Unifying $\mathcal{W}$-Algebras:
A New Class of Quantum $\mathcal{W}$-Algebras

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This is a very short summary of the main results of refs. [1−3]. A slightly longer summary but with different emphasis can be found in [4]. The results reported here were obtained in collaboration with R. Blumenhagen, W. Eholzer, K. Hornfeck and R. Hübel.

One of the main aims of explicit constructions of quantum $\mathcal{W}$-algebras was to gain some insight into the structure of quantum $\mathcal{W}$-algebras and the associated rational conformal field theories (RCFTs). The rational models of those $\mathcal{W}$-algebras existing for isolated c only can be either interpreted as extensions or truncations of $\mathcal{W}$-algebras obtained by Drinfeld-Sokolov (DS) reduction, or the effective central charge is integer (see [5] for a summary). However, among the $\mathcal{W}$-algebras existing for generic c two algebras of types $\mathcal{W}(2,4,6)$ and $\mathcal{W}(2,3,4,5)$ were unexplained for some time (see e.g. [5] for a description of the problem). These two algebras were finally explained by studying generic classical reduction methods [6] which in turn lead to the emergence of '*unifying $\mathcal{W}$-algebras*'. Although many interesting results can be obtained studying classical $\mathcal{W}$-algebras, the concept of unifying $\mathcal{W}$-algebras can be explained from a purely quantum point of view.

The *Kac determinant* can be used to predict *truncations of Casimir $\mathcal{W}$-algebras.* The Kac determinant of the vacuum Verma module $\mathcal{M}_N$ at level N related to $\mathcal{WL}_k$ is given by [7]:

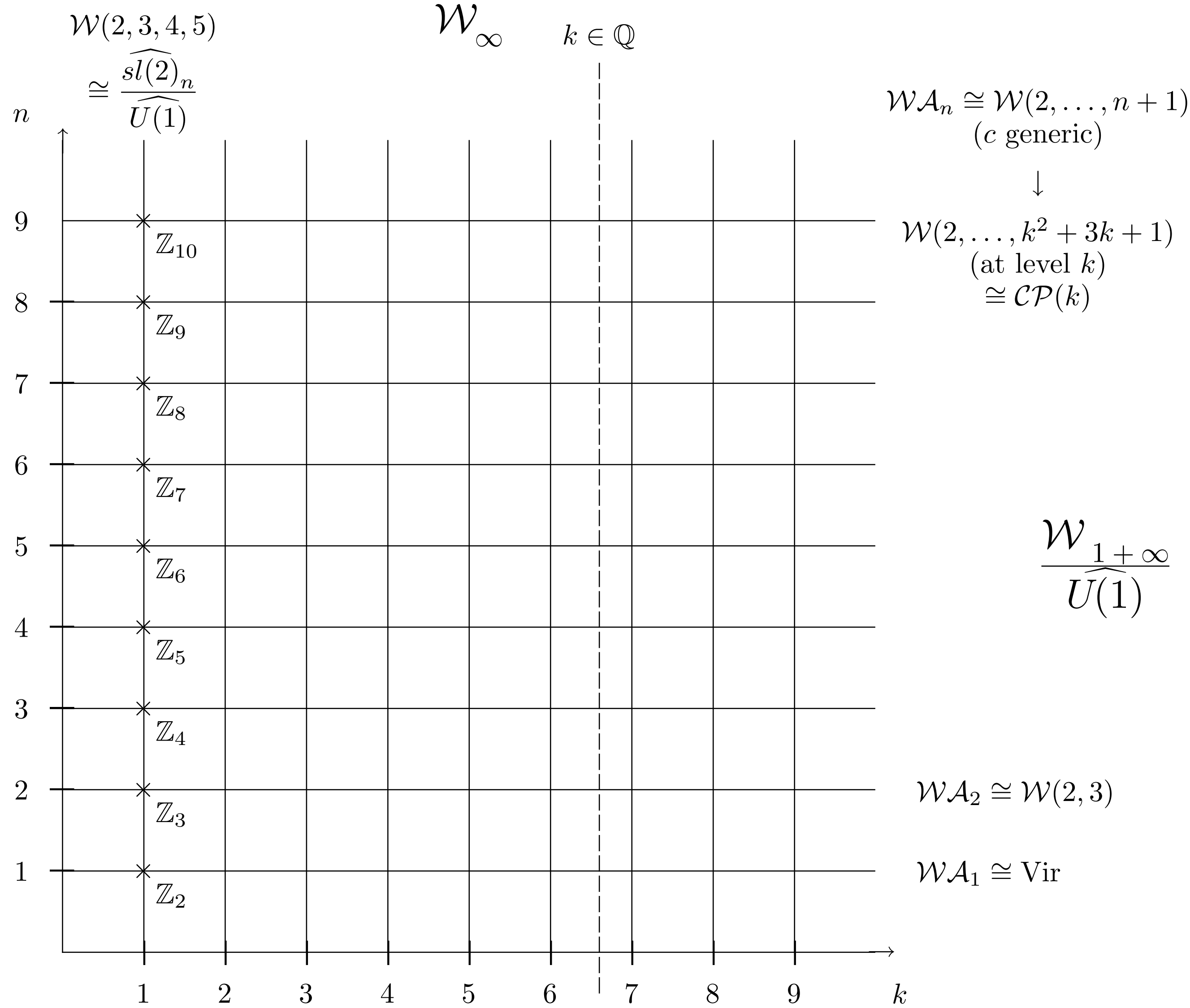
$$\det\mathcal{M}_N \sim \prod_{\beta\in\Delta} \prod_{\substack{mn\leq N \\ m,n\in\mathbb{N}}} \Big((\alpha_+\rho + \alpha_-\rho^\vee, \beta) + \big(\tfrac{1}{2}(\beta,\beta)m\alpha_+ + n\alpha_-\big)\Big)^{p_k(N-mn)} \qquad (1)$$

where $p_k(x)$ is the number of partitions of x into k colours. α_+ and α_- are related to the central charge. For degenerate values of the central charge they can be parametrized by $\alpha_+ = \frac{q}{\sqrt{pq}}$, $\alpha_- = -\frac{p}{\sqrt{pq}}$. The integers p and q are related to the central charge by

$$c_{\mathcal{L}_k}(p,q) = k - 12\frac{(q\,\rho - p\,\rho^\vee)^2}{pq}\,. \qquad (2)$$

Equating one of the non-descendent factors of (1) to zero leads to $(\alpha_+\rho+\alpha_-\rho^\vee,\beta)+\big(\frac{1}{2}(\beta,\beta)m\alpha_+ + n\alpha_-\big) = 0$. Truncations can be expected for solutions to this equation such that $N = mn$ is as small as possible with a suitable root β. The result of this computation suggests that there should be truncations for $m = p - h^\vee + 1$, $n = q - h + 1$. This means that Casimir $\mathcal{W}$-algebras truncate to algebras of fixed field content for their minimal models if the integers p and q are chosen at a fixed offset with respect to the (dual) Coxeter number. A more explicit desciption of this fact is obtained after inserting the coxeter number h and dual coxeter number $h^\vee$ – compare Table 1 of [1].

This gives rise to the structure indicated for the Casimir $\mathcal{W}$-algebras $\mathcal{WA}_n$ in the figure below. The horizontal lines correspond to the algebras $\mathcal{WA}_n$ that exist for generic central charge c. For the kth unitary minimal model all generators of $\mathcal{WA}_n$ with dimension greater than k^2+3k+1 become null fields leading to a truncation of $\mathcal{WA}_n$ to an algebra of type $\mathcal{W}(2,\ldots,k^2+3k+1)$ for all sufficiently large n. One can interpolate these truncations at fixed k to non-integer n giving rise to a new $\mathcal{W}$-algebra existing for generic c – a '*unifying $\mathcal{W}$-algebra*'. In general, unifying $\mathcal{W}$-algebras exist for generic c and can be thought of as continuations of Casimir $\mathcal{W}$-algebras $\mathcal{WL}_n$ to real values of the rank n for certain values of the central charge. Usually, unifying $\mathcal{W}$-algebras are *non-freely* generated.



Also *negative dimensional groups* are related to unifying $\mathcal{W}$-algebras. From the theory of representations of classical groups it is known that e.g. $SU(-n)$ can be formally related to $SU(n)$ and $SO(-2n)$ to $Sp(2n)$. These relations hold for representations where e.g. the dimension formula of one algebra can be obtained by that with the transposed Young tableaux for the other algebra [8] and the Casimir operators are formally equal upon interchanging symmetrization and antisymmetrization [9]. Equating the central charges of the Sugawara energy-momentum tensors for the corresponding Kac-Moody algebras one obtains the following identifications:

$$\widehat{sl(-n)}_k := \widehat{sl(n)}_{-k}\,, \qquad \widehat{so(-2n)}_k := \widehat{sp(2n)}_{-\frac{k}{2}}\,. \tag{3}$$

This leads to the following coset realizations for 'Casimir' $\mathcal{W}$-algebras with negative rank [2, 3]:

$$\begin{aligned} \mathcal{WA}_{-n-1} &\cong \frac{\widehat{sl(n)}_k \oplus \widehat{sl(n)}_{-1}}{\widehat{sl(n)}_{k-1}} \cong \mathcal{W}(2,3,\ldots,(n+1)^2-1)\,, \\ \mathcal{WD}_{-k} &\cong \frac{\widehat{sp(2k)}_\kappa \oplus \widehat{sp(2k)}_{-\frac{1}{2}}}{\widehat{sp(2k)}_{\kappa-\frac{1}{2}}} \cong \mathcal{W}(2,4,\ldots,2k(k+2))\,. \end{aligned} \tag{4}$$

Both $\mathcal{WA}_{-n-1}$ and $\mathcal{WD}_{-k}$ have generic null fields – the lowest one for $\mathcal{WD}_{-k}$ occurs at dimension $2k^2+4k+5$. These algebras appear as unifying $\mathcal{W}$-algebras, e.g. $\mathcal{WD}_{-k} \cong \mathcal{WC}_n$ at $c_{\mathcal{C}_n}(n+k+1, 2n+2k+1)$.

Many unifying $\mathcal{W}$-algebras can be given a *coset realization*. For example, level-rank duality suggests that [10, 11]

$$\mathcal{WA}_{n-1} \cong \frac{\widehat{sl(n)}_k \oplus \widehat{sl(n)}_1}{\widehat{sl(n)}_{k+1}} \cong \frac{\widehat{sl(k+1)}_n}{\widehat{sl(k)}_n \oplus \widehat{U(1)}} = \mathcal{CP}(k)\,. \tag{5}$$

Since there are claims to the contrary in the literature, we would like to emphasize that the coset realization $\widehat{sl(2)}_n/\widehat{U(1)}$ for the unifying $\mathcal{W}$-algebra for the first minimal models of $\mathcal{WA}_{n-1}$ (the $\mathbb{Z}_n$ parafermions) is a finitely generated algebra of type $\mathcal{W}(2,3,4,5)$. Eq. (5) can be generalized to

$$\mathcal{WA}_{n-1} \cong \frac{\mathcal{W}^{sl(r)}_{r-k,1^k}}{\widehat{sl(k)} \oplus \widehat{U(1)}} \cong \mathcal{W}(2,3,\ldots,(k+1)r+k) \qquad \text{at} \qquad c_{\mathcal{A}_{n-1}}(n+k, n+r) \tag{6}$$

with two generic null fields at dimension $(k+1)r+k+3$. The algebra $\mathcal{W}^{sl(r)}_{r-k,1^k}$ in eq. (6) arises by DS reduction for the principal embedding of $sl(2)$ into $sl(r-k) \subset sl(r)$ and is of type $\mathcal{W}(1^{k^2}, 2, 3, \ldots, r-k, \left(\frac{r-k+1}{2}\right)^{2k})$. The k^2 currents of $\mathcal{W}^{sl(r)}_{r-k,1^k}$ form a $\widehat{sl(k)} \oplus \widehat{U(1)}$ Kac-Moody algebra. Eq. (6) gives a coset realization for all unifying $\mathcal{W}$-algebras related to minimal models of $\mathcal{WA}_{n-1}$. Some unifying $\mathcal{W}$-algebras for $\mathcal{WC}_n$ can be interpreted as $\mathcal{WD}_{-k}$ which have the coset realization (4). Also the unfiying $\mathcal{W}$-algebras for certain minimal models of $\mathcal{WD}_n$ and $\mathcal{WB}_n$ can be given a coset realization – see Table 7 of [2] for a complete list of known coset realizations.

The coset realizations of unifying $\mathcal{W}$-algebras agree with level-rank duality where applicable. It is also easily checked that the central charges map nicely onto each other. Furthermore, in some cases equality of structure constants has been checked. One can also check equality of minimal models in those cases where they are known on both sides. Further evidence can be obtained from character computations.

Also the truncations of *linear* $\mathcal{W}_\infty$ *algebras* can be nicely fitted in this picture observing that (see also [12])

$$\mathcal{W}_\infty \cong \lim_{n\to\infty} \mathcal{WA}_{n-1}\,, \qquad \mathcal{W}_{1+\infty} \cong \lim_{k\to\infty} \widehat{U(1)} \oplus \mathcal{CP}(k) = \lim_{k\to\infty} \frac{\widehat{sl(k+1)}_n}{\widehat{sl(k)}_n}\,. \tag{7}$$

For positive integer $c = n$ the algebra $\mathcal{W}_{1+\infty}$ truncates to $\mathcal{W}_n^{gl(n)} \cong \widehat{U(1)} \oplus \mathcal{WA}_{n-1}$ [13] which is a unifying $\mathcal{W}$-algebra for $\widehat{U(1)} \oplus \mathcal{CP}(k)$. Thus, these truncations can be recovered from the right border of the above picture. Also the truncations of $\mathcal{W}_{1+\infty}$ at negative integer $c = -n$ [14] can be understood in a similar manner, one simply needs the algebras $\mathcal{WA}_{-n}$ that we have discussed before.

One can even define [3] '*universal $\mathcal{W}$-algebras*' that depend on *two* parameters (e.g. k and n) and include all minimal models of a family of Casimir $\mathcal{W}$-algebras. This is so because the above figure is densely covered by unifying $\mathcal{W}$-algebras such that one can continue $\mathcal{WA}_n$ to real n for all k. For generic irrational $k \notin \mathbb{Q}$ this universal $\mathcal{W}$-algebra will have infinitely many generators.

To summarize, the space of all $\mathcal{W}$-algebras gives rise to complicated structures including a '*unifying structure*'. However, there are indications [2] that the only rational models of unifying $\mathcal{W}$-algebras are located at intersection points with the Casimir $\mathcal{W}$-algebras. This implies that unifying $\mathcal{W}$-algebras might not give rise to new RCFTs, and that the classification problem of RCFTs (which is far from being solved) could in fact be simpler than a classification of $\mathcal{W}$-algebras. One might hope that (super-symmetric) quantized DS exhausts all possible RCFTs (with exceptions at (half-) integer effective central charge). The concept of unifying $\mathcal{W}$-algebras could still be useful because on the one hand they might lead to conceptual simplifications and on the other hand not all physical phenomena are necessarily described by RCFTs. This applies in particular to string theories and indeed examples for $N = 2$ supersymmetric unifying $\mathcal{W}$-algebras have been found [15].

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