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Representations of W-Algebras with Two Generators and New Rational Models

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Abstract

In conformal field theory we investigate the representations of recently discovered W-algebras with a single generator in addition to the Virasoro field. We show that many of these W-algebras have only a finite number of highest weight representations. We describe methods for the classification and give complete lists. In a sporadic case we determine characters and fusion rules. Different methods are used for W-algebras with continously variable and for those with fixed central extension.

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1. Introduction

Conformally invariant field theories in 2D have gained much interest in theoretical physics since minimal models have been discovered by BPZ in 1984 [1] because they have important applications in 2D statistical mechanics and string theory. Soon afterwards the attention was drawn to $\mathcal{W}$-algebras by Zamolodchikov in 1985 [2]. They not only serve as a useful tool in the investigation of integrable systems [3] [4], but also provide a promising approach to the problem of classifying all rational conformal field theories (RCFT), which is one of the outstanding questions in theoretical physics. Furthermore, this problem is of mathematical interest, too, because of the connection of RCFT with certain invariants of 3-manifolds via topological quantum field theory [5].

As far as the classification problem is concerned, the study of $\mathcal{W}$-algebras themselves in some sense provides a concept complementary to that of fusion algebras. These deal with abstract properties of representations of conformally invariant operator algebras, leaving the latter more or less unspecified. In contrast, when investigating $\mathcal{W}$-algebras one tries at first to construct an algebra of local fields explicitly and then to get insight into RCFT from its irreducible highest weight representations. Since each primary field of a $\mathcal{W}$-algebra invariant chiral conformal field theory yields a highest weight representation the determination of all allowed highest weight representations of the $\mathcal{W}$-algebra already determines the field content of the theory. Furthermore, the fusion algebra can in principle be read off from these representations.

$\mathcal{W}$-algebras are extensions of the Virasoro algebra and therefore also called 'extended conformal algebras' [6]. They correspond to the operator product expansion (OPE) of conformally invariant chiral fields. The singular part of such an OPE yields a Lie bracket structure, the regular part an operation of forming normal ordered products. Implementing the so-called 'conformal bootstrap' the spin 4 algebra has been investigated by K. Hamada et al. [7] and D.H. Zhang [8], the spin 6 algebra by J.M. Figueroa-O'Farrill et al. [9] and some lower spin cases by P. Bouwknegt [10]. Recently many new $\mathcal{W}$-algebras have been constructed using the Lie bracket approach by R. Blumenhagen et al. [11] as well as by H.G. Kausch et al. [12]. For this method there are more structure constants to be calculated and the notion of normal ordered products needed is slightly more involved compared to the conformal bootstrap, but it directly leads to a Lie algebra structure and thus admits the definition of highest weight representations.

In this paper we want to discuss the highest weight representations of these algebras in more detail. On the basis of references [11] and [13] we give restrictions for the existence of consistent highest weight representations with some additional structure for the bosonic algebras $\mathcal{W}(2,4)$ to $\mathcal{W}(2,8)$ as well as for the fermionic algebras $\mathcal{W}(2,\frac{5}{2})$ to $\mathcal{W}(2,\frac{15}{2})$.

The paper is organized as follows: In the next section we present some general results concerning $\mathcal{W}$-algebras. In the third chapter we give the general outline for highest weight representations of $\mathcal{W}$-algebras. We will discuss the $\mathcal{W}$-algebras that can be interpreted in terms of Virasoro-minimal models in the fourth chapter. The fifth chapter contains our explicit results about $\mathcal{W}(2,\delta)$-algebras. The sixth chapter attempts a systematic discussion of these results.

2. General theorems about $\mathcal{W}$-algebras

In this chapter we will review the basic results about $\mathcal{W}$-algebras for the reader's convenience. The presentation closely follows that of [11]; for proofs as well as for more details we refer the reader to [14].

Let $\mathcal{F}$ be the algebra of local chiral fields of a conformal field theory defined on 2-dimensional spacetime (with compactified space). Because of SU(1,1)-invariance $\mathcal{F}$ carries a natural grading by the conformal dimension and is spanned by the non-derivative (i.e. quasi-primary) fields together with their derivatives.

We define the Fourier decomposition of a left chiral field by $\phi(z) = \sum_{n-d(\phi)\in\mathbb{Z}} z^{n-d(\phi)}\phi_n$. We will call the Fourier-components ϕ_n the 'modes' of ϕ.

Denote the vacuum of the theory by $|v\rangle$. Requiring the regularity of $\phi(z)\,|v\rangle$ at the origin implies

$$\phi_n\,|v\rangle = 0 \qquad \forall\; n < d(\phi) \tag{2.1}$$

For $\phi = L$ this implies invariance of the vacuum under rational conformal transformations, also known as 'SU(1,1)-invariance of the vacuum'.

It is well known that the modes of the energy momentum tensor, a descendant of the identity operator, satisfy the Virasoro algebra

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{2.2}$$

with central charge c. A primary field ϕ of conformal dimension δ is characterized by the commutator of its modes with the Virasoro algebra (2.2):

$$[L_m, \phi_n] = (n-(\delta-1)m)\phi_{n+m} \tag{2.3}$$

Iff a field ϕ satisfies (2.3) for $m \in \{-1,0,1\}$ this field is called 'quasi-primary'.

One has a more general formula for the commutator of two local quasi-primary fields whereof (2.2) and (2.3) are special cases:

Let $\{\phi_i \mid i \in I\}$ be a set of non-derivative fields of integer or half-integer conformal dimensions $d(\phi_i) = h(i)$, which together with their derivatives span $\mathcal{F}$. Define the following constants:

$$\begin{aligned} d_{ij} &= \langle v|\,\phi_{i,-h(i)}\phi_{j,h(j)}\,|v\rangle\,, \\ C_{ijk} &= \langle v|\,\phi_{k,-h(k)}\phi_{i,h(k)-h(j)}\phi_{j,h(j)}\,|v\rangle\,. \end{aligned} \tag{2.4a}$$

Then the Lie algebra of the Fourier components of left chiral fields has the form

$$[\phi_{i,m}, \phi_{j,n}]_\pm = \sum_{k\in I} C_{ij}^k\, p_{ijk}(m,n)\phi_{k,m+n} + d_{ij}\,\delta_{n,-m}\binom{n+h(i)-1}{2h(i)-1} \tag{2.4b}$$

where $C_{ij}^l d_{lk} = C_{ijk}$. Using the notation $h(ijk) = h(i) + h(j) - h(k)$ the universal polynomials p_{ijk} are given by

$$p_{ijk}(m,n) = \sum_{\substack{r,s\in\mathbb{Z}_+\\ r+s=h(ijk)-1}} c_{r,s}^{ijk} \binom{m+h(i)-1}{r}\binom{n+h(j)-1}{s} \tag{2.4c}$$

with

$$c_{r,s}^{ijk} = (-1)^r \frac{(2h(k)-1)!}{(h(i)+h(j)+h(k)-2)!} \prod_{t=0}^{s-1}(2h(i)-2-r-t) \prod_{u=0}^{r-1}(2h(j)-2-s-u)\,. \tag{2.4d}$$

The universal polynomials satisfy

$$p_{ijk}(-m,-n) = (-1)^{(h(ijk)-1)} p_{ijk}(m,n) \tag{2.4e}$$

The C_{ijk} are invariant under even permutations of their indices and change under odd permutations by a factor $(-1)^{([h(i)+\frac{1}{2}]+[h(j)+\frac{1}{2}]+[h(k)+\frac{1}{2}])}$.

It will be convenient to define an involution $\phi \mapsto \phi^+$ on the quasi-primary fields

$$\phi_n^+ = (-1)^{[d(\phi)+\frac{1}{2}]} \phi_{-n} \tag{2.5a}$$

Like any involution on the quasi-primary fields this involution will uniquely extend to an involution on $\mathcal{F}$ and implies that the structure constants d_{ij} and C_{ijk} are real. Sometimes our choice of basis will yield imaginary C_{jjj} for a field ϕ_j, then one many choose $\tilde{\phi}_j := i\phi_j$. This field satisfies

$$\tilde{\phi}_n^+ = -(-1)^{[d(\tilde{\phi})+\frac{1}{2}]} \tilde{\phi}_{-n} \tag{2.5b}$$

The structure constant $\tilde{C}_{jjj}$ will be real. All algebras in this paper do indeed have an (anti-) involution of this form.

In addition to their Lie bracket structure $\mathcal{W}$-algebras admit another important operation, namely that of forming normal ordered products (NOPs) of chiral fields. Usually the NOP of two chiral fields ϕ, χ is defined in terms of Fourier components as follows

$$N(\phi,\chi)_n := \epsilon_{\phi\chi} \sum_{k<d(\chi)} \phi_{n-k}\chi_k + \sum_{k\geq d(\chi)} \chi_k \phi_{n-k} \tag{2.6}$$

$\epsilon_{\phi\chi}$ is defined as -1 for ϕ and χ both fermions and 1 otherwise.

In this form it occurs in the OPE of ϕ and χ, but it is not a non-derivative field, so that e.g. equation (2.4) cannot be used to gain any information about its commutator with other fields. For the NOP to be 'well behaved' under SU(1,1)-transformations we have to add some corrections to $N(\phi,\chi)$ defined in (2.6). With the assumptions and notations from above we define the normal ordered product of two chiral fields by

$$\begin{aligned}\mathcal{N}(\phi_j,\partial^n\phi_i) :=& \sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2(h(i)+h(j)+n-1)}{r}^{-1}\binom{2h(i)+n-1}{r}\\ &\times\partial^r N(\phi_j,\partial^{n-r}\phi_i)\\ &-(-1)^n\sum_{\{k:h(ijk)\geq 1\}} C_{ij}^k\binom{h(ijk)+n-1}{n}\binom{2(h(i)+h(j)+n-1)}{n}^{-1}\\ &\times\binom{2h(i)+n-1}{h(ijk)+n}\binom{\sigma(ijk)-1}{h(ijk)-1}^{-1}\frac{\partial^{h(ijk)+n}\phi_k}{(\sigma(ijk)+n)(h(ijk)-1)!}\end{aligned} \tag{2.7}$$

where $\sigma(ijk)=h(i)+h(j)+h(k)-1$.
This field is quasi-primary and has conformal dimension $h(i)+h(j)+n$.
The operation of normal ordering is (anti)commutative – e.g. $\mathcal{N}(\phi,\chi)=\mathcal{N}(\chi,\phi)$ for two bosonic fields ϕ,χ –, but not associative. It satisfies $\mathcal{N}(\phi,\partial\chi)=-\mathcal{N}(\partial\phi,\chi)$.

Formula (2.7) looks complicated but it simplifies if one considers NOPs of quasi-primary fields with derivatives of L. Since all fields turning up in the commutators of a $\mathcal{W}(2,\delta)$-algebra can be written in this form the simplified form of (2.7) is quite useful:

$$\begin{aligned}\mathcal{N}(\phi_j,\partial^n L) =& \sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2(h(j)+n+1)}{r}^{-1}\binom{n+3}{r}\partial^r N(\phi_j,\partial^{n-r}L)\\ &-(-1)^n\binom{2(h(j)+n+1)}{n}^{-1}\frac{(n+1)(n+3)}{2(2h(j)+1+n)}\partial^{n+2}\phi_j\end{aligned} \tag{2.7a}$$

To derive formula (2.7a) first note that only fields ϕ_k with $\mid d(\phi_j)-d(\phi_k)\mid\leq 1$ can appear in the last sum of (2.7). Comparing (2.4) with (2.3) one can read off the structure constants C_{ij}^k.

So far the field content of $\mathcal{F}$ is infinite. Therefore one introduces the notion of non-composite, 'simple' fields. To be more precise, if a basis for $\mathcal{F}$ can be obtained from a set of fields ϕ_i using the operation of forming normal ordered products and derivation we will say that the fields ϕ_i generate $\mathcal{F}$. If all fields ϕ_i are quasi-primary and all of them are orthogonal to any normal ordered product (with respect to the bilinear form defined by d in (2.4a)) we will call these fields 'simple'. In the following we will denote the algebra $\mathcal{F}$ of local chiral fields generated by a set of simple fields ϕ_1, ... , ϕ_n by $\mathcal{W}(d(\phi_1),...,d(\phi_n))$, where $d(\phi)$ denotes the conformal dimension of ϕ. If we additionally assume that ϕ_1 is the generator of the Virasoro algebra having conformal dimension 2 we will speak of a $\mathcal{W}(2,d(\phi_2),...,d(\phi_n))$ algebra.

The reader should note that the above definition implies that any simple field must be either primary or equal to the energy-momentum operator. Thus, in case of a $\mathcal{W}(2,d(\phi_2),...,d(\phi_n))$ algebra the fields ϕ_2, ... , ϕ_n are primary.

The commutators of normal ordered products are completely determined by the commutators of the simple fields involved. This means that the whole Lie algebra structure of the $\mathcal{W}$-algebra is already fixed by the commutation relations of the simple fields it contains.

Thus a construction of a $\mathcal{W}$-algebra with given simple primary fields $\{\phi_i\}$ proceeds as follows:

First construct all linearly independent NOPs which may occur in the commutators of the simple fields. Then calculate the structure constants appearing in these commutators. At this stage the structure constants connecting three additional simple fields will remain free parameters. Finally the validity of the Jacobi identity has to be checked for three simple fields. It is sufficient to check the coefficients of the primary fields on the right hand side of these expressions.

This task has been carried out by R. Blumenhagen et al. [11] for one or two additional primary fields with dimension up to 8 and additionally by A. Kliem up to dimension 10 [15] under the assumption of vanishing self coupling constant.

There is still some freedom in the normalization of the simple fields. Throughout this paper we normalize the simple fields $\{\phi_i\}$ in such a way that

$$d_{\phi_i\phi_j} = \delta_{i,j}\frac{c}{d(\phi_i)} \tag{2.8}$$

Since $d_{LL} = \frac{c}{2}$, equation (2.4a) is consistent with $\langle v\,|\,v\rangle = 1$.

3. General remarks about highest weight representations of $\mathcal{W}(2,\delta)$-algebras

In this chapter we will give the general outline for highest weight representations (HWRs) of $\mathcal{W}(2,\delta)$-algebras starting with some well known properties. For $\mathcal{W}(2,\delta)$-algebras there are only two simple fields. One of them is of course the energy-momentum operator L and the second one will be denoted by 'W'.

For a bosonic $\mathcal{W}(2,\delta)$-algebra it is easy to see that the Cartan subalgebra is generated by L_0 and W_0. Therefore we will postulate for a highest weight representation that the representation space contains a highest weight vector $|\,h,w\rangle$ with the following properties:

$$L_0\,|\,h,w\rangle = h\,|\,h,w\rangle \tag{3.1a}$$
$$W_0\,|\,h,w\rangle = w\,|\,h,w\rangle \tag{3.1b}$$
$$L_n\,|\,h,w\rangle = 0 \quad \forall\, n<0 \tag{3.1c}$$
$$W_n\,|\,h,w\rangle = 0 \quad \forall\, n<0 \tag{3.1d}$$

In order to be able to interpret the L_0-eigenvalue as energy we will require $h \in \mathbb{R}$, while for the W_0-eigenvalue we will admit $w \in \mathbb{C}$.

Since negative modes annihilate the highest weight vector we should really speak of a 'lowest weight representation' because normally a 'highest weight representation' is defined by the vanishing of positive modes when applied to the highest weight vector. However, due to the existence of the involution (2.5) we can neglect this subtelty. Positive modes applied to the highest weight vector give non-trivial states that need not all be linearly independent. The space spanned by these states divided by the linear dependences will be called a $\mathcal{W}$-algebra Verma module $V(c,h,w)$, generalizing the notion of a Virasoro Verma module $V(c,h)$. In the language of the fusion algebra this is equivalent to building families using the complete $\mathcal{W}$-algebra and not only the Virasoro algebra.

The $\mathcal{W}$-algebra Verma module is L_0-graded; it is a direct sum of L_0-eigenspaces. We will call the difference of the L_0-eigenvalue of a L_0-eigenstate and h the 'level' of the state.

Let us now turn to the fermionic case. While in the vacuum representation one has to choose the modes in $\mathbb{Z}+\frac{1}{2}$ for a fermionic field, one can now choose all indices of fermionic fields in $\mathbb{Z}+\frac{1}{2}$ or in $\mathbb{Z}$, corresponding to periodic or anti-periodic boundary conditions respectively. Taking indices in $\mathbb{Z}+\frac{1}{2}$ yields the Neveu-Schwarz sector of the algebra, whereas indices in $\mathbb{Z}$ give the Ramond sector. For an irreducible representation of a $\mathcal{W}$-algebra in the Ramond sector one can assume that W_0 also satisfies (3.1b) though in general fermionic generators cannot be included in the Cartan subalgebra.

We note that a priori more modes vanish on the vacuum (c.f. (2.1)) than on the highest weight vector, even if h and w are zero.

The representation of a $\mathcal{W}$-algebra must respect the commutator. Therefore in the representation the commutator (2.4b) is equal to the following expression:

$$[\phi_m, \chi_n]_\pm = \phi_m \chi_n - \epsilon_{\phi\chi} \chi_n \phi_m \tag{3.2}$$

By abuse of notation we have not explicitly written the linear representation map here.

Applying this formula to W_0^2 of a fermionic $\mathcal{W}$-algebra in the Ramond sector yields:

$$\begin{aligned} W_0^2 \, |h,w\rangle &= \frac{1}{2}[W_0, W_0]_+ \, |h,w\rangle \\ &= f(c,h) \, |h,w\rangle \end{aligned} \tag{3.3}$$

since the anticommutator of a fermionic field with itself contains no fermionic fields. For c fixed $f(c,h)$ is a polynomial in h of order $\delta - \frac{1}{2}$.

The first approach to the study of the HWRs of $\mathcal{W}(2,\delta)$-algebras is based on the isomorphism between the space of states and the space of fields. BPZ have used this isomorphism in order to reduce the question of rationality to a study of null states in the Verma module [1]. We will now show how one can implement this approach for $\mathcal{W}$-algebras.

For $\mathcal{W}$(2,8) it has been shown explicitly by R. Blumenhagen et al. [16], [15] that for the rational c-values for which the $\mathcal{W}$-algebra is consistent, $\mathcal{N}(W,W)$ is a non-trivial linear combination of fields containing at most one W-field. This has been achieved by calculating the d-matrix of all quasi-primary fields with conformal dimension 16.

In such a case one gets a quadratic relation for w with polynomial coefficients in h for c fixed by considering:

$$
\begin{aligned}
0 &= (\mathcal{N}(W,W)_0 - \sum_X \alpha_X X_0)\,|\,h,w\rangle \\
&= (N(W,W)_0 + \sum_X (\beta_{d(X)} C^X_{WW} - \alpha_X) X_0)\,|\,h,w\rangle \\
&= (W_0 W_0 + \sum_X (\tilde{\beta}_{d(X)} C^X_{WW} - \alpha_X) X_0)\,|\,h,w\rangle \\
&= (w^2 + p_1(h) w + p_2(h))\,|\,h,w\rangle
\end{aligned}
\tag{3.4}
$$

where X is built up from L's and at most one W and p_i polynomials in h (c fixed). Factors from commutations of two W-modes have been absorbed in the coefficients $\tilde{\beta}$. Denoting the highest dimension of all X by γ (which here is equal to 2δ), the degree of p_1 is bounded by $\frac{\gamma-\delta}{2}$ and the degree of p_2 will not exceed $\frac{\gamma}{2}$. In this case one can expect at most two branches of representations of the $\mathcal{W}$-algebra, each one with h possibly a free parameter.

If one has a second null field one may find further conditions restricting the possible values of h to a finite set. For a bosonic $\mathcal{W}$-algebra $\mathcal{N}(W,\partial W)$ vanishes due to symmetry considerations. In case of a fermionic algebra $\mathcal{N}(W,W)$ is zero such that examination of these null states will not yield any nontrivial conditions. An interesting candidate for a second null field in a bosonic algebra is a linear combination of $\mathcal{N}(W,\partial^2 W)$ with other fields. For a field of type $\mathcal{N}(W,\partial^n W)$ the highest dimension of all fields appearing in the generalization of (3.4) will be given by $\gamma = 2\delta + n$.

Since this approach is based on the isomorphism of the space of states and the space of fields it designates the physically relevant HWRs of the $\mathcal{W}$-algebra, but does not exclude that there might be other not completely reducible ones.

The second approach is more fundamental. It is based on the observation of one of us (R.V.) [13] that 4-point correlation functions taken between highest weights are not associative. Equivalently, checking Jacobi identities of consistent $\mathcal{W}$-algebras in a HWR instead of the vacuum representation yields non-trivial results. This leads to restrictions on the set of HWRs which can be consistently defined for the $\mathcal{W}$-algebra, which is stronger than the results obtained using the first method.

In order to make this task manageable one needs some more structure on the HWRs.

The $\mathcal{W}$-algebra Verma module can be equipped with a sesquilinear form using the dual $\mathcal{W}$-algebra Verma module. The involution will then turn into the adjoint operation with respect to this sesquilinear form. We will denote the linear form dual to $|\,h,w\rangle$ by $\langle h,w\,|$. By definition $\langle h,w\,|$ operates on the $\mathcal{W}$-algebra Verma module in the following manner:

$$\langle h,w\,|\,h,w\rangle = 1 \tag{3.5a}$$

$$\langle h,w\,|\,s\rangle = 0 \quad \text{otherwise} \tag{3.5b}$$

for all states $|s\rangle$ in L_0-eigenspaces with eigenvalues greater than h.
Since L_0 and W_0 leave L_0-eigenspaces invariant and $\langle h, w\,|$ is nonzero only for the L_0-eigenspace with eigenvalue h, the following equalities hold:

$$\langle h, w|\ L_0 = h\langle h, w| \tag{3.5c}$$

$$\langle h, w|\ W_0 = w\langle h, w| \tag{3.5d}$$

The linear form is defined such that any correlation-function is nonzero only if the sum of the indices is zero. Additionally, the highest weight property implies that any monomial in modes of fields applied to the highest weight vanishes if the sum of the indices is negative. Combining these two remarks implies:

$$\langle h, w|\ L_n = 0 \quad \forall\ n > 0 \tag{3.5e}$$

$$\langle h, w|\ W_n = 0 \quad \forall\ n > 0 \tag{3.5f}$$

Suppose that the following relation holds:

$$\langle h, w|\ \phi_{1,n_1}\phi_{2,n_2}...\phi_{k,n_k}\ |h, w\rangle\ = \langle h, w|\ [..[[\phi_{1,n_1}, \phi_{2,n_2}]_\pm...\phi_{k,n_k}]_\pm\ |h, w\rangle$$

Then one can calculate $\langle h, w\ |\ \ \phi_{1,n_1}...\phi_{k,n_k}\ \ |\ h, w\rangle$ and $\langle h, w\ |\ \ \phi_{k,-n_k}...\phi_{1,-n_1}\ \ |\ h, w\rangle$ using the commutator-formula (2.4b). The symmetry property (2.4e) of the universal polynomials and symmetry of the central term in (2.4b) implies:

$$\langle h, w|\ \phi_{1,n_1}...\phi_{k,n_k}\ |h, w\rangle\ = \prod_{i=1}^{k}(-1)^{[d(\phi_i)+\frac{1}{2}]}\ \langle h, w|\ \phi_{k,-n_k}...\phi_{1,-n_1}\ |h, w\rangle \tag{3.5g}$$

If the structure constants are real, the r.h.s. of (3.5g) is real and one can use the involution (2.5a) for an easy derivation of equations (3.5a) – (3.5g). In general, however, even the existence of an involution is not known a priori but (3.5a) – (3.5g) are always valid. Thus, one should not use the involution for the calculation but contend oneself with (3.5a) – (3.5g).

Obviously, similar remarks apply to the vacuum representation. As a consequence the d-matrix is always symmetric.

When checking the Jacobi identities for the HWRs one has to use both (2.4b) and (3.2). The Jacobi identity for three modes of the field W is:

$$0 = [[W_k, W_l]_\pm, W_m] + cycl. \tag{3.6}$$

A three point correlator is nonzero only if $m = -(k + l)$ - which is possible only for a bosonic algebra (or for a fermionic algebra in the Ramond sector). Let $r, s > 0$ then (3.6) implies for a three point correlator:

$$\begin{aligned} 0 &= \langle h, w|\ [W_{-r}, W_{-s}]_\pm W_{r+s}\ |h, w\rangle \\ &- \langle h, w|\ W_{-s}[W_{r+s}, W_{-r}]_\pm\ |h, w\rangle \\ &- \langle h, w|\ W_{-r}[W_{-s}, W_{r+s}]_\pm\ |h, w\rangle \end{aligned} \tag{3.7}$$

For a four point correlator we obtain, setting $k = -n$, $l = -m$ and multiplying from the right with W_n and assuming $n, m > 0$ ($n \neq m$ assumed for a bosonic algebra):

$$\begin{aligned}
0 =&\langle h, w|\ [[W_{-n}, W_{-m}]_\pm, W_m]W_n\ |h, w\rangle\ + cycl. \\
=&+\langle h, w|\ [W_{-n}, W_{-m}]_\pm W_m W_n\ |h, w\rangle \\
&-\langle h, w|\ W_{-m}[W_m, W_{-n}]_\pm W_n\ |h, w\rangle \\
&+\langle h, w|\ [W_m, W_{-n}]_\pm W_{-m} W_n\ |h, w\rangle \\
&-\langle h, w|\ W_{-n}[W_{-m}, W_m]_\pm W_n\ |h, w\rangle \\
&+\langle h, w|\ [W_{-m}, W_m]_\pm W_{-n} W_n\ |h, w\rangle \\
=&+\langle h, w|\ W_{-n}W_{-m}[W_m, W_n]_\pm\ |h, w\rangle \\
&-\langle h, w|\ W_{-m}[W_m, W_{-n}]_\pm W_n\ |h, w\rangle \\
&+\langle h, w|\ W_{-n}W_m[W_n, W_{-m}]_\pm\ |h, w\rangle \\
&-\langle h, w|\ W_{-n}[W_{-m}, W_m]_\pm W_n\ |h, w\rangle \\
&+\langle h, w|\ W_{-m}W_m[W_{-n}, W_n]_\pm\ |h, w\rangle
\end{aligned} \tag{3.8}$$

where we have used (3.5g) after inserting the commutators as well as $[W_{-n}, W_n]_\pm\ |h, w\rangle = W_{-n}W_n\ |h, w\rangle$.

Now it is clear what has to be verified for highest weight representations. First we write down an expression of the form (3.7) or (3.8) with fixed indices. Then we use the commutator-formula (2.4b) and insert the normal ordered products according to (2.7a) as well as the structure constants. Next one commutes out the L's and W's as prescribed by (2.2) and (2.3). If there still remain correlators containing W-modes, one has to insert the commutator-formula (2.4b) again and carry on. Finally one ends up with a (hopefully) non-trivial condition for the existence of the corresponding HWR. Of course, one would like to work with arbitrary indices, but in practice this is very difficult since the summation limits of the normal ordered products depend on them. Since the generic expression would be polynomial in the indices one can expect to obtain sufficient conditions if one inserts several different combinations of indices. The degree of the generic condition can be estimated by consideration of the degree of the universal polynomials (2.4c). In fact we have never encountered new conditions after studying correlators with m and n greater than 3 (or 4 for $\mathcal{W}(2,8)$). Thus the number of correlators for a complete study in most cases is much smaller than the degree of the generic expression.

As it should be, explicit calculation shows that all conditions are trivial if h and w are equal to zero. Thus, a highest weight vector $|\,0,0\rangle$ has indeed the properties of the vacuum vector $|\,v\rangle$ described by equation (2.1).

4. The Virasoro-minimal case

It has been noticed by R. Blumenhagen et al. [11] that many $\mathcal{W}(2,\delta)$-algebras can be interpreted as the algebra of modes of primary fields in a Virasoro-minimal model. The HWRs in this case have already been studied by one of us (R. V.) [13].

Minimal representations of the Virasoro algebra are given by $p, q \in \mathbb{Z}_+$, coprime and

$$c = 1 - 6\frac{(p-q)^2}{pq} \tag{4.1a}$$

$$h(p,q;r,s) = \frac{(pr-qs)^2 - (p-q)^2}{4pq}\,, \quad 1 \le r \le q-1\,, \; 1 \le s \le p-1\,. \tag{4.1b}$$

If the field algebra of a minimal model has a local chiral subalgebra, this subalgebra may be interpreted as a $\mathcal{W}$-algebra; fusion considerations yield hints when it might exist. A.N. Schellekens et al. have discussed this question in detail from the point of view of modular invariant partition functions [17]. Since the conformal family of a primary field is a representation of the Virasoro algebra, any primary field of a minimal model is a candidate for a highest weight vector in a representation of the complete $\mathcal{W}$-algebra. The central charge, the dimension of the simple fields and all values of h must in this case be given by (4.1) for some p,q's. In the sequel we will choose p even and q odd, which is always possible for Virasoro-minimal models that contain $\mathcal{W}$-algebras and are discussed here.

In all cases where $\mathcal{W}$-algebras can be related to Virasoro-minimal models the characters of the $\mathcal{W}(2,\delta)$-algebra can be written as a finite sum of Virasoro characters. Now it is possible to calculate the modular transformations and to deduce the fusion rules. This has been done in [13] for the fermionic case where in addition to the modular transformations in both sectors the fusion rules have been deduced for the Neveu-Schwarz sector.

We recall that the character χ_h of a representation of the Virasoro algebra is defined as follows:

$$\chi_h(\tau) := \mathrm{tr}_{V(c,h)}\left(e^{2\pi i (L_0 - \frac{c}{24})\tau}\right) \tag{4.2}$$

where $V(c,h)$ is the Virasoro Verma module. The character χ_h^W of a HWR of a $\mathcal{W}$-algebra is defined analogously using the $\mathcal{W}$-algebra Verma module:

$$\chi_h^W(\tau) := \mathrm{tr}_{V(c,h,w)}\left(e^{2\pi i (L_0 - \frac{c}{24})\tau}\right) \tag{4.3}$$

In cases where h can be written as $h(p,q;r,s)$ we will also write $h_{r,s}$ and $\chi_{r,s}$ instead of χ_h (analogously for χ_h^W). If a character of a $\mathcal{W}$-algebra can be written as a sum of Virasoro-characters the value of h for the HWR of the $\mathcal{W}$-algebra must – according to these definitions – be the smallest one of the Virasoro-HWRs.

All $\mathcal{W}$-algebras discussed in this chapter are related to the A-D-E classification of A. Cappelli et al. [18]. The simplest case is described by the proposition in chapter 4 of [11]. These $\mathcal{W}(2,\delta)$-algebras are related to the (A_{q-1}, D_{2n}) series where n is a half integral positive number. They have $\delta = (q-2)(n-1)$ and with $p = 4n-2$ their central charge is given by (4.1a). In this case the characters of all HWRs of the $\mathcal{W}(2,\delta)$-algebra are given by:

$$\begin{aligned} &\chi_{i,j}^W = \chi_{i,j} + \chi_{q-i,j}\ , \quad 1 \le i \le \frac{q}{2}\ , \quad 1 \le j < \frac{p}{2}\ , \quad i, \frac{j+1}{2} \in \mathbb{Z} \\ &\chi_{i,\frac{p}{2}}^W = \chi_{i,\frac{p}{2}}\ , \quad 1 \le i \le \frac{q}{2}\ , \quad i \in \mathbb{Z} \end{aligned} \tag{4.4}$$

with odd j (for a fermionic $\mathcal{W}$-algebra even j yields the Ramond-sector). This implies that for the $\mathcal{W}$-algebra we have HWRs with Virasoro h-values $h^W_{i,j}$:

$$h^W_{i,j} = h_{i,j} \ , \quad 1 \leq i \leq \frac{q}{2} \ , \quad 1 \leq j \leq \frac{p}{2} \ , \quad i, \frac{j+1}{2} \in \mathbb{Z} \tag{4.5}$$

The $(D_{\frac{q}{2}+1}, E_6)$ series contains fermionic $\mathcal{W}(2,\delta)$-algebras with $\delta = \frac{q-4}{2}$ and $c = 1 - \frac{(12-q)^2}{2q}$. The characters of the HWRs are given by:

$$\begin{aligned} \text{Neveu} - \text{Schwarz sector :} \quad & \chi^W_i = \chi_{i,1} + \chi_{i,5} + \chi_{i,7} + \chi_{i,11} \ , \ 1 \leq i \leq \frac{q-1}{2} \\ \text{Ramond sector :} \quad & \chi^W_i = \chi_{i,4} + \chi_{i,8} \ , \ 1 \leq i \leq \frac{q-1}{2} \end{aligned} \tag{4.6}$$

Consequently the values of h^W of the HWRs of the $\mathcal{W}$-algebra must be:

$$\begin{aligned} \text{Neveu} - \text{Schwarz sector :} \quad & h^W_i = \min(h_{i,1}, h_{i,5}) \ , \ 1 \leq i \leq \frac{q-1}{2} \\ \text{Ramond sector :} \quad & h^W_i = h_{i,4} \ , \ 1 \leq i \leq \frac{q-1}{2} \end{aligned} \tag{4.7}$$

$\mathcal{W}(2,8)$ at $c = -\frac{224}{65}$ can be interpreted as a member of the (A_{q-1}, E_8) series which predicts a $\mathcal{W}(2, q-5)$–algebra at $c = 1 - \frac{(30-q)^2}{5q}$ for q and 30 coprime. The characters of the HWRs of these models are given by one of the two following linear combinations of four Virasoro-characters:

$$\begin{aligned} \chi^W_{i,1} &= \chi_{i,1} + \chi_{i,11} + \chi_{i,19} + \chi_{i,29} \ , \ 1 \leq i \leq \frac{q-1}{2} \\ \chi^W_{i,2} &= \chi_{i,7} + \chi_{i,13} + \chi_{i,17} + \chi_{i,23} \ , \ 1 \leq i \leq \frac{q-1}{2} \end{aligned} \tag{4.8}$$

Consequently the values of h^W of the HWRs of the $\mathcal{W}$-algebra must be:

$$\begin{aligned} h^W_{i,1} &= \min(h_{i,1}, h_{i,11}) \ , \ 1 \leq i \leq \frac{q-1}{2} \\ h^W_{i,2} &= \min(h_{i,7}, h_{i,13}) \ , \ 1 \leq i \leq \frac{q-1}{2} \end{aligned} \tag{4.9}$$

Furthermore $\mathcal{W}(2,8)$ at $c = \frac{21}{22}$ can be interpreted as the bosonic sector of $\mathcal{W}(2, \frac{7}{2})$ which in turn is a member of the $(D_{\frac{q}{2}+1}, E_6)$ series with $p = 12$ and $q = 11$ (c.f. [11] and [13]). The characters of the HWRs of a $\mathcal{W}$-algebra that can be interpreted as the bosonic sector of a fermionic member of the $(D_{\frac{q}{2}+1}, E_6)$ series are given by one of the following three expressions:

$$\begin{aligned} \chi^W_{i,1} &= \chi_{i,1} + \chi_{i,7} \ , \ 1 \leq i \leq \frac{q-1}{2} \\ \chi^W_{i,2} &= \chi_{i,4} + \chi_{i,8} \ , \ 1 \leq i \leq \frac{q-1}{2} \\ \chi^W_{i,3} &= \chi_{i,5} + \chi_{i,11} \ , \ 1 \leq i \leq \frac{q-1}{2} \end{aligned} \tag{4.10}$$

The different characters correspond to representations of the corresponding fermionic $\mathcal{W}$-algebra in the Ramond and Neveu-Schwarz sector where the Neveu-Schwarz sector splits into $\chi^W_{i,1}$ and $\chi^W_{i,3}$. This implies for the corresponding values of h^W:

$$\begin{aligned} h^W_{i,1} &= \min(h_{i,1}, h_{i,7}) \ , \ 1 \leq i \leq \frac{q-1}{2} \\ h^W_{i,2} &= h_{i,4} \qquad\qquad\quad , \ 1 \leq i \leq \frac{q-1}{2} \\ h^W_{i,3} &= \min(h_{i,5}, h_{i,11}) \ , \ 1 \leq i \leq \frac{q-1}{2} \end{aligned} \tag{4.11}$$

Most of the HWRs of fermionic $\mathcal{W}$-algebras and many of the HWRs of bosonic $\mathcal{W}$-algebras are explained by this argument. For a generically existent algebra the models discussed in this chapter may be imbedded in a continuum of HWRs. Otherwise we found that the only admitted values of h are the ones listed here. In particular, for all isolated values of c that can be parametrized by (4.1a), the Jacobi identity (3.8) restricts the values of h to the ones of this chapter.

For more details we refer the reader to [13]. Although the formula given there for the S-matrix as well as for the fusion rules were primarily deduced for the fermionic case they generalize to the bosonic case without change.

5. Explicit results about HWRs of some $\mathcal{W}(2,\delta)$-algebras

In this section both algorithms explained in section 3 are applied to the known few cases of $\mathcal{W}(2,\delta)$-algebras. We will first check Jacobi identities and in some cases look for null fields afterwards. The calculations were performed with a computer. The non-time-intensive calculations were performed in REDUCE and MATHEMATICA™ while for commutations and expansion of quasi-primary NOPs a special C-program had to be written.

We use the same fields and coupling constants like those given in [11]. Listing them again would take too much place so we refer the interested reader to this reference (the definitions (3.18) in [11] of the fields $D5$ and $D6$ with dimension 12 have to be interchanged in order to be in agreement with the coupling constants given in the appendix of the reference).

When checking Jacobi identities (which we described as the second approach to the study of HWRs of $\mathcal{W}(2,\delta)$-algebras) it is very convenient to calculate

$$d_{W_{-n}W_n}(c,h,w) := \langle h,w|\ [W_{-n},W_n]_\pm\ |h,w\rangle$$

once for arbitrary n and to substitute this expression for all two point functions $\langle h,w|\ W_{-n}W_n\ |h,w\rangle$ with $n>0$ afterwards. This saves one step when calculating (3.7) or (3.8).

We will not list data for Virasoro-minimal cases in this chapter. The values of h we obtained with the aid of the computer are exactly those described in the last chapter. Nonetheless we will mention all $\mathcal{W}$-algebras that correspond to Virasoro-minimal models for the reader's convenience and in a few instances we will stress some observations in the Virasoro-minimal case.

Bosonic $\mathcal{W}$-algebras

$\mathcal{W}(2,4)$

First one has to check if $\mathcal{W}(2,4)$ admits arbitrary HWRs – or equivalently if the Jacobi identities are satisfied for the HWRs of $\mathcal{W}(2,4)$. Calculating (3.8) for several different combinations of indices first yields non-trivial conditions which all become trivial as soon as the relation of c and the self coupling constant C_{WW}^W is inserted. Thus $\mathcal{W}(2,4)$ most probably admits arbitrary HWRs (like $\mathcal{W}(2,3)$).

Next one may check for which values of the central charge one can construct a null field involving $\mathcal{N}(W,W)$. Calculating the d-matrix of $\mathcal{N}(W,W)$ and the five other quasi-primary fields with dimension 8 shows that there is such a null field for $c=1$, $c=-11$, $c=-76$, $c=-\frac{444}{11}$ and $c=-\frac{11}{14}$. Except for $c=-76$ there also is an additional null field involving $\mathcal{N}(W,\partial^2 W)$. According to (3.4) this yields two different conditions that are quadratic in w. So one can eliminate w^2, determine w and insert it into both conditions. For $c=-11$, $c=-\frac{444}{11}$ and $c=-\frac{11}{14}$ this yields a finite set of rational values of h where in the corresponding HWRs these two fields are indeed null fields. Remember that this does not exclude the existence of other HWRs for $\mathcal{W}(2,4)$ at these three values of the central charge for which these two fields are not null fields. The following table contains those values of h where they are null fields:

$\mathcal{W}(2,4)$					
$c=-11$		$c=-\frac{444}{11}$		$c=-\frac{11}{14}$	
h	$w\ C_{WW}^{W}{}^{-1}$	h	$w\ C_{WW}^{W}{}^{-1}$	h	$w\ C_{WW}^{W}{}^{-1}$
0	0	0	0	0	0
0	$\frac{11}{1222}$	$-\frac{9}{11}$	$-\frac{13905}{641861}$	$\frac{11}{14}$	0
$-\frac{1}{2}$	$-\frac{47}{1012}$	$-\frac{10}{11}$	$\frac{834300}{18613969}$	$\frac{11}{16}$	$-\frac{9945}{10304}$
$-\frac{3}{8}$	$\frac{611}{8096}$	$-\frac{12}{11}$	$\frac{250290}{19897691}$	$-\frac{3}{112}$	$\frac{1989}{113344}$
$\frac{3}{2}$	$-\frac{1833}{44}$	$-\frac{14}{11}$	$-\frac{1752030}{52457549}$	$\frac{13}{112}$	$\frac{13005}{113344}$
$-\frac{11}{24}$	$\frac{47}{864}$	$-\frac{15}{11}$	$-\frac{6859800}{577033039}$	$\frac{3}{2}$	$-\frac{49725}{3542}$
$-\frac{1}{3}$	$-\frac{611}{6831}$	$-\frac{16}{11}$	$\frac{23175}{2280763}$	$\frac{2}{7}$	$-\frac{1989}{7084}$
$-\frac{1}{8}$	$\frac{1833}{8096}$	$-\frac{17}{11}$	$\frac{50985}{52457549}$	$-\frac{1}{14}$	$-\frac{153}{7084}$
$\frac{1}{6}$	$-\frac{3055}{2484}$	$-\frac{18}{11}$	$\frac{13905}{52457549}$	$\frac{3}{7}$	$-\frac{765}{644}$
$\frac{13}{24}$	$-\frac{130331}{9504}$	$-\frac{19}{11}$	$\frac{55620}{577033039}$		

We have included $c=-\frac{11}{14}$ in this table although it is a Virasoro-minimal model belonging to the $(D_{\frac{q}{2}+1}, E_6)$ series because its existence was not obvious before studying null fields. For this value of the central charge the two null field conditions we studied also admit $h=\frac{3}{14}$ but we have not included it in the above table because it obviously is an artefact which would vanish when studying more null fields.

For $c=1$ both fields are null fields if w and h satisfy the following relation:

$$w = -\frac{\sqrt{3}(4h-1)h}{18\sqrt{2}} \tag{5.1}$$

For $c=-76$ the single null field implies that w and h satisfy one of the following relations:

$$w = \frac{\sqrt{88247}(103h+307)h}{352988\sqrt{39}} \tag{5.2a}$$

$$w = \frac{\sqrt{88247}(103h^2+665h+1074)}{352988\sqrt{39}} \tag{5.2b}$$

The fact that we have found 'good' relations between h and w for $c=1$ and $c=-76$ which do not restrict the HWRs to a finite set suggests that the corresponding models of $\mathcal{W}(2,4)$ are degenerate but not rational.

It has been shown by H.G. Kausch and G.M.T. Watts [19] that there is a realization of $\mathcal{W}(2,4)$ in terms of B_2. They have also deduced a determinant formula for $\mathcal{W}(2,4)$. Reformulating the way we proved rationality here, one has to study the determinant formula for the vacuum representation in order to show the existence of at least two independent, non-trivial null states. It turns out that for each value of c where one has at least one null state there are in fact infinetely many null states. Thus the minimal values of c can be parametrized by p, q coprime and:

$$c = -\frac{2(5p-6q)(3p-5q)}{pq} \tag{5.3}$$

The three rational models we have found as well as both degenerate models have indeed such a parametrization. For $c = 1$ and $c = -76$ a detailed examination of the determinant formula shows that most of the null states it predicts are dependent. Thus for $c = 1$ and $c = -76$ we have sufficient null states for a degenerate, but nor for a minimal model.

With the aid of the determinant formula one can also examine null states in a general Verma module und show that one can parametrize h for the three minimal models above as follows:

$$h = \frac{((r_1+r_2)^2 + r_2^2)p^2 - 2((2s_1+s_2)r_1 + 2(s_1+s_2)r_2)pq + 2(s_1^2 + (s_1+s_2)^2)q^2}{4pq} + \frac{c-2}{24} \tag{5.4}$$

Of course, this is not an exhaustive study of the minimal series of $\mathcal{W}(2,4)$ yet, but one may expect that a complete understanding can be achieved by a generalization of the arguments used here.

$\mathcal{W}$(2,5)

Since C_{WW}^{W} is zero for $\mathcal{W}(2,5)$ (as for all $\mathcal{W}(2,\delta)$-algebras with δ odd), the Jacobi identity for the four point correlator (3.8) can be calculated in one step. The resulting expression is non-trivial. Obviously, if w appears in this expression it can appear only quadratic because for vanishing self coupling constant the commutator of two W-modes will not contain any W's. The result is therefore a linear expression in w^2 with polynomial coefficients in h (for c fixed). It is not difficult to eliminate w^2 using two results obtained with different indices. So one can calculate the values of h for which HWRs of $\mathcal{W}(2,5)$ can exist.

The results for the values of $c = -\frac{350}{11}$ and $c = \frac{6}{7}$ are not listed here, because they are contained in the (A_{q-1}, D_{2n}) series and the h-values obtained are indeed those predicted in chapter 4.

For $c = -7$ we were not able to exclude any HWR satisfying the following condition:

$$w^2 = -\frac{(4h+1)^2(3h+1)h^2}{500} \tag{5.5}$$

One can also explicitly construct null fields involving either $\mathcal{N}(W,W)$ or $\mathcal{N}(W,\partial^2 W)$ for $c = -7$. It turns out that each of these two fields is a null field exactly in the HWRs satisfying (5.5).

For $c = 134 \pm 60\sqrt{5}$ no restrictions could be found. For these values of the central charge all HWRs seem to be admitted.

One can also calculate the three point correlator (3.7) for $\mathcal{W}(2,5)$. For $c = -\frac{350}{11}$ and w^2 nonzero this yields the condition $h = -\frac{10}{11}$, which is not trivial, but contained in the results obtained by calculating (3.8).

$\mathcal{W}$(2,6)

In the case of $\mathcal{W}(2,6)$ evaluation of the Jacobi identity (3.8) first yields conditions for generic c which all become trivial as soon as the correct self coupling constant C_{WW}^{W} is inserted. For the irrational values of the central charge where C_{WW}^{W} is zero, one has to insert c as well before the condition becomes trivial. Thus $\mathcal{W}(2,6)$ should have arbitrary HWRs for generic c (like $\mathcal{W}(2,4)$).

Surprisingly, for both values of the central charge where c is rational and C_{WW}^{W} vanishes ($\mathcal{W}(2,6)$ is inconsistent for $c=-2$), there are restrictions on the HWRs of $\mathcal{W}(2,6)$. Even though the HWRs at $c=-\frac{516}{13}$ were already discussed, we believe this result to be remarkable enough that we also list the values of h here although they could easily be evaluated according to the results of chapter 4.

For $\mathcal{W}(2,6)$ we have also studied restrictions on the HWRs coming from the presence of null fields. It is possible to construct exactly two null fields involving either $\mathcal{N}(W,W)$ or $\mathcal{N}(W,\partial^2 W)$ for $c=-17$, $c=-\frac{306}{55}$ and $c=-\frac{590}{9}$. Postulating that they are null fields also in a HWR restricts the HWRs to a finite set. For $c=-\frac{1420}{17}$ there is a null field involving $\mathcal{N}(W,W)$, but there is none involving $\mathcal{N}(W,\partial^2 W)$. Nonetheless it has been possible to construct a null field involving $\mathcal{N}(W,\partial^4 W)$ as well as all 28 fields with dimension 16. This also restricts the values of the representations in which these two fields are null fields to a finite set.

We list the rational values of h that belong to these four minimal models in the following table (we believe that the irrational solutions of these conditions would vanish when studying more conditions). The two values of c with vanishing C_{WW}^{W} that we have discussed above are also listed in this table.

$\mathcal{W}(2,6)$												
$c=-\frac{516}{13}$		$c=-47$		$c=-17$		$c=-\frac{306}{55}$		$c=-\frac{590}{9}$		$c=-\frac{1420}{17}$		
h	w^2	h	w^2	h		h		h		h		
0	0	0	0	0	$-\frac{17}{28}$	0	$\frac{5}{11}$	0	$-\frac{19}{9}$	0	$-\frac{49}{17}$	$-\frac{59}{17}$
$-\frac{10}{13}$	0	-2	0	0	$-\frac{3}{7}$	$-\frac{13}{55}$	$\frac{17}{11}$	$-\frac{73}{27}$	$-\frac{17}{9}$	$-\frac{27}{17}$	$-\frac{50}{17}$	$-\frac{60}{17}$
$-\frac{17}{13}$	0	$-\frac{5}{3}$	0	$-\frac{3}{4}$	$-\frac{5}{28}$	$-\frac{8}{55}$	$\frac{7}{5}$	$-\frac{64}{27}$	$-\frac{13}{9}$	$-\frac{30}{17}$	$-\frac{52}{17}$	
$-\frac{21}{13}$	0	$-\frac{15}{8}$	0	$-\frac{2}{3}$	$\frac{1}{7}$	$-\frac{3}{55}$		$-\frac{55}{27}$	$-\frac{8}{3}$	$-\frac{37}{17}$	$-\frac{53}{17}$	
$-\frac{22}{13}$	0	$-\frac{23}{12}$	0	$-\frac{5}{12}$	$\frac{15}{28}$	$\frac{7}{55}$		$-\frac{25}{9}$	$-\frac{7}{3}$	$-\frac{39}{17}$	$-\frac{55}{17}$	
$-\frac{20}{13}$	0	$-\frac{3}{2}$	$-\frac{5}{15022464}$	$\frac{9}{4}$		$-\frac{3}{11}$		$-\frac{23}{9}$		$-\frac{46}{17}$	$-\frac{57}{17}$	
$-\frac{15}{13}$	$-\frac{7}{7366034}$	$-\frac{5}{4}$	$-\frac{5}{3755616}$	$-\frac{5}{7}$		$\frac{4}{11}$		$-\frac{20}{9}$		$-\frac{48}{17}$	$-\frac{58}{17}$	

For $c=-17$, $c=-\frac{306}{55}$, $c=-\frac{590}{9}$ and $c=-\frac{1420}{17}$ we have also determined the values of w itself, but we have not presented them here because they are too complicated. Except for the vacuum representation in all four cases all values of w are nonzero except for $h=\frac{17}{11}$ and

$c = -\frac{306}{55}$. Especially for $c = -17$ one has two HWRs with $h = 0$, but only for one of them also $w = 0$ holds.

Although the model belonging to $c = -\frac{306}{55}$ is a Virasoro minimal model we have also listed it in the above table because in order to see that it exists one had to study null fields first. This model is a member of the (A_{q-1}, E_8) series.

Additionally one can construct a null field involving $\mathcal{N}(W, W)$ for $c = -\frac{1242}{5}$. This field is a null field in the HWRs satisfying one of the following two relations:

$$w = -\frac{11\sqrt{436412491110865}(65725h^2 + 1323915h + 6665504)h}{26184749466651900\sqrt{209}} \tag{5.6a}$$

$$w = -\frac{\sqrt{436412491110865}(144595h^3 + 4554174h^2 + 47823641h + 167439222)}{5236949893330380\sqrt{209}} \tag{5.6b}$$

Since there neither is an additional null field with dimension 14 involving $\mathcal{N}(W, \partial^2 W)$ nor even one with dimension 16 involving $\mathcal{N}(W, \partial^4 W)$ and in contrast to the other values of c the null field condition with $\mathcal{N}(W, W)$ has a rational polynomial as solution we doubt that there is a minimal model corresponding to $\mathcal{W}(2, 6)$ at $c = -\frac{1242}{5}$.

In complete analogy to $\mathcal{W}(2, 4)$ one can derive a minimal series for $\mathcal{W}(2, 6)$ by consideration of G_2. Insertion into the formulae derived by J.M. Figueroa-O'Farrill [20] for Casimir algebras in general yields the following parametrization of the minimal series of $\mathcal{W}(2, 6)$:

$$c = -\frac{2(12p - 7q)(7p - 4q)}{pq} \tag{5.7}$$

We have observed that for all values of c contained in the above table there is indeed such a parametrization, but also the non-minimal value $c = -\frac{1242}{5}$ has such a parametrization. We suppose that the explanation of such degenerate but not minimal models is analogous to the case of $\mathcal{W}(2, 4)$.

$\mathcal{W}$(2,7)

Since $\mathcal{W}(2, 7)$ is consistent only for $c = -\frac{25}{2}$ one can easily calculate (3.8) for c fixed which simplifies calculations. The following condition was deduced for HWRs of $\mathcal{W}(2, 7)$ at $c = -\frac{25}{2}$:

$$w^2 = \frac{(2h + 1)^2(16h + 5)^2(16h + 9)h^2}{4167450} \tag{5.8}$$

This indicates that there are two (possibly isomorphic) branches of representations of $\mathcal{W}(2, 7)$, each with infinitely many HWRs.

$\mathcal{W}(2,8)$

While for $c = -\frac{944}{17}$ the $\mathcal{W}(2,8)$ is consistent with vanishing self coupling constant and the same procedure when checking a Jacobi identity as for $\mathcal{W}(2,4)$ – $\mathcal{W}(2,7)$ can be applied, it is also consistent for several values of c with non-vanishing self coupling constant. In these cases the Jacobi identity (3.8) yields results with cubic polynomials in w. Elimination of w is therefore more complicated than for any other $\mathcal{W}$-algebra we studied. One has to take three different results, eliminate first w^3, then w^2 and has to insert the resulting expression for w in a fourth result of (3.8). Note that this procedure gives the values of w and not only those of w^2.

For six values of c where $\mathcal{W}(2,8)$ is consistent the Jacobi identity argument shows that only finitely many HWRs exist. Three of them are Virasoro-minimal ($c = -\frac{944}{17}$, $c = -\frac{224}{65}$ and $c = \frac{21}{22}$) and therefore not listed here. Except for $c = -\frac{944}{17}$ the self-coupling constant C_{WW}^{W} does not vanish. These two algebras belong to the $(D_{\frac{q}{2}+1}, E_6)$ and (A_{q-1}, E_8) series. The structure of these algebras is very similar to those of the (A_{q-1}, D_{2n}) series as was shown in chapter 4.

For the remaining three values of the central charge the values of h for their HWRs are listed in the following table.

$\mathcal{W}(2,8)$						
$c=-23$		$c=-\frac{712}{7}$		$c=-\frac{3164}{23}$		
0	$-\frac{23}{32}$	0	$-\frac{27}{7}$	0	$-\frac{111}{23}$	$-\frac{125}{23}$
0	$-\frac{1}{2}$	-4	$-\frac{29}{7}$	$-\frac{54}{23}$	$-\frac{112}{23}$	$-\frac{129}{23}$
-1	$-\frac{7}{32}$	$-\frac{19}{5}$	$-\frac{30}{7}$	$-\frac{67}{23}$	$-\frac{116}{23}$	$-\frac{130}{23}$
$-\frac{15}{16}$	$\frac{1}{8}$	$-\frac{21}{5}$	$-\frac{108}{35}$	$-\frac{81}{23}$	$-\frac{118}{23}$	$-\frac{131}{23}$
$-\frac{3}{4}$	$\frac{17}{32}$	$-\frac{17}{7}$	$-\frac{122}{35}$	$-\frac{91}{23}$	$-\frac{119}{23}$	$-\frac{132}{23}$
$-\frac{7}{16}$	3	$-\frac{23}{7}$	$-\frac{127}{35}$	$-\frac{94}{23}$	$-\frac{120}{23}$	$-\frac{133}{23}$
$-\frac{31}{32}$		$-\frac{24}{7}$	$-\frac{148}{35}$	$-\frac{98}{23}$	$-\frac{122}{23}$	
$-\frac{7}{8}$		$-\frac{26}{7}$		$-\frac{103}{23}$	$-\frac{124}{23}$	

It is no error that for $c = -23$ we have listed $h = 0$ twice. There are indeed two HWRs for $h = 0$ one with vanishing w and one with $w \neq 0$. We will come back to this observation in chapter 6.

We have not listed the values of w here since they are not only very large numbers but also irrational and sometimes even imaginary. Although we have calculated them we will not even give the formulae that could be used for calculating w, because they are nasty, too. We simply note that for $h = 0$ there is a HWR with $w = 0$ for all values of c. For the specific values of c listed above these are the only values of h yielding vanishing w.

In the basis of $\mathcal{F}$ we used the self coupling constant C_{WW}^{W} is imaginary for the values of

c listed above. Thus one should replace the field W by iW and then one may define a consistent involution $(iW)_j^+ = -(iW)_{-j}$.

For $c = -\frac{1015}{2}$ it is necessary that h and w satisfy one of the following relations:

$$w = \frac{\sqrt{391391}(938816h^3 + 58511028h^2 + 1215416554h + 8414752437)h}{67248090\sqrt{19246721816706711}} \tag{5.9a}$$

$$w = \frac{\sqrt{391391}(20653952h^4 + 1766242840h^3 + 56646740674h^2 + 807540438431h + 4317513425295)}{1479457980\sqrt{19246721816706711}} \tag{5.9b}$$

We are quite confident that there are indeed two non-isomorphic branches of HWRs for $\mathcal{W}(2,8)$ at $c = -\frac{1015}{2}$ corresponding to (5.9a) or (5.9b) respectively. Note that this is the maximum amount of representations that can exist if one uses the knowledge (3.4) about $\mathcal{N}(W,W)$.

For $c = 350 \pm 252\sqrt{2}$ one finds no restrictions for the HWRs. So, for these values of the central charge all HWRs seem to be admitted, like for the irrational values of c where $\mathcal{W}(2,4)$, $\mathcal{W}(2,5)$ and $\mathcal{W}(2,6)$ are consistent with vanishing C_{WW}^W although now the self coupling constant is non-zero.

Fermionic $\mathcal{W}$-algebras

Most of the fermionic $\mathcal{W}$–algebras are contained in some series mentioned in chapter 4. Therefore we will list only a few results on fermionic $\mathcal{W}$–algebras here. Since all fermionic $\mathcal{W}$-algebras discussed here exist only for isolated values of c it is sufficient to check the Jacobi identities. This already yields the proper minimal models.

The algebra $\mathcal{W}(2, \frac{5}{2})$ is the simplest $\mathcal{W}(2,\delta)$–algebra that has restrictions for the central charge; it is consistent only for one value of c. As it is even possible to perform all calculations for the verification of the Jacobi identities 'by hand' it is a good example to get acquainted with the procedure. This algebra has already been studied by one of us (R. V.) [13] earlier along with $\mathcal{W}(2, \frac{7}{2})$ at $c = \frac{21}{22}$. There also have been earlier studies of $\mathcal{W}(2, \frac{9}{2})$ at $c = -35$ [21]. Our results are compatible with the original ones although the four point correlator originally examined was not of the form (3.8) but of the form

$$0 = \langle h, w |\ [[W_{-n}, W_{-m}]_+, X_{m+n}]\ | h, w\rangle\ + cycl. \tag{5.10}$$

with $X_{m+n} = [W_m, W_n]_+$.

$\mathcal{W}(2, \frac{9}{2})$ and $\mathcal{W}(2, \frac{15}{2})$ both exist for $c = 1 - 8\delta$ in addition to Virasoro-minimal values of c. Therefore we will discuss these algebras in more detail below. The algebra $\mathcal{W}(2, \frac{11}{2})$ is consistent only for $c = -\frac{712}{26}$ and $\mathcal{W}(2, \frac{13}{2})$ is consistent for $c = \frac{9}{34}$, $c = -\frac{611}{14}$ and $c = -\frac{111}{10}$. All these algebras correspond to Virasoro-minimal models. Thus the corresponding values of h are indeed given by (4.5).

$\mathcal{W}(2,\frac{9}{2})$

The algebra $\mathcal{W}(2,\frac{9}{2})$ is consistent for five values of the central charge. The values $c=\frac{25}{26}$, $c=-\frac{7}{20}$, $c=-\frac{125}{22}$ and $c=-\frac{279}{10}$ belong to Virasoro-minimal models and therefore the argument of chapter 4 yields the correct values of h.

$c=-35$ is interesting because it belongs to the $(1-8\delta)$ series. Here we list the HWRs which are allowed for $c=-35$ after calculating several four point correlators.

$\mathcal{W}(2,\frac{9}{2})$ $\quad c=-35$		
Neveu-Schwarz sector	*Ramond sector*	
h	h	w^2
0	$-\frac{7}{8}$	$\frac{49}{577728}$
$-\frac{3}{2}$	$-\frac{9}{8}$	$\frac{49}{2310912}$
$-\frac{4}{3}$	$-\frac{35}{24}$	0
$-\frac{7}{5}$	$-\frac{51}{40}$	0
$-\frac{11}{10}$	$-\frac{59}{40}$	0

$\mathcal{W}(2,\frac{15}{2})$

There are six values of c altogether for which $\mathcal{W}(2,\frac{15}{2})$ is consistent. The values $c=\frac{25}{28}$, $c=-\frac{11}{38}$, $c=-\frac{39}{10}$, $c=-\frac{473}{34}$ and $c=-\frac{825}{16}$ are Virasoro-minimal and therefore not listed explicitly here. The following table contains only the values of h for which HWRs of $\mathcal{W}(2,\frac{15}{2})$ might exist in the Neveu-Schwarz-sector as well as in the Ramond sector for $c=-59$.

$\mathcal{W}(2,\frac{15}{2})$ $\quad c=-59$			
Neveu-Schwarz sector		*Ramond sector*	
0	$-\frac{17}{7}$	$-\frac{99}{40}$	$-\frac{131}{56}$
$-\frac{5}{2}$	$-\frac{31}{14}$	$-\frac{91}{40}$	$-\frac{115}{56}$
$-\frac{12}{5}$	$-\frac{13}{7}$	$-\frac{15}{8}$	$-\frac{13}{8}$
$-\frac{21}{10}$		$-\frac{139}{56}$	

Since the values of w^2 are important for the explanation of the RCFT, we give the formula that can be used to calculate w^2 in the Ramond sector although we do not list the values of w^2 explicitly:

$$w^2=\frac{(91+40h)(99+40h)(115+56h)(131+56h)(139+56h)(177+400h+192h^2)}{66399626487398400} \tag{5.11}$$

6. Interpretation

We first consider $\mathcal{W}$-algebras which exist for generic c and contain rational models imbedded in continuous families of irrational ones.

Finite dimensional Lie algebras as well as the Virasoro algebra have arbitrary HWRs (which are generally not irreducible). Obviously, this is not true for all $\mathcal{W}$-algebras. This fact demonstrates once more that $\mathcal{W}$-algebras should not be regarded as Lie algebras (compare e.g. A. Bilal [22]). If however a $\mathcal{W}$-algebra can be related to a Lie algebra one expects neither restrictions on the central charge nor restrictions on h. Consequently, we have been able to deduce conditions for the existence of consistent HWRs of a $\mathcal{W}$-algebra only if the corresponding value of c was isolated, but not for the generically existent $\mathcal{W}$-algebras $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$. It had been conjectured by P. Bouwknegt [10] that these algebras are related to affine Lie algebras and a classical version of $\mathcal{W}(2,6)$ had been constructed by J. Balog et al. [23]. Recently, H.G. Kausch and G.M.T. Watts have shown that one can indeed construct the quantum algebras $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$ as algebras of B_2 and G_2 respectively [19]. If for a $\mathcal{W}$-algebra an explicit construction in terms of a Lie algebra is possible one can also construct arbitrary HWRs of this algebra using the universal enveloping algebra of this Lie algebra. One should however keep in mind that the models have special properties for certain discrete values of c. This is a pure quantum effect which does not show up in the classical versions. In particular, there are two values of c yielding vanishing C_{WW}^{W} for which there are only finitely many HWRs of $\mathcal{W}(2,6)$.

More generally, both $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$ have minimal series of which these two RCFTs are simple examples. The corresponding values of c for the minimal models should be given by (5.3) or (5.7) for $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$ respectively. Studying null fields we have been able to construct some of these minimal models explicitly.

For $\mathcal{W}(2,4)$ we have seen that $c=-11$, $c=-\frac{11}{14}$ and $c=-\frac{444}{11}$ yield minimal models while the models corresponding to $c=1$ and $c=-76$ most probably are only degenerate. Since for $c=1$ there is a free field construction in terms of one free boson (c.f. [12]) one knows much about this model which lies beyond the scope of this paper.

For $\mathcal{W}(2,6)$ the representations at $c=-17$, $c=-\frac{306}{55}$, $c=-\frac{590}{9}$ and $c=-\frac{1420}{17}$ are those minimal models whose null fields we have been able to construct explicitly. The HWRs for $c=-\frac{1242}{5}$ seem to correspond to a degenerate but not minimal model of $\mathcal{W}(2,6)$.

In the cases where a $\mathcal{W}$-algebra is consistent for irrational values of the central charge we have not been able to deduce any conditions. This applies to $\mathcal{W}(2,4)$ at $c=86\pm60\sqrt{2}$, $\mathcal{W}(2,5)$ at $c=134\pm60\sqrt{5}$, $\mathcal{W}(2,6)$ at $c=194\pm112\sqrt{6}$ and even $\mathcal{W}(2,8)$ at $c=350\pm252\sqrt{2}$ although here the self-coupling constant is nonzero. This is in remarkable contrast to the rational isolated values of c, for all of which at least one condition was found. This result, however, is not unexpected because it has been shown by C. Vafa [24] and G. Anderson et al. [25] that RCFTs should yield rational values of the central charge as well as rational conformal dimensions.

Two cases of field theories with a degenerate but apparently not rational $\mathcal{W}(2,\delta)$-algebra are closely related to the Virasoro-minimal case discussed in chapter 4. More generally, degenerate models of the Virasoro algebra are given by:

$$c = 1 - 24\alpha_0^2 \tag{6.1a}$$

$$\alpha_\pm = \alpha_0 \pm \sqrt{1+\alpha_0^2} \tag{6.1b}$$

$$h(c;r,s) = \frac{1}{4}((\alpha_- r + \alpha_+ s)^2 - (\alpha_- + \alpha_+)^2) \tag{6.1c}$$

For $\mathcal{W}(2,5)$ at $c = -7$ and $\mathcal{W}(2,7)$ at $c = -\frac{25}{2}$ equation (4.1) still gives the correct parametrization of c, but with $p = 1$. This observation generalizes to Virasoro-degenerate models with $p = 1$ and $2 \leq q \in \mathbb{Z}$ because H.G. Kausch [26] has shown that a BRST-construction for the $\mathcal{W}(2,\delta)$-algebras belonging to this $(1,q)$ series is possible where – using the notation of (4.1) – $\delta = h(1,q;1,3)$ holds. So one can expect an infinite number of HWRs of these $\mathcal{W}$-algebras like for the generically consistent algebras. Our explicit calculations based on either Jacobi identities or on null fields suggest that for these $\mathcal{W}$-algebras one does indeed have infinitely many HWRs, the only difference between the generically existent algebras and them being that we have two branches here with one free parameter h in each branch instead of two parameters. For $w = 0$, h takes on exactly those values that correspond to Virasoro-degenerate models, thus motivating the linear factors of (5.5) and (5.8).

There are three more examples of $\mathcal{W}$-algebras which are degenerate but apparently not rational, because only $\mathcal{N}(W,W)$ is a linear combination of the other fields with dimensio 2δ. They include $\mathcal{W}(2,4)$ at $c = -76$, $\mathcal{W}(2,6)$ at $c = -\frac{1242}{5}$ and $\mathcal{W}(2,8)$ at $c = -\frac{1015}{2}$. Here again there are two branches of HWRs each of them with h as free parameter (w is determined by h). Since here C_{WW}^W is nonzero the formulae for these two branches are distinct. One of the branches does not even contain the vacuum representation and therefore these two branches of HWRs cannot be isomorphic. We guess that this observation can be generalized to a series for all $\mathcal{W}(2,\delta)$–algebras with δ even, but not even a proper parametrization for c is known yet.

Now let us consider the rational $\mathcal{W}$–algebras. These permit a good criterion for the classification of their RCFTs. Let h_{min} be the smallest possible h-value and define

$$\tilde{c} := c - 24h_{min} \tag{6.2}$$

One always has $\tilde{c} > 0$, since $\tilde{c}$ describes the asymptotic behaviour of the dimension of the L_0 eigenspaces, as discussed in Appendix B.

Calculating $\tilde{c}$ for all sets of HWRs we have studied one finds a class of $\mathcal{W}$-algebras that yield $\tilde{c} < 1$. These are exactly the $\mathcal{W}$-algebras that correspond to Virasoro-minimal models. $\tilde{c}$ is of the form $\tilde{c} = 1 - \frac{6}{n}$.

Another group of HWRs of $\mathcal{W}$-algebras yields $\tilde{c} = 1$. Apart from the well known unitary models (see P. Ginsparg [27] and E.B. Kiritsis [28]), this group consists exactly of those $\mathcal{W}$-algebras that exist for $c = 1 - 8\delta$ or $c = 1 - 3\delta$. The members of these series admit finitely many HWRs that can be parametrized according to (6.1) using rational r and s. This family contains $\mathcal{W}(2,4)$ at $c=-11$, $\mathcal{W}(2,\frac{9}{2})$ at $c=-35$, $\mathcal{W}(2,6)$ at $c=-47$ and $c=-17$, $\mathcal{W}(2,\frac{15}{2})$ at $c=-59$ and $\mathcal{W}(2,8)$ at $c=-23$. To be more specific let us look at two special cases of (6.1):

$$h(c;r,r) = r^2\alpha_0^2 - \alpha_0^2 \tag{6.3a}$$
$$h(c;r,-r) = r^2\alpha_0^2 + r^2 - \alpha_0^2 \tag{6.3b}$$

Let $m := 2\alpha_0^2$. Note that for the cases listed above $m \in \frac{\mathbb{Z}_+}{2}$ holds. With this definition we obtain from (6.3):

$$h\big(c;\frac{n}{2m},\frac{n}{2m}\big) = \frac{n^2}{8m} - \frac{m}{2} \tag{6.4a}$$
$$h\big(c;\frac{n}{2m+4},-\frac{n}{2m+4}\big) = \frac{n^2}{8m+16} - \frac{m}{2} \tag{6.4b}$$
$$h\big(c;\frac{n}{m},\frac{n}{m}\big) = \frac{n^2}{2m} - \frac{m}{2} \tag{6.5a}$$
$$h\big(c;\frac{n}{m+2},-\frac{n}{m+2}\big) = \frac{n^2}{2m+4} - \frac{m}{2} \tag{6.5b}$$

Empirically, our calculations for the $(1-8\delta)$ – series yield in the fermionic case the h-values of (6.4) with $n \in \mathbb{Z}_+$, $n \le m$ and $n = 2m$ and in the bosonic case the h-values of (6.5) with $n \le \frac{m}{2}$ and $n = m$. For the fermionic algebras even and odd n respectively yields HWRs in the Neveu-Schwarz and Ramond sector. For the bosonic $(1-3\delta)$ – series one has to take (6.4) with $n \in \mathbb{Z}_+$, $n \le 2m$ or $n = 4m$.

Here the parametrization using rational parameters is pure phenomenology. One of us (M. F.) has studied these algebras in detail [21]. He shows that the characters of the representations of these $\mathcal{W}$-algebras can be written in terms of Jacobi-Riemann-Theta-functions, namely $\Theta_{\lambda,k}(\tau,0,0) = \sum_{r\in\mathbb{Z}} e^{2\pi i\tau\frac{(2kr-\lambda)^2}{4k}}$ with λ closely related to n and the modulus k given by m as appearing in (6.4) and (6.5). Moreover, these characters form a finite dimensional unitary projective representation of the modular group and yield proper fusion rules. This explains the rational indices used above.

Most of the minimal models of $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$ have $\tilde{c} > 1$, in particular $\mathcal{W}(2,4)$ at $c = -\frac{444}{11}$, $\mathcal{W}(2,6)$ at $c = -\frac{590}{9}$ and $c = -\frac{1420}{17}$. This leaves two examples that do not fit into any of the above patterns, namely $\mathcal{W}(2,8)$ at $c = -\frac{712}{7}$ and $c = -\frac{3164}{23}$. The latter model apparently belongs to a series whose first members coincide with particular minimal $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$ models. As a first observation, $\mathcal{W}(2,4)$ at $c = -\frac{444}{11}$, $\mathcal{W}(2,6)$ at $c = -\frac{1420}{17}$ and $\mathcal{W}(2,8)$ at $c = -\frac{3164}{23}$ seem to belong to a series because they comprise exactly $n-1$ HWRs if one writes $c = \frac{m}{n}$ with $m,n \in \mathbb{Z}$. More convincing is the following consideration:

First, let us look at $\mathcal{W}(2,4)$ and $c = -\frac{444}{11}$, i.e. $\tilde{c} = \frac{12}{11}$. We can use the 10 possible values of h to calculate:

$$T^{11} = -\mathbf{1} \tag{6.6a}$$
$$\mathrm{tr}(T^N) = -e^{2\pi i\frac{N}{2}} \tag{6.6b}$$

where T is the representation matrix in the space of characters of the modular transformation that translates the modular parameter τ to $\tau + 1$. (6.6b) is valid for all N that are not multiples of 11. For $\mathcal{W}(2,6)$ and $c = -\frac{1420}{17}$, or equivalently $\tilde{c} = \frac{20}{17}$ an analogous calculation leads to:

$$T^{17} = e^{2\pi i \frac{1}{6}} \mathbf{1} \tag{6.7a}$$
$$\mathrm{tr}(T^N) = -e^{2\pi i \frac{5N}{6}} \tag{6.7b}$$

(6.7b) will hold for all N that are not multiples of 17. Finally, for $\mathcal{W}(2,8)$ at $c = -\frac{3164}{23}$ ($\tilde{c} = \frac{28}{23}$) one can use the 22 values of h we obtained to calculate:

$$T^{23} = e^{2\pi i \frac{5}{6}} \mathbf{1} \tag{6.8a}$$
$$\mathrm{tr}(T^N) = -e^{2\pi i \frac{N}{6}} \tag{6.8b}$$

(6.8b) is valid for all N that are not multiples of 23. This 'nice' behaviour of T suggests that it should be possible to find the models corresponding to these three RCFTs. The modular group can be characterized by the relations $S^2 = (ST)^3 = \mathbf{1}$. Because (6.6) – (6.8) yield additional relations for T itself, it is probable that these HWRs correspond to a (possibly projective) representation of some subgroup of the full modular group. For $\mathcal{W}(2,4)$ and $\tilde{c} = \frac{12}{11}$ this might be $\Gamma(11)$.
Since the values of $\tilde{c}$ can be parametrized by the conformal dimension δ and

$$\tilde{c} = \frac{4(\delta - 1)}{3\delta - 1} \tag{6.9}$$

it is plausible to assume that a complete series of $\mathcal{W}(2,\delta)$-algebras with δ even and $\tilde{c}$ given by (6.9) exists.

It is remarkable that there exists a $\mathcal{W}(2,2)$-algebra with $\tilde{c} = \frac{4}{5}$, namely the tensor product of two Virasoro algebras with $\tilde{c} = \frac{2}{5}$. Here at least $T^5 = e^{2\pi i \frac{5}{6}} \mathbf{1}$. The behaviour of $\mathrm{tr}(T^N)$ is slightly different; the pattern of the phase is the same, but the modulus behaves differently. Nevertheless, $\mathcal{W}(2,2)$ also has two null fields with dimension 2δ and $3\delta - 2$; like the other members of this series. Thus, it is possible that this $\mathcal{W}$-algebra should be considered as a member of the same series.

Finally there is one HWR with $\tilde{c} > 1$ which we have not discussed yet. The HWRs of $\mathcal{W}(2,8)$ at $c = -\frac{712}{7}$ yield $\tilde{c} = \frac{8}{7}$. Remembering that $\mathbb{Z}_5$-symmetric models have $c = \frac{8}{7}$ we can guess that $\mathcal{W}(2,8)$ at $c = -\frac{712}{7}$ is related to a $\mathbb{Z}_5$-symmetric model with the ground-state-energy shifted by $h_{min} = -\frac{30}{7}$. Under this assumption it is possible to identify nine of our h-values with primary fields in a $\mathbb{Z}_5$-parafermionic model (c.f. V.A. Fateev et al. [29]). If we now consider the restriction $\tilde{T}$ of T to the remaining six representations we obtain:

$$\tilde{T}^7 = e^{2\pi i \frac{1}{3}} \mathbf{1} \tag{6.10a}$$
$$\mathrm{tr}(\tilde{T}^N) = -e^{2\pi i \frac{2}{3} N} \tag{6.10b}$$

(6.10b) is valid for all N that are not multiples of 7. Here, a model producing the remaining six values of h is easy to find. It is just a tensor product of two Virasoro-minimal models with $p = 2$ and $q = 7$.

The parafermionic model yields a 15-dimensional representation of the full modular group with 'good' fusion rules. The tensor product of the Virasoro-minimal models gives a 9-dimensional representation. Taking them together yields a 24-dimensional representation of the modular group. On this representation one can define naturally a representation of $\mathcal{S}_2$ that commutes with the representation of the modular group. The symmetric and antisymmetric spaces are 15 and 9 dimensional invariant subspaces. The 15-dimensional symmetric space (or 'uncharged' space) yields our model; it is not identical with the original parafermionic model. In terms of characters this action of $\mathcal{S}_2$ determines some of the characters χ^W as linear combinations of parafermionic and Virasoro characters completely. The remaining linear combinations have to be determined by explicitly calculating the dimensions of the first levels of the Verma-module $V(c, h, w)$. In terms of the partition functions Z^P of the parafermionic model and Z^L of the Virasoro model the new model yields

$$Z = \frac{1}{2}\big(Z^P + (Z^L)^2\big) \tag{6.11}$$

Explicit expressions for the characters of this model, its S-matrix and fusion rules are given in Appendix A. Since $\mathbb{Z}_5$-symmetric models are special minimal models of $\mathcal{W}(2, 3, 4, 5)$, there should be a connection of $\mathcal{W}(2, 8)$ and $\mathcal{W}(2, 3, 4, 5)$ at $c = -\frac{712}{7}$. It is worthwhile noticing that the representation of $\mathcal{W}(2, 3, 4, 5)$ at $\tilde{c} = \frac{8}{7}$ is unitary.

Obviously the classification of the HWRs of $\mathcal{W}$-algebras according to their value of $\tilde{c}$ is a very natural way to reproduce families which share the same structure. In order to elucidate the importance of $\tilde{c}$ we will give an upper bound for it in the case of semirational theories. We call a conformal field theory 'semirational' if S transforms the characters into finite sums of characters. Of course, all rational theories are semirational. Since one may assume that the vacuum representation exists for any $\mathcal{W}$-algebra we have $h_{min} \leq 0$ which implies $c \leq \tilde{c}$. Thus any upper bound for $\tilde{c}$ is also an upper bound for c.

Proposition: For any algebra $\mathcal{W}(d(\phi_1), ..., d(\phi_k))$ generated by k fields that corresponds to a semirational theory the following inequality holds:

$$0 < \tilde{c} < k \tag{6.12}$$

or $c = 0$ and $h = 0$.

The proof is an easy adaption of an argument of Cardy [30]. For completeness it will be spelled out in Appendix B.

For unitary representations $h_{min} = 0$ holds, implying:

$$0 < c < k \tag{6.13}$$

In the case of a $\mathcal{W}(2,\delta)$-algebra this proposition implies that rational theories can exist only for those values of the central charge where

$$c \leq \tilde{c} < 2 \tag{6.14}$$

holds. Indeed $0 < \tilde{c} < 2$ is true for our rational theories and the isolated rational values of c always satisfy $c < 2$ (in fact even $c < 1$).

7. Conclusions

We have proved that many of the new $\mathcal{W}(2,\delta)$-algebras discovered in [11] and [12] are rational and argued that others are not. For the rational cases, we have obtained necessary conditions on the HWRs, which restrict them to a finite set. We are confident that all of these candidate HWRs do indeed exist. On the one hand, in the cases where a conceptual understanding of the algebras is possible we have found exactly the expected HWRs. This is a check both for our algorithms and for the theoretical arguments, which often are not yet quite rigorous. On the other hand, our list of h values satisfy the stringent consistency conditions which follow from modular invariance (as described by S.D. Mathur et al. [31]).

For $\mathcal{W}$-algebras with isolated central extensions c, our method consists of the study of Jacobi identities in four point correlators. In principle, three and five point correlators, but no higher ones, could give further restrictions. In a few cases, however, we explicitly verified that this does not happen.

For $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$ we had to use a different method. Here interesting RCFTs are hidden in the continuum of c values. To find their permitted HWRs, we imposed the physical condition that null fields vanish in all representations. Our explicit results will contribute to a complete understanding of the minimal series for these two $\mathcal{W}$-algebras.

Unfortunately, our $\mathcal{W}(2,\delta)$–algebras do not include new unitary ones. Nevertheless, they yield universality classes of continuous phase transitions in statistical mechanics, and they help to complete the classification of all RCFTs, which is of general importance.

The structure of some of our new RCFTs was so unexpected that computer data were necessary before one could try to understand these models. Now most of the HWRs have a good classification. For some, however, among them some belonging to rational theories, the understanding still is rather rudimentary. Open questions include the determination of the characters of $\mathcal{W}(2,4)$ with $\tilde{c} = \frac{12}{11}$, $\mathcal{W}(2,6)$ with $\tilde{c} = \frac{20}{17}$ and $\mathcal{W}(2,8)$ with $\tilde{c} = \frac{28}{23}$ and their representations under the modular group as well as the interpretation of most theories that are not rational.

Although one may assume that with the results of this paper the classification of the $\mathcal{W}(2,\delta)$-algebras and their representations and of the corresponding RCFTs is well under way, the many open questions even in this special case show that there is still much work to be done until all RCFTs are classified. The study of $\mathcal{W}$-algebras and their HWRs will be an important tool in this context.

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Appendix A: The parafermionic model and $\mathcal{W}(2,8)$

We will give explicit expressions for the characters of the HWRs of $\mathcal{W}(2,8)$ at $c = -\frac{712}{7}$ here. It is important to notice that the characters do not change if the 'energy' is shifted as described in chapter 6.

In chapter 6 we stated that the values of p and q are fixed to 2 and 7 respectively. Thus we will neglect the dependences on p and q in the following. It is well known (see e.g. A. Rocha-Caridi [32]) that the characters of Virasoro-minimal models are given by the following expression:

$$\chi^L_{r,s}(\tau) = q^{\frac{c-1}{24}} \eta(\tau) \sum_{k \in \mathbb{Z}} (q^{a_k} - q^{b_k}) \tag{a.1}$$

with $q = e^{2\pi i\tau}$, $a_k = h_{r+2qk,s}$ and $b_k = h_{r+2qk,-s}$ and $\eta(\tau)$ defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1 - q^i) \tag{a.2}$$

The representation matrix of S (the modular transformation $\tau \to -\tau^{-1}$) in this case is:

$$S_{r,s}^{r',s'} = \sqrt{\frac{2}{pq}} (-1)^{rs'+r's+1} \sin \frac{\pi p r r'}{q} \sin \frac{\pi q s s'}{p} \tag{a.3}$$

The characters for the tensor-product of two Virasoro-minimal models are given by the product of the characters of each part. Consequently the full matrix of S is given by:

$$S_{(r_1,s_1),(r_2,s_2)}^{(r_1',s_1'),(r_2',s_2')} = S_{r_1,s_1}^{r_1',s_1'} S_{r_2,s_2}^{r_2',s_2'} \tag{a.4}$$

According to D. Gepner et al. [33] the characters of the parafermionc $\mathbb{Z}_n$-models are given by:

$$\chi_{l,m}^{P}(\tau) = \eta(\tau) c_m^l(\tau) \quad l \in \{0, ...[\frac{n}{2}]\}, m \in \{-l, ..., 2n-l\}, l - m \in 2\mathbb{Z} \tag{a.5}$$

where the c_m^l are the so-called string functions (c.f. e.g. V. Kac [34]). The matrix of S for the parafermionic model is given by:

$$\begin{aligned} b_{l,m}^{l',m'} &= \frac{1}{\sqrt{n(n+2)}} e^{\frac{i\pi m m'}{n}} \sin \frac{\pi(l+1)(l'+1)}{n+2} \\ S_{l,m}^{l',m'} &= b_{l,m}^{l',m'} + b_{l,m}^{n-l',n-m'} \end{aligned} \tag{a.6}$$

The characters χ^W of the $\mathcal{W}(2,8)$-model can now be expressed as linear combinations of the characters χ^P and χ^L. Most of the linear combinations are determined by examining symmetry under $\mathcal{S}_2$. In order to determine the remaining free coefficients one has to calculate the dimension of $V(c,h,w)$ on level one (in two cases) and on level four (in one case). Additionally, we checked equality of the characters and the dimensions of $V(c,h,w)$ up to order two in all cases, up to order seven for the vacuum character and up to order four for χ_{13}^W. Note that the characters of the anti-symmetric space are zero if all $\mathcal{S}_2$-charges are neglected.

$$
\begin{aligned}
\chi_1^W &:= \chi_{h=0}^W = \frac{1}{2}(\chi_{1,2}^{L,1}\chi_{1,2}^{L,2} - \chi_{2,0}^P) \\
\chi_2^W &:= \chi_{h=-\frac{24}{7}}^W = \frac{1}{2}(\chi_{1,5}^P - \chi_{1,1}^{L,1}\chi_{1,1}^{L,2}) \\
\chi_3^W &:= \chi_{h=-\frac{122}{35}}^W = \frac{1}{2}(\chi_{0,2}^P + \chi_{0,-2}^P) \\
\chi_4^W &:= \chi_{h=-\frac{108}{35}}^W = \frac{1}{2}(\chi_{0,4}^P + \chi_{0,-4}^P) \\
\chi_5^W &:= \chi_{h=-\frac{30}{7}}^W = \frac{1}{2}(\chi_{1,3}^{L,1}\chi_{1,3}^{L,2} - \chi_{0,0}^P) \\
\chi_6^W &:= \chi_{h=-\frac{17}{7}}^W = \frac{1}{2}(\chi_{1,5}^P + \chi_{1,1}^{L,1}\chi_{1,1}^{L,2}) \\
\chi_7^W &:= \chi_{h=-\frac{26}{7}}^W = \frac{1}{2}(\chi_{1,1}^{L,1}\chi_{1,2}^{L,2} + \chi_{1,2}^{L,1}\chi_{1,1}^{L,2}) \\
\chi_8^W &:= \chi_{h=-\frac{27}{7}}^W = \frac{1}{2}(\chi_{1,1}^{L,1}\chi_{1,3}^{L,2} + \chi_{1,3}^{L,1}\chi_{1,1}^{L,2}) \\
\chi_9^W &:= \chi_{h=-\frac{148}{35}}^W = \frac{1}{2}(\chi_{1,1}^P + \chi_{1,-1}^P) \\
\chi_{10}^W &:= \chi_{h=-\frac{127}{35}}^W = \frac{1}{2}(\chi_{1,3}^P + \chi_{1,-3}^P) \\
\chi_{11}^W &:= \chi_{h=-\frac{21}{5}}^W = \frac{1}{2}(\chi_{2,2}^P + \chi_{2,-2}^P) \\
\chi_{12}^W &:= \chi_{h=-\frac{19}{5}}^W = \frac{1}{2}(\chi_{2,4}^P + \chi_{2,-4}^P) \\
\chi_{13}^W &:= \chi_{h=-4}^W = \frac{1}{2}(\chi_{1,2}^{L,1}\chi_{1,2}^{L,2} + \chi_{2,0}^P) \\
\chi_{14}^W &:= \chi_{h=-\frac{23}{7}}^W = \frac{1}{2}(\chi_{1,3}^{L,1}\chi_{1,3}^{L,2} + \chi_{0,0}^P) \\
\chi_{15}^W &:= \chi_{h=-\frac{29}{7}}^W = \frac{1}{2}(\chi_{1,2}^{L,1}\chi_{1,3}^{L,2} + \chi_{1,3}^{L,1}\chi_{1,2}^{L,2})
\end{aligned}
\tag{a.7}
$$

The characters belonging to different parts of the tensor product of the Virasoro-minimal model have been distinguished here by an additional upper index in order to show explicitly the change of basis that has been performed.

Using the explicit knowledge of the matrix of S in the basis of χ^P and χ^L it is now a simple change of basis that has to be performed in order to obtain the matrix of S for the $\mathcal{W}(2,8)$-model (the matrix for the change of basis is given by (a.7)). For $\mathcal{W}(2,8)$ at $c = -\frac{712}{7}$ one has explicitly $S = S^+ = S^{-1}$.

According to the famous formula of E. Verlinde [35] one can now calculate the fusion constants $N_{i,j}^k$:

$$N_{i,j}^k = \sum_m \frac{S_m^i S_m^j S_k^m}{S_0^m} \tag{a.8}$$

For the special case discussed here the fusion constants are indeed integers in the range from 1 to 8. Not all of the fusion constants are independent. They obey the following relations:

$$\begin{aligned} N^k_{i,j} &= N^k_{j,i} \\ N^k_{i,j} &= N^i_{j,k} \\ N^k_{i,j} &= N^{\sigma(k)}_{\sigma(i),\sigma(j)} \end{aligned} \tag{a.9}$$

The second equality is due to the trivial charge conjugation matrix in our case while the last equality is given by the only non-trivial automorphism σ of the fusion algebra:

$$\sigma = (3\ \ 4)(9\ \ 10)(11\ \ 12) \in \mathcal{S}_{15} \tag{a.10}$$

where $(i\ \ j)$ denotes the transposition of i and j.

The automorphism σ of the fusion algebra commutes with S, but not with T. It does commute with T^5. It can be used to construct a statistical model with a fault line whose position can be shifted by gauge transformations.

Finally we list the fusion constants $N^k_{i,j}$. We have not used all their symmetries. The notation is $[i] \times [j] = \sum_k N^k_{i,j}[k]$; zeros are not listed.

$$[1] \times [\Phi_i] = 1[\Phi_i]$$

$$[4] \times [\Phi_i],\ [10] \times [\Phi_i],\ [12] \times [\Phi_i] \ \text{ are given by the automorphism } \sigma$$

$[2]\times[2]=1[1]+1[2]+1[3]+1[4]+1[8]$

$[2]\times[3]=1[2]+1[3]+1[8]+1[9]+1[15]$

$[2]\times[4]=1[2]+1[4]+1[8]+1[10]+1[15]$

$[2]\times[5]=1[5]+1[6]+1[7]+1[11]+1[12]+1[13]$

$[2]\times[6]=1[5]+1[6]+1[11]+1[12]+1[15]$

$[2]\times[7]=1[5]+1[11]+1[12]+1[13]+1[14]$

$[2]\times[8]=1[2]+1[3]+1[4]+1[8]+1[9]+1[10]+1[14]+1[15]$

$[2]\times[9]=1[3]+1[8]+1[9]+1[11]+1[13]+1[14]+1[15]$

$[2]\times[10]=1[4]+1[8]+1[10]+1[12]+1[13]+1[14]+1[15]$

$[2]\times[11]=1[5]+1[6]+1[7]+1[9]+1[11]+1[12]+1[13]+1[14]+1[15]$

$[2]\times[12]=1[5]+1[6]+1[7]+1[10]+1[11]+1[12]+1[13]+1[14]+1[15]$

$[2]\times[13]=1[5]+1[7]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+1[15]$

$[2]\times[14]=1[7]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+1[15]$

$[2]\times[15]=1[3]+1[4]+1[6]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+2[15]$

$$[3]\times[3]=1[1]+1[2]+1[3]+1[4]+1[6]+1[8]+1[9]+1[11]+1[14]+1[15]$$
$$[3]\times[4]=1[3]+1[4]+1[8]+1[9]+1[10]+1[13]+1[14]+1[15]$$
$$[3]\times[5]=1[5]+1[6]+1[7]+1[9]+2[11]+1[12]+1[13]+1[14]+1[15]$$
$$[3]\times[6]=1[3]+1[5]+1[6]+1[7]+1[9]+1[11]+1[12]+1[13]+1[14]+1[15]$$
$$[3]\times[7]=1[5]+1[6]+1[7]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+1[15]$$
$$[3]\times[8]=1[2]+1[3]+1[4]+2[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+2[15]$$
$$[3]\times[9]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+2[15]$$
$$[3]\times[10]=1[4]+1[7]+1[8]+1[9]+2[10]+1[11]+1[12]+1[13]+2[14]+2[15]$$
$$[3]\times[11]=1[3]+2[5]+1[6]+1[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+1[14]+2[15]$$
$$[3]\times[12]=1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[3]\times[13]=1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[3]\times[14]=1[3]+1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+2[10]+1[11]+2[12]+2[13]+2[14]+2[15]$$
$$[3]\times[15]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$

$$[5]\times[5]=1[1]+1[2]+1[3]+1[4]+1[6]+1[7]+2[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+2[15]$$
$$[5]\times[6]=1[2]+1[3]+1[4]+1[5]+1[6]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+2[15]$$
$$[5]\times[7]=1[2]+1[3]+1[4]+1[5]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+2[14]+2[15]$$
$$[5]\times[8]=2[5]+1[6]+1[7]+1[9]+1[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[5]\times[9]=1[3]+1[5]+1[6]+1[7]+1[8]+2[9]+1[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[5]\times[10]=1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+2[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[5]\times[11]=1[2]+2[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[5]\times[12]=1[2]+1[3]+2[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[5]\times[13]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+3[14]+3[15]$$
$$[5]\times[14]=1[3]+1[4]+1[5]+1[6]+2[7]+2[8]+2[9]+2[10]+2[11]+2[12]+3[13]+2[14]+3[15]$$
$$[5]\times[15]=1[3]+1[4]+2[5]+2[6]+2[7]+2[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$

$$[6]\times[6]=1[1]+1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+1[15]$$
$$[6]\times[7]=1[3]+1[4]+1[6]+1[7]+1[8]+1[9]+1[10]+1[11]+1[12]+1[13]+1[14]+2[15]$$
$$[6]\times[8]=1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+1[14]+2[15]$$
$$[6]\times[9]=1[3]+1[5]+1[6]+1[7]+1[8]+2[9]+1[10]+2[11]+1[12]+2[13]+2[14]+2[15]$$
$$[6]\times[10]=1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+2[10]+1[11]+2[12]+2[13]+2[14]+2[15]$$
$$[6]\times[11]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+1[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[6]\times[12]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+1[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[6]\times[13]=1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[6]\times[14]=1[3]+1[4]+1[5]+1[6]+1[7]+1[8]+2[9]+2[10]+2[11]+2[12]+2[13]+3[14]+3[15]$$
$$[6]\times[15]=1[2]+1[3]+1[4]+2[5]+1[6]+2[7]+2[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+3[15]$$

$$[7]\times[7]=1[1]+1[3]+1[4]+1[6]+1[7]+2[8]+1[9]+1[10]+1[11]+1[12]+2[13]+1[14]+1[15]$$
$$[7]\times[8]=1[5]+1[6]+2[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+1[14]+2[15]$$
$$[7]\times[9]=1[3]+1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[7]\times[10]=1[3]+1[4]+1[5]+1[6]+1[7]+1[8]+1[9]+1[10]+2[11]+2[12]+2[13]+2[14]+2[15]$$
$$[7]\times[11]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[7]\times[12]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[7]\times[13]=1[2]+1[3]+1[4]+1[5]+1[6]+2[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[7]\times[14]=1[2]+1[3]+1[4]+2[5]+1[6]+1[7]+1[8]+2[9]+2[10]+2[11]+2[12]+2[13]+3[14]+3[15]$$
$$[7]\times[15]=1[3]+1[4]+2[5]+2[6]+1[7]+2[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$

$$[8]\times[8]=1[1]+1[2]+2[3]+2[4]+1[6]+1[7]+2[8]+2[9]+2[10]+1[11]+1[12]+2[13]+2[14]+3[15]$$
$$[8]\times[9]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[8]\times[10]=1[2]+1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+2[13]+2[14]+3[15]$$
$$[8]\times[11]=1[3]+1[4]+2[5]+2[6]+2[7]+1[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+3[15]$$
$$[8]\times[12]=1[3]+1[4]+2[5]+2[6]+2[7]+1[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+3[15]$$
$$[8]\times[13]=1[3]+1[4]+2[5]+2[6]+2[7]+2[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+3[15]$$
$$[8]\times[14]=1[2]+1[3]+1[4]+2[5]+1[6]+1[7]+2[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$
$$[8]\times[15]=1[2]+2[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+3[11]+3[12]+3[13]+4[14]+5[15]$$

$$[9]\times[9]=1[1]+1[2]+1[3]+2[4]+2[5]+2[6]+1[7]+2[8]+1[9]+2[10]+2[11]+3[12]+2[13]+2[14]+3[15]$$
$$[9]\times[10]=1[3]+1[4]+1[5]+1[6]+1[7]+2[8]+2[9]+2[10]+2[11]+2[12]+3[13]+3[14]+3[15]$$
$$[9]\times[11]=1[2]+1[3]+1[4]+2[5]+2[6]+2[7]+2[8]+2[9]+2[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$
$$[9]\times[12]=1[3]+1[4]+2[5]+1[6]+2[7]+2[8]+3[9]+2[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$
$$[9]\times[13]=1[2]+1[3]+1[4]+2[5]+2[6]+2[7]+2[8]+2[9]+3[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$
$$[9]\times[14]=1[2]+1[3]+2[4]+2[5]+2[6]+2[7]+2[8]+2[9]+3[10]+3[11]+3[12]+3[13]+3[14]+4[15]$$
$$[9]\times[15]=1[2]+2[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+4[11]+4[12]+4[13]+4[14]+5[15]$$

$$[11]\times[11]=1[1]+1[2]+2[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+4[11]+3[12]+3[13]+4[14]+5[15]$$
$$[11]\times[12]=1[2]+2[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+3[11]+3[12]+4[13]+4[14]+5[15]$$
$$[11]\times[13]=1[2]+2[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+3[11]+4[12]+4[13]+4[14]+5[15]$$
$$[11]\times[14]=1[2]+1[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+4[11]+4[12]+4[13]+4[14]+5[15]$$
$$[11]\times[15]=1[2]+2[3]+2[4]+3[5]+3[6]+3[7]+3[8]+4[9]+4[10]+5[11]+5[12]+5[13]+5[14]+6[15]$$

$$[13]\times[13]=1[1]+1[2]+2[3]+2[4]+2[5]+2[6]+2[7]+3[8]+3[9]+3[10]+4[11]+4[12]+4[13]+4[14]+5[15]$$
$$[13]\times[14]=1[2]+2[3]+2[4]+3[5]+2[6]+2[7]+3[8]+3[9]+3[10]+4[11]+4[12]+4[13]+4[14]+5[15]$$
$$[13]\times[15]=1[2]+2[3]+2[4]+3[5]+3[6]+3[7]+3[8]+4[9]+4[10]+5[11]+5[12]+5[13]+5[14]+7[15]$$

[14]×[14]=1[1]+1[2]+2[3]+2[4]+2[5]+3[6]+3[7]+3[8]+3[9]+3[10]+4[11]+4[12]+4[13]+4[14]+5[15]

[14]×[15]=1[2]+2[3]+2[4]+3[5]+3[6]+3[7]+4[8]+4[9]+4[10]+5[11]+5[12]+5[13]+5[14]+7[15]

[15]×[15]=1[1]+2[2]+3[3]+3[4]+4[5]+3[6]+4[7]+5[8]+5[9]+5[10]+6[11]+6[12]+7[13]+7[14]+8[15]

Appendix B: Proof of the Proposition

Proposition: For any algebra $\mathcal{W}(d(\phi_1), ..., d(\phi_k))$ generated by k fields that corresponds to a semirational theory the following inequality holds:

$$0 < \tilde{c} < k \tag{6.12}$$

or $c = 0$ and $h = 0$.

The proof is based on the behaviour of the character at infinity. For a semirational theory modular invariance implies that under the transformation S χ_h^W (defined by (4.3)) satisfies:

$$\chi_h^W\left(-\frac{1}{\tau}\right) = \sum_{h' \in I} a_{h'h} \chi_{h'}^W(\tau) \qquad | I | < \infty \tag{b.1}$$

Let $q := e^{2\pi i \tau}$ and $\tilde{q} := e^{-2\pi i \frac{1}{\tau}}$. We will now consider the limit $\tau \to i\infty$, i.e. $q \to 0$ and $\tilde{q} \to 1$. Evaluation of (4.3) in the case of $\chi_h^W\left(-\frac{1}{\tau}\right)$ leads to the following equality:

$$\chi_h^W\left(-\frac{1}{\tau}\right) = \tilde{q}^{h-\frac{c}{24}} \sum_{n=0}^{\infty} c_n \tilde{q}^n = (1 + o(1)) \sum_{n=0}^{\infty} c_n \tilde{q}^n \qquad \tau \to i\infty \tag{b.2}$$

with $c_n \in \mathbb{Z}_+$. c_n is the number of linearly independent states in the $\mathcal{W}$-algebra-Verma-module on level n. Since the series (b.2) generally is divergent we will have to give an estimate for c_n. Since all states in the module can be generated by applying modes of the fields ϕ_i in lexicographic order to the highest weight vector an upper bound for c_n is given by:

$$1 \le c_n \le (\underbrace{p * p * ... * p}_{k})(n) \tag{b.3}$$

where $p(n)$ is the number of partitions of n whose generating function is the Euler-η-function. The '*' denotes the discrete convolution. The lower bound follows from the theory of representations of the Virasoro algebra and is true for almost all n except for

$h = 0$ and $c = 0$. Of course we have counted many null states and in fact for many cases a better estimate for c_n can be given. Let $P(q) := \sum_{n=0}^{\infty} p(n) q^n$. Now we can evaluate (b.2) further:

$$\begin{aligned} \chi_h^W\big(-\frac{1}{\tau}\big) &\leq (1+o(1)) \sum_{n=0}^{\infty} (\underbrace{p * p * \dots * p}_{k})(n) \tilde{q}^n \\ &= (1+o(1)) \big(P(\tilde{q})\big)^k \\ &= (1+o(1)) \big(b \ \mid \tau \mid^{-\frac{1}{2}} \ q^{-\frac{1}{24}} P(q)\big)^k \\ &= (b^* + o(1)) \ \mid \tau \mid^{-\frac{k}{2}} \ q^{-\frac{k}{24}} \end{aligned} \tag{b.4}$$

with b and b^* certain constants and $P(q)^k$ absorbed in b^*. The second step can easily be inferred from the behaviour of $\eta(\tau) = q^{\frac{1}{24}} P^{-1}(q)$ under modular transformations:

$$\eta\big(-\frac{1}{\tau}\big) = \sqrt{-i\tau}\,\eta(\tau) \tag{b.5}$$

Of course, one could also use the behaviour of $P(q)$ itself under modular transformations (see e.g. G. Andrews [36] ch. 5.2).

On the other hand we can insert the character in (b.1) and obtain:

$$\begin{aligned} \chi_h^W\big(-\frac{1}{\tau}\big) &= (1+o(1)) \sum_{h' \in I} a_{h'h} q^{h' - \frac{c}{24}} \\ &= (1+o(1)) \ a_{h_{min}h} q^{h_{min} - \frac{c}{24}} \end{aligned} \tag{b.6}$$

Finally we obtain:

$$\mathcal{O}(1) < (1+o(1)) a_{h_{min}h} q^{h_{min} - \frac{c}{24}} \leq (b^* + o(1)) \mid \tau \mid^{-\frac{k}{2}} \ q^{-\frac{k}{24}} \tag{b.7}$$

This can only be true if:

$$0 < c - 24 h_{min} < k \tag{b.8}$$

which completes the proof.

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