Correlation Functions of $\mathbb{Z}_n$-Quantum Spin Chains

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Abstract

We discuss numerical and perturbative methods to study correlation functions in massive phases of $\mathbb{Z}_n$-quantum spin chains. Explicit computations are performed for the $\mathbb{Z}_3$-chiral Potts model - a model that violates parity. We find that also the massive phases are oscillating and empirically observe a relation between the oscillation length and the parity breaking asymmetry.

Finally, we present a form factor approach which allows us in some special cases to prove the validity of the empirical relation.

1. The $\mathbb{Z}_n$-chiral Potts model

The $\mathbb{Z}_3$-chiral Potts spin quantum chain was introduced in [1] in almost the same form as we are going to use. Soon afterwards, this chain and its generalizations to higher $\mathbb{Z}_n$ were shown to be integrable because the Hamiltonian satisfies Onsager's algebra [2] (a property nowadays referred to as 'superintegrability'). Later on, integrability was established for even more general values of the parameters by showing that the associated 2D spin model satisfies a generalized Yang-Baxter equation (see e.g. [3] and references therein).

The $\mathbb{Z}_n$-chiral Potts spin quantum chain with N sites is defined by the Hamiltonian:

$$H_N^{(n)} = -\sum_{j=1}^{N}\sum_{k=1}^{n-1} \frac{1}{\sin\frac{\pi k}{n}} \left\{ e^{i\varphi(\frac{2k}{n}-1)}\sigma_j^k + \lambda e^{i\phi(\frac{2k}{n}-1)}\Gamma_j^k\Gamma_{j+1}^{n-k} \right\}. \tag{1.1}$$

For $n=2$ eq. (1.1) defines the well-known Ising model. In this case the operators σ_j and Γ_j are the Pauli spin matrices σ_z and σ_x acting in a vector space $\mathbb{C}^2$ located at site j. For $n=3$ we have

$$\begin{aligned}
\sigma_j &= \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{j-1\text{ times}} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \otimes \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{N-j\text{ times}}, \\
\Gamma_j &= \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{j-1\text{ times}} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \underbrace{\mathbb{1}\otimes\ldots\otimes\mathbb{1}}_{N-j\text{ times}}.
\end{aligned} \tag{1.2}$$

We assume periodic boundary conditions, i.e. $\Gamma_{N+1} = \Gamma_1$. The Hamiltonian (1.1) commutes with the translation operator and the charge operator $\prod_{j=1}^{N}\sigma_j$. Their eigenvalues have the form e^{iP} respective $e^{\frac{2\pi i Q}{n}}$ leading to conservation of momentum $0 \le P < 2\pi$ and charge $Q = 0, 1, \ldots, n-1$.

The Hamiltonian (1.1) is 'superintegrable' for $\phi = \varphi = \frac{\pi}{2}$, self-dual for $\phi = \varphi$ and integrable for $\cos\varphi = \lambda\cos\phi$.

Below, we will study the *massive high-temperature phase* of the chiral Potts quantum chain. For the $\mathbb{Z}_3$-chain it occurs for $-\pi < \varphi < \pi$. It extends to $\lambda = 1, 0.901292\ldots, 0$ for $\phi = \varphi = 0$, $\frac{\pi}{2}$, π respectively [3 − 5].

The results in $[6-9]$ indicate that the spectrum of the general $\mathbb{Z}_n$-Hamiltonian (1.1) can be explained in terms of quasiparticles in the complete massive high-temperature: all excitations $\Delta E_{Q,r}(P,\phi,\varphi)$ above the groundstate can be composed in the large chain limit of $n-1$ fundamental excitations $\mathsf{E}_1(P),\ldots,\mathsf{E}_{n-1}(P)$ according to the following rules:

$$\Delta E_{Q,r}(P,\phi,\varphi)=\sum_{k=1}^{m_r}\mathsf{E}_{Q^{(k)}}(P^{(k)})\,,\quad P=\sum_{k=1}^{m_r}P^{(k)}\bmod 2\pi\,,\quad Q=\sum_{k=1}^{m_r}Q^{(k)}\bmod n. \quad (1.3)$$

Additionally, the fundamental quasiparticles satisfy a Pauli principle, i.e. $Q^{(i)}=Q^{(j)}$ implies $P^{(i)}\neq P^{(j)}$. The dispersion relations of the fundamental particles violate parity and therefore their minimum is in general not located at zero momentum.

2. Perturbative computation of the correlation functions

In this section we study a correlation function for the $\mathbb{Z}_3$-chiral Potts model perturbatively. For the massive *low-temperature* phase a non-vanishing wave vector has been predicted in [5] where also its critical exponent was calculated from level crossings. Perturbative calculations of the correlation function $C_\Gamma(x)$ for the massive *low-temperature phase* have been performed in [10] and direct indications for an oscillatory contribution have been obtained.

The correlation function for the operator Γ is defined by the following expression:

$$C_\Gamma(x):=\langle v|\ \Gamma_x^+\Gamma_0\ |v\rangle \qquad 0\leq x<\frac{N}{2} \quad (2.1)$$

where $|v\rangle$ is the normalized eigenvector of the Hamiltonian to lowest energy.

In order to be able to calculate the correlation functions we need to know the groundstate $|v\rangle$. We will calculate it from the free groundstate using a Wigner-Brilloiun perturbation. The expansion of the groundstate $|v\rangle$ provides us with an expansion for the correlation functions in powers of λ

$$C_\Gamma(x)=\sum_{\nu=0}^{\infty}\lambda^\nu C_\Gamma^{(\nu)}(x)\,. \quad (2.2)$$

For the correlation function $C_\Gamma(x)$ one finds the following final result for the lowest orders (results up to the 4th order have been presented in [9]):

$$C_\Gamma(x)=\delta_{x,0}+\lambda\delta_{x,1}\frac{e^{i\frac{\phi}{3}}}{3\cos\left(\frac{\varphi}{3}\right)}+\lambda^2\left[\frac{1}{6\cos\left(\frac{\varphi}{3}\right)^2}\left\{\delta_{x,1}\frac{e^{-i\frac{2\phi}{3}}}{2}+\delta_{x,2}e^{i\frac{2\phi}{3}}\right\}\right]+\mathcal{O}(\lambda^3)\,. \quad (2.3)$$

In order to obtain reliable statements about the correlation function one has to specialize to compute even further orders. In [9] a sixth order expansion of $C_\Gamma(x)$ was computed e.g. on the superintegrable line $\phi=\varphi=\frac{\pi}{2}$. We fit the results obtained for $C_\Gamma(x)$ by a complex exponential function:

$$C_\Gamma(x)=a\ e^{\left(\frac{2\pi i}{L}-\frac{1}{\xi_\Gamma}\right)x}+(1-a)\delta_{x,0}\,. \quad (2.4)$$

Results for the superintegrable case are given by the values in table 1.

λ	ξ_Γ	a	L	ξ_σ	p	q	$P_{\min}$	$\frac{LP_{\min}}{2\pi}$
0.25	0.55(3)	0.55(5)	14.3(2)	0.25(2)	0.35(4)	0.32(4)	0.471	1.07(2)
0.50	0.9(1)	0.59(3)	16.5(8)	0.38(4)	0.35(3)	0.24(3)	0.401	1.05(5)
0.75	1.5(6)	0.64(3)	18.3(8)	0.55(6)	0.36(2)	0.09(2)	0.308	0.90(4)

Table 1: Parameters for the correlation functions (2.4) at $\phi=\varphi=\frac{\pi}{2}$.

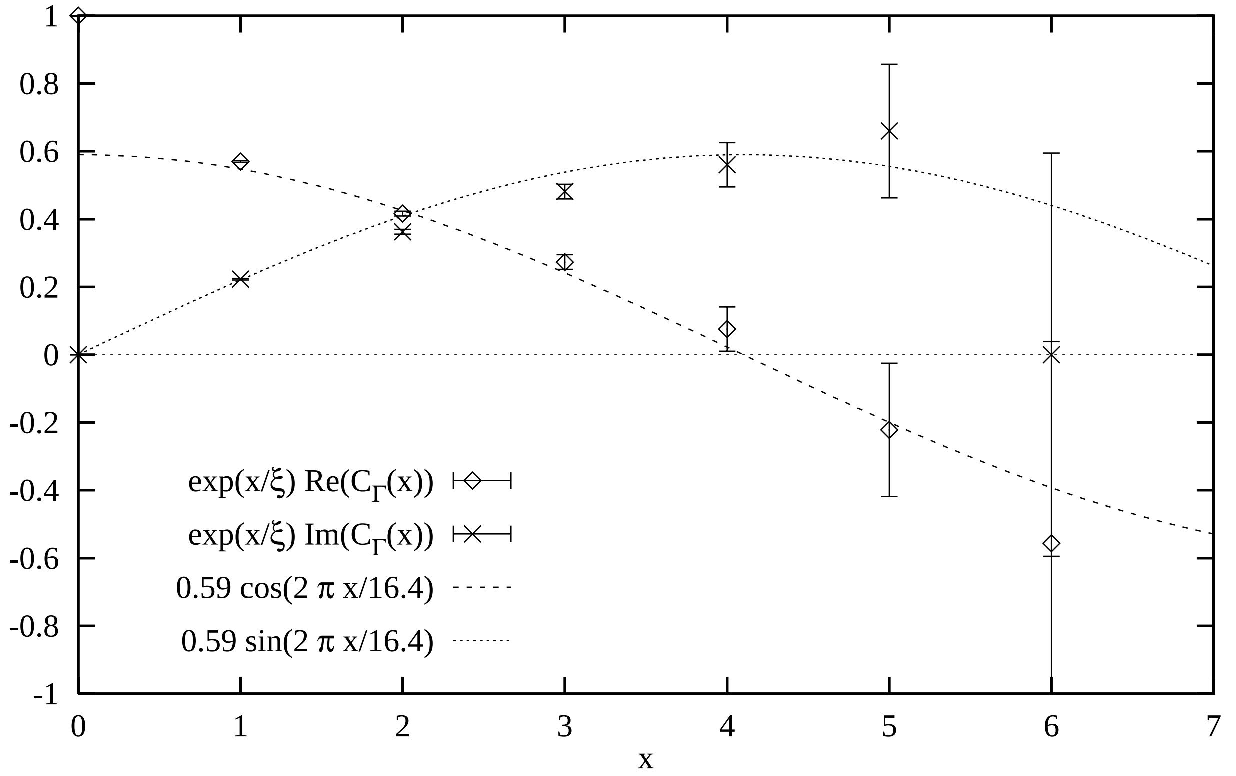

Fig. 1: Correlation function $C_\Gamma(x)$ stretched by $e^{\frac{x}{\xi_\Gamma}}$ in comparison to the fit (2.4) at $\phi = \varphi = \frac{\pi}{2}$, $\lambda = \frac{1}{2}$. The 'error bars' are given by $ae^{\frac{6-x}{\xi_\Gamma}}$ which conveys an idea how much the values have actually been stretched. The oscillatory contribution to $C_\Gamma(x)$ is clearly visible.

Fig. 1 shows the stretched correlation function $e^{\frac{x}{\xi_\Gamma}} C_\Gamma(x)$ in comparison to the fits. The 'error bars' are not really error bars but given by $ae^{\frac{6-x}{\xi_\Gamma}}$ which gives an idea how much the values have actually been stretched and what might be the contribution of the next orders in the perturbation expansion. The agreement for all x not only in the real part but also in the imaginary part is convincing.

Let us now discuss the implications of (2.3) under the assumption that (2.4) is the correct form for general values of the chiral angles. From the leading orders in (2.3) we read off the following identity for the ratio of $C_\Gamma(1)$ and $C_\Gamma(2)$:

$$\frac{C_\Gamma(2)}{C_\Gamma(1)} = \frac{e^{i\frac{\phi}{3}}}{2\cos\left(\frac{\varphi}{3}\right)}\lambda + \mathcal{O}(\lambda^2). \tag{2.5}$$

On the other hand we immediately obtain from (2.4) that $C_\Gamma(2)/C_\Gamma(1) = e^{-\frac{1}{\xi_\Gamma}} e^{\frac{2\pi i}{L}}$. Comparison with (2.5) leads to

$$L = \frac{6\pi}{\mathrm{Re}(\phi)}\,, \qquad\qquad \frac{1}{\xi_\Gamma} = -\ln\left(\frac{\lambda}{2\cos\left(\frac{\varphi}{3}\right)}\right) - \frac{\mathrm{Im}(\phi)}{3} \tag{2.6}$$

for small values of λ. Thus, the oscillation vanishes smoothly for $\phi \to 0$. For λ small, the minimum of the dispersion relation of the fundamental particles is also proportional to ϕ – for $n = 3$ and $Q = 1$ we have $P_{\min} = \frac{\mathrm{Re}(\phi)}{3}$. Thus, we obtain from (2.6) for very high temperatures

$$P_{\min} L\,|_{\lambda\to 0} = 2\pi \qquad \forall \phi, \varphi. \tag{2.7}$$

Furthermore, the minimal momentum $P_{\min}$ decreases with increasing λ [8]. Similarly, we read off from (2.3) that the inverse oscillation length L^{-1} also decreases with increasing inverse temperature λ. Thus, (2.7) has a chance to be valid for all values of λ in the massive high-temperature phase. Indeed, using the values of $P_{\min}$ given in table 8 of [7] we see that $P_{\min}L = 2\pi$ holds quite accurately for $\lambda = 0.25, 0.5, 0.75$ at $\phi = \varphi = \frac{\pi}{2}$ (compare table 1). Nevertheless, we suspect that in general this relation is not exact but an excellent approximation.

3. Numerical computation of the correlation functions

In this section we study the correlation function for the operator Γ of the $\mathbb{Z}_3$-chain numerically. We will in particular check the relation (2.7) numerically.

The groundstate of the $\mathbb{Z}_3$-chain is easily obtained numerically for general values of the parameters (even in the non-hermitian case $\phi \in \mathbb{C}$) using vector iteration up to $N = 13$ sites. The more difficult point is to calculate the matrix elements of $\Gamma_x^+\Gamma_0$ because this operator does not conserve momentum and thus does not leave a space of momentum eigenstates invariant. It is crucial to really work with $\Gamma_x^+\Gamma_0$ and *not* to replace it by $\frac{1}{N}\sum_{r=0}^{N-1}\Gamma_{x+r}^+\Gamma_r$ because this would destroy the oscillation. Table 2 collects results of numerical calculations at $N = 12$ sites. Note that in the integrable case and for small values of λ one has to admit complex ϕ in order to satisfy the integrability-condition (compare also [11]). For complex choices of ϕ the Hamiltonian becomes non-hermitean such that the excitation energies become complex. Thus, $P_{\min}$ is not strictly the minimum of the dispersion relations but instead a point of symmetry – compare [11, 9, 8] and section 4 below.

φ	ϕ	λ	ξ_Γ	L	a	$P_{\min}$	$\frac{LP_{\min}}{2\pi}$
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{1}{4}$	0.62(8)	10(1)	0.4(2)	0.746	1.18 ± 0.12
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{1}{2}$	1.2(6)	12(1)	0.4(1)	0.685	1.34 ± 0.14
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{3}{4}$	1.4(1)	16(1)	0.65(3)	0.602	1.54(8)
$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{1}{4}$	0.6(1)	14.05(6)	0.44(8)	0.470	1.051(4)
$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{1}{2}$	0.60(5)	16(1)	0.59(3)	0.401	1.07(8)
$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3}{4}$	1.3(2)	29(6)		0.189	0.9(2)
$\frac{3\pi}{8}$	$\frac{3\pi}{8}$	$\frac{1}{4}$	0.53(3)	18(2)	0.62(1)	0.341	1.0(1)
$\frac{3\pi}{8}$	$\frac{3\pi}{8}$	$\frac{1}{2}$	0.8(2)	25(1)		0.283	1.14(6)
$\frac{3\pi}{8}$	$\frac{3\pi}{8}$	$\frac{3}{4}$	1.8(6)	50 ± 23		0.211	1.7(8)
$\frac{3\pi}{8}$	$-i\,0.98942$	$\frac{1}{4}$	0.7(2)	∞		0	
$\frac{3\pi}{8}$	0.69919	$\frac{1}{2}$	1.1(3)	49(7)		0.159	1.2(2)
$\frac{3\pi}{4}$	$\pi + i\,1.70004$	$\frac{1}{4}$	1.0(5)	5.9(1)	0.3(1)	$\frac{\pi}{3}$	0.99(2)

Table 2: Parameters for the correlation function (2.4) calculated numerically for $N = 12$ sites.

The agreement with the prediction $LP_{\min} = 2\pi$ is good for all values in table 2 bearing in mind that we have ignored systematic errors. Fig. 2 shows the stretched correlation function $e^{\frac{x}{\xi_\Gamma}}C_\Gamma(x)$ obtained numerically for $N = 13$ sites at the integrable point $\varphi = \frac{3\pi}{4}$, $\lambda = \frac{1}{4}$ in comparison to the fits.

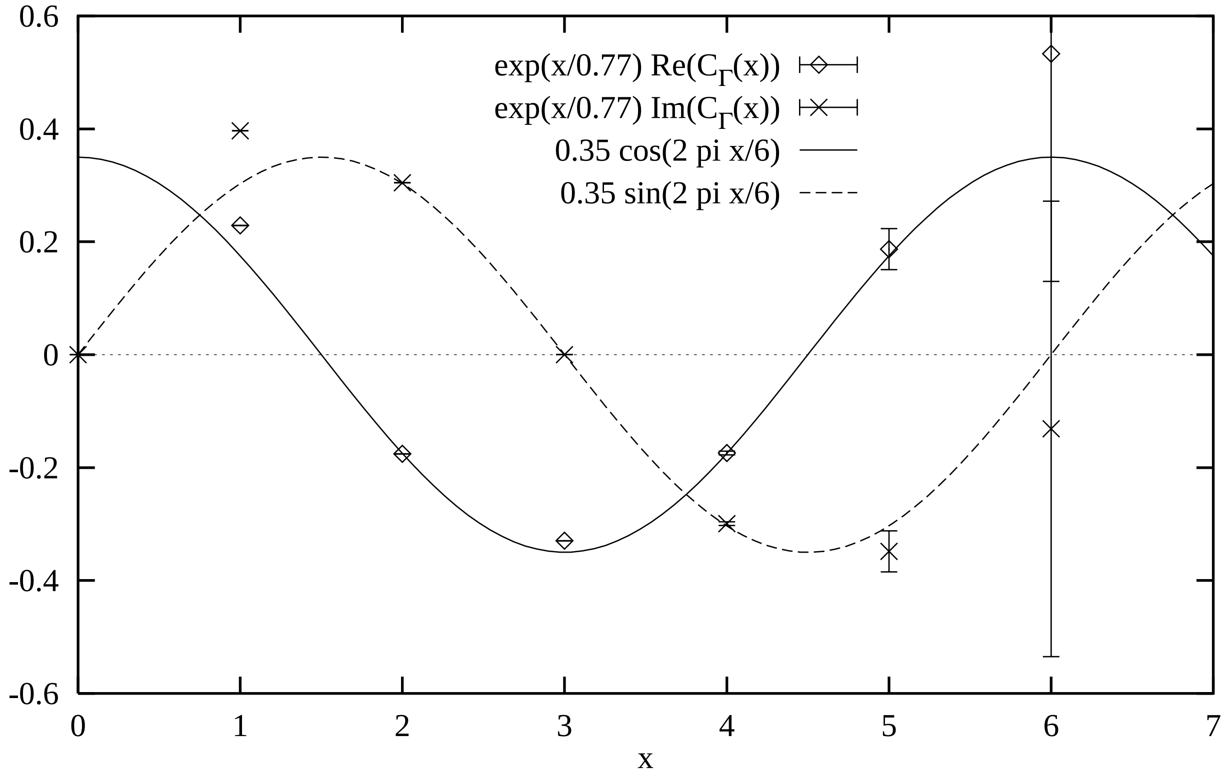

Fig. 2: Numerical values for the correlation function $C_\Gamma(x)$ at $N = 13$ sites stretched by $e^{\frac{x}{\xi_\Gamma}}$ in comparison to the fit (2.4) at $\phi = \frac{3\pi}{4}$, $\lambda = \frac{1}{4}$, $\phi = \pi + i1.70004$. The error bars are given by the differences between $N = 12$ and $N = 13$ sites. One observes that the agreement with these fits is good and the oscillatory contribution to $C_\Gamma(x)$ is clearly visible.

4. Form factors and correlation functions

Finally we indicate how to use form factors in order to determine the oscillation length L exactly using symmetries of the Hamiltonian. One can use the quasiparticle picture (1.3) in order to rewrite a correlation function $C_\Xi(x)$ as follows:

$$C_\Xi(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} \left(\prod_{i=1}^{n} \mathrm{d}p_i \right) \, e^{ix\left(\left(\sum_{j=1}^{n} p_j \right) - P_{|v\rangle} \right)} \, \frac{|\langle p_1, \ldots, p_n | \; \Xi_1 \; | v \rangle \,|^2}{\langle v \,|\, v \rangle} \tag{4.1}$$

where we have inserted a complete set of normalized n-particle states $|\, p_1, \ldots, p_n \rangle$.

In order to gain some intuition we perform some simple computations for the form factors of the operator Γ for the $\mathbb{Z}_3$-chain. Considering perturbation series for the states one sees that

$$\langle p_1, \ldots, p_r | \;\; \Gamma_1 \;\; | v \rangle \; = \mathcal{O}(\lambda^{[\frac{r}{2}]}) \tag{4.2}$$

and one finds the single-particle form factor for $n = 3$ up to order λ:

$$\langle p_1 | \;\; \Gamma_1 \;\; | v \rangle \; = \frac{1}{\sqrt{\int 1 \mathrm{d}p}} \left\{ 1 + \lambda \frac{\cos\left(p_1 - \frac{\phi}{3} \right)}{3 \cos\left(\frac{\varphi}{3} \right)} + \mathcal{O}(\lambda^2) \right\} . \tag{4.3}$$

The maximum of the form factor is obtained when $\mathsf{E}_1(p_1)$ has a minimum and vice versa. Furthermore, multi-particle states have higher energy than single-particle states and due to

(4.2) the contributions of the form factors decrease with the number of particles for small λ. In summary, one obtains the qualitative statement that the contributions of the form factors decrease with increasing energy.

According to (4.2) the single-particle form factor (4.3) is the only one contributing to $C_\Gamma(x)$ up to order λ. Indeed upon inserting (4.3) into (4.1) one recovers (2.3) up to first order in λ – as it should be. Note however that the direct computation of (2.3) is simpler than the one using a form factor expansion.

Now we indicate a result for the oscillation length L obtained with the form factor expansion (4.1) from symmetries of the Hamiltonian. For this end we introduce a parity operator $\mathfrak{P}$ in the natural manner such that $\mathfrak{P}\sigma_x\mathfrak{P} = \sigma_{-x}$, $\mathfrak{P}\Gamma_x\mathfrak{P} = \Gamma_{-x}$. The following statement is based on results obtained via numerical diagonalization in [11] which showed that the Hamiltonian (1.1) has exact symmetries for particular values of the parameters. Denote the restriction of the Hamiltonian $H_N^{(n)}$ in eq. (1.1) to the spaces with momentum P and charge Q by '$H_N^{(n)}(P,Q)$'. Then one has the following identity (see also [11]):

$$\varphi \in \mathbb{R}, \mathrm{Re}(\phi) = \pi \quad \Rightarrow \quad \mathfrak{P}H_N^{(n)}(P_{\mathrm{m},Q} + P, Q)\mathfrak{P} = \left(H_N^{(n)}(P_{\mathrm{m},Q} - P, Q)\right)^+ \tag{4.4}$$

for any Q that is invertible in $\mathbb{Z}_n$ and $P_{\mathrm{m},Q} = \pi(1 - \frac{2Q}{n})$. The restriction to Q invertible in $\mathbb{Z}_n$ (those Q whose greatest common divisor with n is 1) was necessary for technical reasons but there are indications that eq. (4.4) is true without this restriction.

Now let the Hamiltonian have a symmetry of the type indicated by (4.4). Then the oscillation length L of the correlation function $C_{\Gamma^Q}(x)$ satisfies

$$LP_{\mathrm{m},Q} = 2\pi\,. \tag{4.5}$$

Note that the result (4.5) is also true for more general Hamiltonians. We conclude that for $\varphi \in \mathbb{R}, \mathrm{Re}(\phi) = \pi$ one has $C_{\Gamma^Q}(x) = e^{\frac{2\pi i x}{L}} f_{Q,r}(x)$ with $f_{Q,r}(x) \in \mathbb{R}$ for all x and $L = \frac{2n}{n-2Q}$.

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