$. For the minimal models, they can be written in the form (this follows with a few manipulations from [11]):

$$\chi_{r,s} = \mathrm{tr}_{V_{c_{p,q},h_{r,s}}} q^{L_0 - \frac{c}{24}} = \frac{1}{\eta(q)} \left(\Theta_{rp-sq,pq} - \Theta_{rp+sq,pq}\right) . \tag{2.18}$$

Here we have used the Dedekind eta function $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ and the Riemann-Jacobi theta functions $\Theta_{\lambda,k} = \sum_{n\in\mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}}$.

From the characters one constructs the (diagonal) torus partition function for the full theory:

$$Z = \sum_{r,s} \chi_{r,s}(q)\chi_{r,s}(\bar{q}) . \tag{2.19}$$

In this context, q should be expressed via the modular parameter τ of the torus as $q = \mathrm{e}^{2\pi i\tau}$. That this is indeed a partition function can be inferred from the fact that for $q = \bar{q}$ $Z = \mathrm{tr}\, q^{L_0+\bar{L}_0-\frac{c}{12}}$ bears a great resemblance to $Z = \mathrm{tr}\, \mathrm{e}^{-\beta_c H}$. Note however, that τ is related to the geometry of the system, not the temperature.

Recall that partition functions of statistical models can be identified with Euclidean path integrals. So, (2.19) is a closed expression for the Euclidean path integral of a full Virasoro-minimal theory! It is also a closed expression for the scaling limit of the partition function of the corresponding statistical model *at its critical point.* The determination of the partition function on the basis of a lattice model is actually rather cumbersome, and consequently the Z predicted by conformal field theory has only been checked in a few cases (see e.g. [15, 16]).

We conclude this section by introducing the operator product expansion (OPE) in the space of local chiral fields

$$\phi(z) \circ \psi(w) = \sum_{m\in\mathbb{Z}} \frac{\chi^{(m)}(w)}{(z-w)^{d(\phi)+d(\psi)-m}}\,, \tag{2.20}$$

where the l.h.s. is supposed to be radially ordered and analytic for $z \neq w$. The OPE can be regarded as a convenient notation for commutation relations and normal ordered products.

The Virasoro algebra (2.4) is equivalent to the following OPE of the energy-momentum tensor with itself:

$$T(z) \circ T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{(z-w)} + reg. \tag{2.21}$$

Here '$reg.$' stands for the non-singular part of the OPE.

Let us verify that from (2.21) we indeed recover (2.4). This can be done using Cauchy integrals and the deformation of contours of integration indicated in Fig. 3:

$$\begin{aligned}
[L_n, L_m] &= L_n L_m - L_m L_n = \\
&= \left(\oint_0 \frac{\mathrm{d}z}{2\pi i} \oint_0 \frac{\mathrm{d}w}{2\pi i} - \oint_0 \frac{\mathrm{d}w}{2\pi i} \oint_0 \frac{\mathrm{d}z}{2\pi i}\right) z^{n+1} w^{m+1} T(z) \circ T(w) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} \oint_w \frac{\mathrm{d}z}{2\pi i}\; z^{n+1} w^{m+1} T(z) \circ T(w) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} \oint_w \frac{\mathrm{d}z}{2\pi i}\; z^{n+1} w^{m+1} \left(\frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{(z-w)}\right) \\
&= \oint \frac{\mathrm{d}w}{2\pi i}\; \frac{c}{12}(n+1)n(n-1)w^{n-2}w^{m+1} \\
&\qquad + 2(n+1)w^n w^{m+1} T(w) + w^{n+1} w^{m+1} T'(w)\,,
\end{aligned} \tag{2.22}$$

where we have used $\frac{1}{k!}\frac{\partial^k}{\partial z^k} z^{n+1}|_{z=w} = \frac{1}{k!}(n+1)\cdots(n+2-k)w^{n+1-k}$. The last term on the last line of this equality has to be integrated by parts which yields $-(m+n+$

$2)w^{m+n+1}T(w)$. Together with the previous term one obtains $(n-m)w^{m+n+1}T(w)$. After performing finally the w-integration one obtains the desired result (2.4).

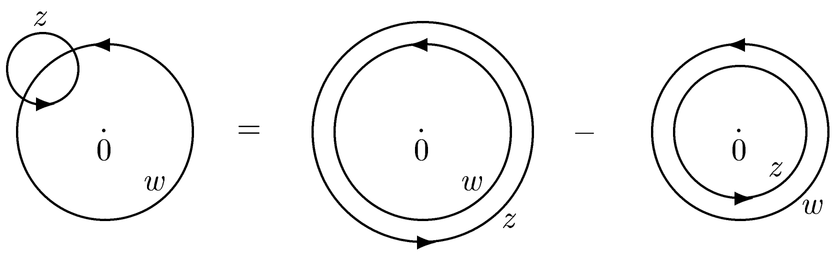

Fig. 3: Deformation of contours of integration for obtaining a mode algebra from the OPE.

The fields appearing as the coefficients of $(z-w)^k$ in the regular part of the OPE (2.20) are normal ordered products. In particular, the constant part in $(z-w)$ of the OPE $\phi(z)\circ\psi(w)$ is the normal ordered product $(\phi\,\psi)(z)$. It can be checked with a similar computation as above that this yields the normal ordering formula (2.13).

To compute with OPEs of normal ordered products, it is sufficient to know that the Wick rule is satisfied. It reads (see e.g. [17]):

$$\underbrace{\phi_1(z)\circ(\phi_2\,\phi_3)(y)} = \frac{1}{2\pi i}\oint_y \frac{\mathrm{d}x}{x-y}\Big\{\big\{\underbrace{\phi_1(z)\circ\phi_2(x)}\big\}\circ\phi_3(y)+\phi_2(x)\circ\big\{\underbrace{\phi_1(z)\circ\phi_3(y)}\big\}\Big\} \tag{2.23}$$

where the underhooks under the OPEs refer to taking only their singular parts.

3. Extended conformal algebras, in particular $\mathcal{W}(2,3)$

It is obvious from (2.17) that all Virasoro-minimal models have central charge $c<1$. However, for applications in string theory one needs $c>1$ (e.g. $c=9$ with $N=2$ supersymmetry). Also statistical models were known corresponding to rational conformal field theories with $c>1$ [18]. This means that just the Virasoro algebra is not always a sufficient symmetry – one needs larger symmetries in order to obtain sufficiently strong constraints also for $c>1$. This brings us to our main subject: Extended conformal algebras, also called '$\mathcal{W}$-algebras' [4]). The presentation to follow will be even more superficial than that of the previous section. It should also be noted that only very few facts about $\mathcal{W}$-algebras have really been proven on a mathematically satisfactory level. Reviews are given e.g. by [17, 19].

First we sketch what a $\mathcal{W}$-algebra is supposed to be, and then discuss an example. The ingredients of a bosonic quantum $\mathcal{W}$-algebra are:

[4]) What the '$\mathcal{W}$' stands for is not clear.

1) A quantum $\mathcal{W}$-algebra is a vector space of local chiral quantum fields, *i.e.* there is a basis of fields $\phi(z)$ depending on a complex coordinate $z \in \overline{\mathbb{C}}$ (including one point at infinity) with definite integer '*conformal dimension*' $d(\phi)$ and mode expansion (2.11).
2) The algebraic structure is encoded in the OPE of two fields $\phi(z)$, $\chi(w)$:

$$\phi(z) \circ \chi(w) = \sum_{\psi} \frac{A^{\psi}_{\phi\chi} \psi(w)}{(z-w)^{d(\phi)+d(\chi)-d(\psi)}}, \tag{3.1}$$

where the sum on the r.h.s. of (3.1) runs over all fields in the $\mathcal{W}$-algebra. The $A^{\psi}_{\phi\chi}$ are the structure constants of the $\mathcal{W}$-algebra.
3) The $\mathcal{W}$-algebra contains a distinguished field, the energy-momentum tensor T with OPE (2.21).
4) The energy-momentum tensor T defines a grading on the $\mathcal{W}$-algebra:

$$T(z) \circ \phi(w) = \frac{d(\phi)\phi(w)}{(z-w)^2} + \sum_{\substack{\psi \\ d(\psi) \neq d(\phi)}} \frac{A^{\psi}_{T\phi} \psi(w)}{(z-w)^{2+d(\phi)-d(\psi)}}, \tag{3.2}$$

where $d(\phi)$ is the conformal dimension of ϕ.
5) Fields transforming as

$$T(z) \circ \phi(w) = \frac{d(\phi)\phi(w)}{(z-w)^2} + \frac{\phi'(w)}{(z-w)} + \sum_{\substack{\psi \\ |d(\psi)-d(\phi)|>1}} \frac{A^{\psi}_{T\phi} \psi(w)}{(z-w)^{2+d(\phi)-d(\psi)}} \tag{3.3}$$

are called '*quasi-primary*'. The field ϕ is called '*primary*' if the remaining sum runs only over fields ψ with $d(\psi) > d(\phi) + 1$. Note that the energy-momentum tensor T itself is quasi-primary with conformal dimension $d(T) = 2$.
6) A basis for all fields in the $\mathcal{W}$-algebra can be obtained from a countable basic set of quasi-primary fields $\{W^{(i)}\}$ using normal ordered products and derivatives. The $W^{(i)}$ are called '*generators*'. If the set of generators is minimal, the $\mathcal{W}$-algebra is called a '$\mathcal{W}(d(W^{(1)}), d(W^{(2)}), \ldots)$'.
7) There is an irreducible representation of the mode algebra, the so-called '*vacuum representation*', *i.e.* the modes ϕ_n of any field $\phi(z)$ act as endomorphisms in the vacuum representation. The vacuum representation has a cyclic highest weight vector $|0\rangle$ with the property [5]

$$\phi_n \,|0\rangle = 0 , \qquad \forall n > -d(\phi) . \tag{3.4}$$

A normal ordered differential polynomial that acts trivially in the vacuum representation is identically equal to zero. Fields with this property are called '*null fields*'.

[5]) It would be sufficient to impose the highest weight condition $\phi_n \,|0\rangle = 0$ for all $n > 0$. The condition given here then follows from $L_{-1} \,|0\rangle = 0$ together with covariance under L_{-1}.

Property 7) gives a precise meaning to the fields $\phi(z)$. The definition of a $\mathcal{W}$-algebra can be made precise using the mathematical language of vertex operator algebras. Then, property 7) is the starting point of the definition and properties 1) - 6) turn out to be reformulations of the axioms of a vertex operator algebra. In general, one will have to impose additional finiteness conditions like requiring the graded components of the vacuum module to be finite dimensional.

One of the first and most famous examples for $\mathcal{W}$-algebras is Zamolodschikov's $\mathcal{W}(2,3)$ [20] (see also [21] for a review). It is the extension of the algebra associated to the field $T(z)$ by an additional field $W(z)$ which has conformal dimension three. The OPEs of the fundamental fields $T(z)$ and $W(z)$ are

$$\begin{aligned}
T(z)\circ T(w) =& \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{(z-w)} + reg. \\
T(z)\circ W(w) =& \frac{3W(w)}{(z-w)^2} + \frac{W'(w)}{(z-w)} + reg. \\
W(z)\circ W(w) =& \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{T'(w)}{(z-w)^3} \\
&+ \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}T''(w)\right) \\
&+ \frac{1}{(z-w)}\left(\frac{16}{5c+22}\Lambda'(w) + \frac{1}{15}T'''(w)\right) + reg.
\end{aligned} \tag{3.5}$$

with the quasi-primary field Λ introduced in (2.15). The first line of (3.5) is just the OPE form (2.21) of the Virasoro algebra, the second line simply states that the field $W(z)$ is a primary field of conformal dimension three. The important part of (3.5) is the third OPE with the non-linear term Λ. OPEs for composite fields can be derived from the basic one (3.5) via the Wick rule (2.23). In particular, the OPE of Λ with itself and the generators will yield further fields, and iteration of this procedure will lead to fields of arbitrarily high dimension. It is therefore crucial to keep the non-linear structure in mind and not very useful to interpret $\mathcal{W}(2,3)$ as a Lie algebra.

In order to discuss highest weight representations of $\mathcal{W}(2,3)$ we should first pass to modes. After a mode expansion of (3.5), we find the following commutation relations for $\mathcal{W}(2,3)$:

$$\begin{aligned}
[L_n, L_m] =& (n-m)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\,, \\
[L_n, W_m] =& (2n-m)W_{m+n}\,, \\
[W_n, W_m] =& \frac{n-m}{30}(2m^2 - mn + 2n^2 - 8)L_{m+n} \\
&+ \frac{16}{5c+22}(n-m)\Lambda_{m+n} + \frac{c}{3}\binom{n+2}{5}\delta_{n+m,0}\,.
\end{aligned} \tag{3.6}$$

In complete analogy to (2.6), we now introduce highest weight representations of $\mathcal{W}(2,3)$ via a cyclic vector $|h,w\rangle$ subject to

$$L_n\,|h,w\rangle = 0\,, \qquad W_n\,|h,w\rangle = 0 \qquad \forall n>0\,, \tag{3.7a}$$

$$L_0\,|h,w\rangle = h\,|h,w\rangle\,, \qquad W_0\,|h,w\rangle = w\,|h,w\rangle\,. \tag{3.7b}$$

A careful analysis of such highest weight representations is even more difficult than in the case of the Virasoro algebra. So, mostly one proceeds by constructing representations for $\mathcal{W}(2,3)$ in terms of those of an underlying Kac-Moody algebra or using a free-field construction of the algebra. This was done first for the unitary minimal series (see e.g. [22, 23, 19]) and then completed with the non-unitary minimal models [24]. Very often, the minimal models are described in terms of the weights of the associated Lie algebra ($su(3)$ in the case of $\mathcal{W}(2,3)$). Here we give a more explicit description of the minimal series of $\mathcal{W}(2,3)$:

$$\begin{aligned} c_{p,q} &= 2\left(1 - 12\frac{(p-q)^2}{pq}\right) , \\ h_{p,q;r_1,s_1,r_2,s_2} &= \frac{3(q(r_1+r_2)-p(s_1+s_2))^2 + (q(r_1-r_2)-p(s_1-s_2))^2}{12pq} + \frac{c_{p,q}-2}{24} . \end{aligned} \tag{3.8}$$

where $q, p > 2$ are two coprime integers. The positive integers r_i and s_i are subject to the conditions $r_1 + r_2 < p$ and $s_1 + s_2 < q$. The precise formula (3.8) is taken from [25] where it was inferred from the unitary case $q = p+1$ and a null-field analysis of some non-unitary minimal models.

From (3.8) we find that $c_{4,5} = \frac{4}{5}$ which we have already encountered as the Virasoro-minimal model corresponding to the three-state Potts model. This is no accident, as can be seen from that fact that 3 appears in the list of Virasoro-minimal weights. This indicates that at $c = \frac{4}{5}$ there is a field of dimension 3 which one can either include in the symmetry algebra yielding a $\mathcal{W}(2,3)$ or regard as a representation in the Virasoro-minimal model.

Constructions of $\mathcal{W}$-algebras in terms of simpler (linear) systems play an important rôle in their analysis. Very often such constructions are based on '*current algebras*' which are generated by currents $J_i(z)$ (fields with dimension $d(J_i) = 1$) with OPEs

$$J_i(z) \circ J_j(w) = \frac{g_{ij}\,k}{(z-w)^2} + \sum_l \frac{f_{ij}^l\,J_l(w)}{(z-w)} + reg. \tag{3.9}$$

where g_{ij} is an invariant metric of some Lie algebra g and the f_{ij}^l are the structure constants of this Lie algebra. The current algebra defined by (3.9) is denoted by '$\widehat{g}_k$' and k is called the '*level*'. Since the OPE of the basic currents J_i is linear, it actually defines an infinite dimensional Lie algebra (analogously to the Virasoro algebra). This not only simplifies the study of their representations, but Kac and Moody first introduced such algebras as infinite-dimensional Lie algebras (see e.g. chapter 2 of [26] for a review). Integer levels ($k > -h^\vee$) give rise to '*integrable representations*' which play a special rôle.

From a QFT point of view, the central term in (3.9) is the well-known (anomalous) Schwinger term.

When we take the non-linear structure into account (as we should in order to construct $\mathcal{W}$-algebras), the Sugawara construction provides us with a realization of the Virasoro algebra

$$T_g = \frac{1}{k+h^\vee}\sum_{i,j} g^{ij}(J_i\,J_j)\,, \tag{3.10}$$

where $h^\vee$ is the dual Coxeter number of g. It can be checked straightforwardly using essentially (2.23) that T_g satisfies the OPE (2.21) with

$$c_g = \frac{k \dim g}{k + h^\vee} \,. \tag{3.11}$$

For a non-trivial subalgebra $h \subset g$ we can now define the coset

$$\frac{\widehat{g}_k}{\widehat{h}_{ik}} \tag{3.12}$$

('i' is the Dynkin index of the embedding of h into g) as those normal ordered polynomials in $\widehat{g}_k$ that commute with $\widehat{h}_{ik}$. A first example of a field in the coset algebra is the coset energy-momentum tensor

$$T_{g/h} = T_g - T_h \,, \tag{3.13}$$

which satisfies the Virasoro algebra with

$$c_{g/h} = c_g - c_h \,. \tag{3.14}$$

However, in general the coset algebra is an extension of the Virasoro algebra. A series of important special cases is given by the cosets [27, 28]

$$\mathcal{WA}_{N-1} = \frac{\widehat{su(N)}_k \oplus \widehat{su(N)}_1}{\widehat{su(N)}_{k+1}} \,. \tag{3.15}$$

Since it is useful for understanding the structure of the coset algebra and since we will also refer back to it later, we note that one can argue on general grounds [29] that

$$\lim_{k\to\infty} \mathcal{WA}_{N-1} = \frac{\widehat{su(N)}_1}{su(N)} \,, \tag{3.16}$$

i.e. in this limit the coset-$\mathcal{W}$-algebra is described as the classical $su(N)$ invariants in $\widehat{su(N)}_1$.

For $N = 2$ we recover from (3.15) the famous GKO construction of the unitary minimal models of the Virasoro algebra [30]. For $su(2)$ we have $\dim g = 3$, $h^\vee = 2$ and then it is straightforward to recover the central charge formula (2.17) with $p = k+2$, $q = k+3$ from the coset construction.

The algebra $\mathcal{W}(2,3)$ is also recovered from (3.15) with $N = 3$ (for $su(3)$ one has $\dim g = 8$ and $h^\vee = 3$). Then not only the energy-momentum tensor T appears in the coset algebra, but also a third order polynomial W in the currents whose leading term is given by the completely antisymmetric tensor.

The coset construction is a useful tool also for the study of representations [28], since the decomposition of the representations of $\widehat{g}_k$ into representations of $\widehat{h}_{ik}$ gives rise to representations of the coset (3.12).

Another very important construction is the so-called '*quantized Drinfeld-Sokolov*' reduction (see e.g. [31, 32]). This reduction is a Hamiltonian reduction that also gives a precise meaning to the 'quantized Miura map' of the Fateev-Luk'yanov construction (see e.g. [19, 23]). We do not enter into the details of this construction, we just wish to mention that it provides another powerful tool for the construction of representations of $\mathcal{W}$-algebras [24].

The type of a $\mathcal{W}$-algebra given in terms of an explicit construction can be controlled quite well for *generic* c. For example, one can argue using e.g. (3.16) that $\mathcal{WA}_{N-1} \cong \mathcal{W}(2, 3, \ldots, N)$ if c is not special. However, at special values of c one can have *truncations* (see e.g. [33]), *i.e.* some generators drop out due to the appearance of null fields. Now one can try to study $\mathcal{W}$-algebras not in terms of constructions, but by solving some general consistency conditions (see e.g. [20, 34, 35]). Then one finds that $\mathcal{W}$-algebras with a prescribed field content can exist for generic c or for some values of c only. Examples for either case were known from the very beginning: $\mathcal{W}(2, 3)$ exists for generic c while an algebra of type $\mathcal{W}(2, \frac{5}{2})$ exists only for $c = -\frac{13}{14}$ [20]. While for all presently known algebras existing for generic c also a construction is known, the non-generic cases seem to be related to truncations (or also extensions) of generically existing algebras. For example, the $\mathcal{W}(2, \frac{5}{2})$ at $c = -\frac{13}{14}$ can be either understood as a truncation of a $\mathcal{W}(2, \frac{5}{2}, 4)$ existing for generic c (see e.g. [36]) or as an extension of the Virasoro algebra arising in the $c_{4,7}$ minimal model [35].

The two constructions mentioned above can be combined with each other and also further with reductions by classical (group) symmetries (see e.g. [29, 33, 37, 38] for examples how such an interplay gives rise to interesting structures in the space of $\mathcal{W}$-algebras).

4. Quasifinite representations of $\mathcal{W}_{1+\infty}$ and the FQHE

In this section we sketch how quasifinite representations of $\mathcal{W}_{1+\infty}$ [39] can be used to describe the fractional quantum Hall effect (FQHE) [40, 41]. Our presentation essentially follows the review [42].

The goal is to see that quantum Hall states are incompressible quantum fluids which means that their symmetry is $\mathcal{W}_{1+\infty}$. As a first step, let us briefly review the symmetry of a classical fluid w_∞ (on the classical level there is basically no difference between w_∞ and $w_{1+\infty}$). A classical two-dimensional fluid is described by a distribution function depending on the coordinates z, $\bar{z}$ of the plane. Incompressibility of this fluid means invariance under area-preserving diffeomorphisms. According to the Liouville theorem such transformations are precisely the canonical transformations. Let us therefore introduce Poisson brackets such that $\{z, \bar{z}\} = \frac{i}{\rho}$ with ρ the density of the fluid. Then a basis of the generators of of area-preserving diffeomorphisms is given by $\mathcal{L}_{n,m} = \rho^{(n+m)/2} z^n \bar{z}^m$. These generators satisfy the Poisson brackets

$$\{\mathcal{L}_{n,m}, \mathcal{L}_{k,l}\} = -i(mk - nl)\mathcal{L}_{n+k-1,m+l-1} \, . \tag{4.1}$$

Depending on the permitted indices (we omit the restrictions here), these Poisson brackets define w_∞ or $w_{1+\infty}$.

The algebras w_∞ and $w_{1+\infty}$ have different quantizations $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$. Here we are interested in the latter. We perform a change of basis $\mathcal{L}_{j-n,j} \to V_n^j$ where the V^j are fields of conformal dimension $j+1 \geq 1$. Schematically, their commutation relations are given by

$$[V_n^i, V_m^j] = (jn - im)V_{n+m}^{i+j-1} + q(i,j,m,n)V_{n+m}^{i+j-3} + \ldots + c^i(n)\delta^{i,j}\delta_{n+m,0}\,. \tag{4.2}$$

In comparison to (4.1) we observe a central extension $c^i(n)$ and also generators of lower conformal dimension. The origin of the latter are normal-ordering corrections which typically arise in the quantization of a classical $\mathcal{W}$-algebra. The full commutation relations can be found e.g. in [43, 44], but they are not essential for our discussion and therefore we omit them here. A crucial feature should be noted nevertheless, namely that the commutation relations (4.2) are linear in the generators V^i which permits one to regard $\mathcal{W}_{1+\infty}$ as a Lie algebra.

The name '$\mathcal{W}_{1+\infty}$' is due to the infinitely many generators which have conformal dimensions 1 and greater. The commutation relations of the first two generators read explicitly

$$[V_n^0, V_m^0] = cn\delta_{n+m,0}\,, \tag{4.3a}$$
$$[V_n^1, V_m^0] = -mV_{n+m}^0\,, \tag{4.3b}$$
$$[V_n^1, V_m^1] = (n-m)V_{n+m}^1 + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\,. \tag{4.3c}$$

From (4.3a) we see that V^0 satisfies the algebra $\widehat{U(1)}$ of an Abelian current. Eq. (4.3c) is nothing else but the Virasoro algebra (2.4) implying that V^1 can be identified with the energy-momentum tensor. Finally, (4.3b) means that V^0 is a primary field of dimension one.

There is an alternate way to arrive at $\mathcal{W}_{1+\infty}$ which resembles the step from the Witt algebra to the Virasoro algebra described above. E.g. the papers [39, 45] are based on this second approach. One starts with the Lie algebra diff$(S^1, \mathbb{C})$ of the diffeomorphisms of the circle into the complex numbers. Upon choosing a suitable basis, the commutation relations of the generators already contain all linear terms of the r.h.s. of (4.2). Then, one just has to add a central extension (which as in the case of the Witt algebra is again unique).

The next step is to analyze the highest weight representations of $\mathcal{W}_{1+\infty}$. The so-called '*quasifinite*' representations [39, 46] are a very important subclass which are also the ones relevant for us. Quasifiniteness is defined by all spaces with a given conformal dimension ($L_0 = V_0^1$ grade) being finite dimensional. Since the algebra has infinitely many generators, there will typically be infinitely many states with a given energy in an irreducible highest weight representation. The requirement of quasifiniteness is therefore a very strong restriction. Such representations do indeed exist, e.g. the vacuum representation is always quasifinite, as one can easily see from the condition (3.4). If one further requires unitarity, the central charge is constrained to be a positive integer $c = N \in \mathbb{N}_+$ [39].

The appearance of unitary quasifinite representation at $c = N$ is closely related to a truncation of the $\mathcal{W}$-algebra $\mathcal{W}_{1+\infty}$ at this value of the central charge: Here it is actually of type

$\mathcal{W}(1, 2, \ldots, N)$ [45], *i.e.* as a $\mathcal{W}$-algebra it has only a *finite* number of generators. There is however a price to pay for this simplification, namely the commutation relations of the truncated algebra are not linear anymore. This truncation clearly ensures quasifiniteness, the interesting point is that quasifiniteness and unitarity require a truncation.

The truncated algebra has several interesting realizations. The result of [45] was actually obtained by identifying $\mathcal{W}_{1+\infty}$ at $c = N$ with $\widehat{U(1)} \oplus \mathcal{WA}_{N-1}$. Together with (3.16) we therefore have

$$\mathcal{W}_{1+\infty} \cong \widehat{U(1)} \oplus \frac{\widehat{su(N)}_1}{su(N)} \tag{4.4}$$

at $c = N$. $\mathcal{W}_{1+\infty}$ at $c = N$ can also be realized in terms of the $U(N)$-invariants of N complex free fermions [45, 47, 48]. This second realization is particularly suitable for understanding why the commutation relations can be written in a linear form (see also [49] for further related comments).

$\mathcal{W}_{1+\infty}$ has no truly minimal models anymore, but quasifinite representation give rise to natural generalizations of quasirational conformal field theories which albeit not finite contain a countable number of highest weight representations with certain nice properties. At $c = N$, the representations in such quasirational models are labeled by an N-dimensional lattice. The simplest case is $c = 1$ where $\mathcal{W}_{1+\infty}$ boils down to the Abelian $\widehat{U(1)}$ algebra. The set of highest weight representations is then a one-dimensional space labeled by the charge.

To apply the preceding discussion to the FQHE, we have to argue that FQHE states are incompressible quantum fluids. Quite generally, incompressibility is accompanied by a spectral gap. However, here a much more precise statement exists which is based on the Laughlin wavefunctions [50]. For example, for filling fractions $\nu = (2p + 1)^{-1}$, the FQHE groundstates are well described by the following many-electron wavefunctions

$$\psi_p = \prod_{i<j} (z_i - z_j)^{2p+1} \mathrm{e}^{-\frac{1}{2}\sum |z_i|^2} \, . \tag{4.5}$$

Using a realization of $\mathcal{W}_{1+\infty}$ at $c = 1$ in terms of a free boson, one can show that these states are highest weight states of $\mathcal{W}_{1+\infty}$ (see e.g. [51, 52]). This highest weight property is the precise algebraic formulation of FQHE states being incompressible quantum fluids.

This can be generalized to other filling fractions using a generalization of the wavefunctions (4.5) (see [52] and references therein)

$$\psi_K = \prod_{I<J} \prod_{i \leq j} (z_i^I - z_j^J)^{K_{I,J}} \prod_{I} \prod_{i<j} (z_i^I - z_j^I)^{K_{I,I}} \mathrm{e}^{-\frac{1}{2}\sum_{i,I} |z_i^I|^2} \, . \tag{4.6}$$

Here K is a symmetric $N \times N$ matrix with integer entries. Then the filling fraction is given by

$$\nu = \sum_{I,J} (K^{-1})_{I,J} \, . \tag{4.7}$$

Using now a realization of $\mathcal{W}_{1+\infty}$ in terms of N free fields, one can again show that the states (4.7) are highest weight states [52]. For matrices $K_{I,J} = 2p \pm \delta_{I,J}$ one finds from (4.7) the filling fractions

$$\nu = \frac{N}{2pN \pm 1}\,. \tag{4.8}$$

So, finally the filling fraction is related to the central charge $c = N$ of $\mathcal{W}_{1+\infty}$. However, it is a bit unsatisfactory that this result involves a special auxiliary $N \times N$ matrix K, *i.e.* that quantities of physical relevance (such as the filling fraction) are not given in terms of representation theoretic data only. The rôle of $K_{I,J}$ is to provide a metric on the lattice of representations which is needed to identify the physical charges [40].

5. Conclusion

It is fair to mention that the study of extended conformal symmetries in their own right is not a mainstream program of research in theoretical physics anymore. This is now being taken over by mathematicians under the name of 'vertex operator algebras', which is probably more natural anyway. Nevertheless, I hope to have illustrated that these concepts were not only initially developed (at least partially) by physicists, but can have important applications in (mathematical) physics, and thus it may be useful to be familiar with them.

Acknowledgments

Financial support by Fundación Antorchas is gratefully acknowledged for the stay during which these lectures were given. My own experience with $\mathcal{W}$-algebras is essentially due to work carried out in the research group of W. Nahm at the Physikalisches Institut der Universität Bonn, Germany, all of whom I would like to warmly thank for the shared experience.

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