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Ghost Systems: A Vertex Algebra Point of View

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Abstract

Fermionic and bosonic ghost systems are defined each in terms of a single vertex algebra which admits a one-parameter family of conformal structures. The observation that these structures are related to each other provides a simple way to obtain character formulae for a general twisted module of a ghost system. The $U(1)$ symmetry and its subgroups that underly the twisted modules also define an infinite set of invariant vertex subalgebras. Their structure is studied in detail from a $\mathcal{W}$-algebra point of view with particular emphasis on $\mathbb{Z}_N$-invariant subalgebras of the fermionic ghost system.

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1. Introduction

Ghost systems (originally introduced in the context of string theory – see e.g. [1]) and their interrelation have recently attracted renewed attention [2, 3, 4] because of their relevance in statistical physics (see e.g. [5, 6, 7, 8]). Ghost systems are also important in conformal field theory since they serve as building blocks in free field (Wakimoto) realizations of current algebras (see e.g. [9, 10, 11] and references therein). Despite their simplicity, these systems support remarkably rich structures on their own as well. For example, at $c = -2$ a structure has been exhibited [12] that is quite analogous to the $c = 1$ classification [13, 14].

The aim of the first part of this paper is to clarify the relation between the different ghost systems and their characters. We will advocate the usefulness of the point of view to regard all fermionic (or bosonic) ghost systems as just one linear system which admits a one-parameter family of inequivalent conformal structures. The main input we use for a mathematically precise formulation is the (twisted) Borcherds identity [15, 16] which implies in particular explicit expressions for the action of the grading operators corresponding to the ghost number and energy in twisted modules. Identities between characters then follow immediately.

The second part of this paper contains a discussion of symmetry transformations of the ghost fields leaving a chosen energy-momentum tensor invariant and vertex subalgebras invariant under such transformations. Many (quasi-)rational conformal field theories with integer central charge c can be obtained by similar projections of simple linear systems. Such constructions are interesting because there is evidence [17] for the conjecture that all rational models of (bosonic) $\mathcal{W}$-algebras can be obtained by quantised Drinfeld-Sokolov reduction [18, 19, 20] with possible exceptions at integer central charge c. Examples of conformal theories which are at least difficult to obtain by quantised Drinfeld-Sokolov reduction are given by the classification of $c = 1$ conformal field theories [13, 14] and the classification of $c = 24$ conformal field theories with a single primary field [21]. Apart from being interesting in their own right, $\mathcal{W}$-algebras arising as invariant subspaces of symmetries may be useful for physical applications, for example in the description of the fractional quantum Hall effect [22, 23] using quasifinite representations of $\mathcal{W}_{1+\infty}$ [24, 25, 26, 16, 27].

In appendix A we recall the definition of twisted modules of a vertex algebra as well as some useful formulae which will be needed in the paper. Appendix B contains the computation of the characters and the partition function of $\mathbb{Z}_N$-orbifolds of the complex fermion.

2. The fermionic ghost system

Let us first briefly review the fermionic ghost system $b(z)$, $c(z)$ (also called b-c system) [1] in operator-product-expansion (OPE) language before we present a more precise definition in the framework of vertex algebras. The b-c system is characterised by the following non-trivial OPE

$$b(z) \circ c(w) = \frac{1}{(z-w)} + reg. \tag{2.1}$$

($b(z) \circ b(w)$ and $c(z) \circ c(w)$ have no singular part). Among the fields that can be built out of the fields $b(z)$ and $c(z)$, a particularly important one is the ghost-number current

$$J(z) = (c\,b)(z)\,. \tag{2.2}$$

This current satisfies the OPE

$$J(z) \circ J(w) = \frac{1}{(z-w)^2} + reg.\,, \tag{2.3}$$

and its OPEs with the basic ghost fields read

$$J(z) \circ b(w) = -\frac{b(w)}{z-w} + reg.\,, \qquad J(z) \circ c(w) = \frac{c(w)}{z-w} + reg. \tag{2.4}$$

The ghost number of a field $\phi(w)$ is defined as the coefficient in the first order pole of the OPE $J(z) \circ \phi(w)$. In particular, $c(z)$ is assigned the ghost number $+1$ and $b(z)$ is assigned the ghost number -1. The ghost number of a normal-ordered product is simply the sum of the ghost numbers of the constituent fields.

The b-c system admits a one-parameter family of energy-momentum tensors

$$T^{\lambda}(z) = (\lambda - 1)(b\,\partial c)(z) - \lambda(c\,\partial b)(z) = T^0(z) - \lambda \partial J(z)\,. \tag{2.5}$$

With respect to this energy-momentum tensor, the ghost fields become primary fields of dimension $\dim(b) = 1 - \lambda$, $\dim(c) = \lambda$. The central charge of the associated Virasoro algebra is $c_\lambda = -2(6\lambda(\lambda - 1) + 1)$.

The conformal dimension of the current (2.2) with respect to the energy-momentum tensor (2.5) is one, but it is not primary for $\lambda \neq \frac{1}{2}$.

A remark may be in place for readers more familiar with a Lagrangian point of view. Quite often, ghost systems are introduced by specifying an action from which one then obtains the OPE (2.1) and a $U(1)$-invariance of the action gives rise to (2.2) as a conserved current. However, the data encoded in an action needs to be supplemented by specifying the transformation laws of the fundamental ghost fields which is essentially equivalent to directly prescribing the energy-momentum tensor (2.5). We have adopted an algebraic point of view, *i.e.* prescribing OPEs in place of an action and specifying T_λ instead of transformation laws, which is a one-to-one correspondence.

The OPE (2.1) is invariant under the transformation $b \mapsto \mathrm{e}^{-2\pi i\alpha} b$ and $c \mapsto \mathrm{e}^{2\pi i\alpha} c$ which enables one to impose twisted boundary conditions

$$\lim_{\theta \to 2\pi} b(\mathrm{e}^{i\theta} z) = \mathrm{e}^{-2\pi i\alpha} b(z)\,, \qquad \lim_{\theta \to 2\pi} c(\mathrm{e}^{i\theta} z) = \mathrm{e}^{2\pi i\alpha} c(z)\,. \tag{2.6}$$

In order to be more precise we first expand the fields $b(z)$ and $c(z)$ into modes. Since the b-c system admits many Virasoro elements (2.5) we use the mathematical mode convention which does not refer to fixed conformal dimensions of the fields, *i.e.* a particular choice of L_0. A field $\phi(z)$ which is a composite in b, c and satisfies the β-twisted boundary condition $\phi(\mathrm{e}^{2\pi i} z) = \mathrm{e}^{-2\pi i\beta}\phi(z)$ is expanded in terms of its modes $\phi_n^{(\alpha)}$ as

$$\phi(z) = \sum_{n \in \mathbb{Z} + \beta} \phi_n^{(\alpha)} z^{-n-1}\,. \tag{2.7}$$

Here, we have included an upper index α for the modes that characterises the twist (2.6) of the fundamental ghost fields. In case of a trivial twist $\alpha = 0$ we will suppress this upper index.

Using standard techniques one can show that the OPE (2.1) gives rise to the following anti-commutation relations [1]) between the modes $b_n^{(\alpha)}$, $c_k^{(\alpha)}$ of the ghost fields with the boundary conditions (2.6)

$$[b_m^{(\alpha)}, c_k^{(\alpha)}]_+ = \delta_{k+m,-1}\,, \qquad [b_m^{(\alpha)}, b_n^{(\alpha)}]_+ = [c_k^{(\alpha)}, c_l^{(\alpha)}]_+ = 0\,, \tag{2.8}$$

where $m, n \in \mathbb{Z} + \alpha$, $k, l \in \mathbb{Z} - \alpha$.

As usual one introduces a Fock space $\mathcal{F}^{(\alpha)}$ for the b-c system by introducing a cyclic vector $|0\rangle$ with the property

$$b_n^{(\alpha)}|0\rangle = 0 = c_k^{(\alpha)}|0\rangle\,, \qquad \forall\, n > -1+\alpha,\ k > -1-\alpha\,. \tag{2.9}$$

A basis for the Fock space $\mathcal{F}^{(\alpha)}$ is given by

$$b_{n_1}^{(\alpha)} \cdots b_{n_r}^{(\alpha)} c_{k_1}^{(\alpha)} \cdots c_{k_s}^{(\alpha)}|0\rangle \qquad \text{with} \quad \begin{cases} n_1 < n_2 < \ldots < n_r \leq -1+\alpha\,, \\ k_1 < k_2 < \ldots < k_s \leq -1-\alpha\,, \end{cases} \tag{2.10}$$

and one assigns a twist $\beta = (r-s)\alpha$ to such a state.

In the untwisted case the Fock space $\mathcal{F}^{(0)}$ of the fermionic ghost system carries the structure of a simple vertex algebra (see e.g. section 5.1 of [28]). This vertex algebra has a $\widehat{U(1)}$ vertex subalgebra generated by the state

$$|J\rangle = c_{-1}b_{-1}|0\rangle \tag{2.11}$$

corresponding to the current (2.2), and a one-parameter family of conformal structures given by the conformal vectors [2])

$$|T^\lambda\rangle = (\lambda-1)b_{-1}c_{-2}|0\rangle - \lambda c_{-1}b_{-2}|0\rangle \tag{2.12}$$

corresponding to (2.5).

In the case of a non-trivial twist $\alpha \notin \mathbb{Z}$, the Fock space $\mathcal{F}^{(\alpha)}$ can be viewed as a twisted module of this vertex algebra. In particular, a twisted version [16] of the Borcherds identity holds (see (M3) in appendix A). If one wants to compute the action of the fields T^λ and J in $\mathcal{F}^{(\alpha)}$, one can exploit a special case of this identity which is also reported from [16] in appendix A.

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[1]) Note that the definition of normal-ordered products for twisted boundary conditions is more subtle than that of commutation relations. We will define an appropriate normal-ordering procedure later using the twisted Borcherds identity.

[2]) In fact, one can show that these states are the most general possible Virasoro elements with respect to which b and c are primary. To see this, one makes an ansatz for $|T^\lambda\rangle$ as a general linear combination in terms of the basis vectors (2.10). Then the primarity condition $b_1|T^\lambda\rangle \sim b_{-1}|0\rangle$, $c_1|T^\lambda\rangle \sim c_{-1}|0\rangle$ implies that the conformal vector must be of the form $|T^\lambda\rangle = Ab_{-1}c_{-2}|0\rangle + Bc_{-1}b_{-2}|0\rangle$. Finally, the coefficients A and B are fixed by the requirement that this vector must give rise to a Virasoro algebra.

In order to comply with standard notations we use conventions slightly different from (2.7) for the action of the field T^λ in $\mathcal{F}^{(\alpha)}$:

$$T^\lambda(z) = \sum_{n\in\mathbb{Z}} L_n^{(\lambda,\alpha)} z^{-n-2} . \tag{2.13}$$

Using the twisted Borcherds identity one finds that the modes J_m^α of the current J (2.2) acting in $\mathcal{F}^{(\alpha)}$ are explicitly given by

$$J_m^{(\alpha)} = \sum_{k<-\alpha} c_k^{(\alpha)} b_{m-1-k}^{(\alpha)} - \sum_{k\geq-\alpha} b_{m-1-k}^{(\alpha)} c_k^{(\alpha)} + \delta_{m,0}\alpha , \tag{2.14}$$

while for the modes of the family of energy-momentum tensors (2.5) one obtains

$$\begin{aligned} L_n^{(\lambda,\alpha)} = (\lambda-1) & \left(\sum_{k<\alpha} (k-n) b_k^{(\alpha)} c_{n-1-k}^{(\alpha)} - \sum_{k\geq\alpha} (k-n) c_{n-1-k}^{(\alpha)} b_k^{(\alpha)} \right) \\ & -\lambda \left(\sum_{k<-\alpha} (k-n) c_k^{(\alpha)} b_{n-1-k}^{(\alpha)} - \sum_{k\geq-\alpha} (k-n) b_{n-1-k}^{(\alpha)} c_k^{(\alpha)} \right) \\ & + \left(\lambda\alpha + \frac{\alpha(\alpha-1)}{2} \right) \delta_{n,0} \end{aligned} \tag{2.15}$$

(see also [12] for analogous formulae in the case $\lambda = 0$). These explicit formulae permit to check our results directly. For example, one can check by elementary manipulations (following e.g. the lines of [29]) that (2.15) yields indeed a representation of the Virasoro algebra with the correct central charge.

Now it is straightforward to check that

$$L_n^{(\lambda,\alpha)} = L_n^{(0,\alpha)} + \lambda(n+1) J_n^{(\alpha)} , \tag{2.16}$$

in agreement with (2.5). The specialisation of (2.16) to $n = 0$ implies immediately the following identity between the characters for the different choices of conformal elements (2.12)

$$\chi^{(\lambda,\alpha)}(q,z) := \mathrm{tr}_{\mathcal{F}^{(\alpha)}} q^{L_0^{(\lambda,\alpha)} - \frac{c_\lambda}{24}} z^{J_0^{(\alpha)}} = q^{-\frac{c_\lambda - c_0}{24}} \chi^{(0,\alpha)}(q, q^\lambda z) . \tag{2.17}$$

In order to obtain an explicit expression for the r.h.s. one needs the commutation relations of $b_n^{(\alpha)}$ and $c_k^{(\alpha)}$ with $J_0^{(\alpha)}$ which encode the ghost number (-1 for b and $+1$ for c) and those with $L_0^{(0,\alpha)}$ which encodes the conformal dimensions $\dim(b) = 1$ and $\dim(c) = 0$ (for an explicit formula compare appendix A). With this information it is easy to see from (2.10) that the character on the r.h.s. of (2.17) equals (see also [12, 2])

$$\chi^{(0,\alpha)}(q,z) = q^{\frac{1}{12}} q^{\frac{\alpha(\alpha-1)}{2}} z^\alpha \prod_{n=1}^{\infty} (1 + zq^{n+\alpha-1})(1 + z^{-1}q^{n-\alpha}) , \tag{2.18}$$

so that the most general form of the character reads

$$\chi^{(\lambda,\alpha)}(q,z) = q^{\frac{1}{12}} q^{\frac{1}{2}(\lambda+\alpha)(\lambda+\alpha-1)} z^\alpha \prod_{n=1}^{\infty} (1 + zq^{n+(\alpha+\lambda)-1})(1 + z^{-1}q^{n-(\alpha+\lambda)}) . \tag{2.19}$$

Note that for $\lambda = \frac{1}{2}$ and $\alpha = 0$ we recover the character of the complex fermion in the Neveu-Schwarz sector, and the $\lambda = \alpha = \frac{1}{2}$ case concerns the Ramond sector. In the latter case there is just a small asymmetry which arises from the asymmetry of the twist (2.6): Inspection of (2.9) shows that precisely one of the zero modes of the complex fermion is represented trivially for $\lambda = \alpha = \frac{1}{2}$.

The character (2.19) depends on λ and α only via the combination $\lambda+\alpha$ (apart from the prefactor z^α). This generalises a striking similarity of the characters for the cases $\lambda = 0$ and $\lambda = \frac{1}{2}$ already observed in [2]. The fact that λ and α appear only in this special combination can actually be interpreted in terms of a spectral flow (which was introduced for the $N = 2$ superconformal algebra in [30]). First note that the automorphism corresponding to the boundary conditions (2.6) is inner and generated by the current (2.2). Furthermore, all $\mathcal{F}^{(\alpha)}$ viewed as $\widehat{U(1)}$-modules are isomorphic independently of α as can e.g. be inferred by inspection of (2.10) and (2.4). The spectral flow [30] then relates the characters of $\mathcal{F}^{(\alpha)}$ to those of $\mathcal{F}^{(0)}$ by subtracting $\alpha\partial J(z)$ from $T^\lambda(z)$ which amounts to replacing λ by $\lambda + \alpha$ in (2.5).

Although the computations presented above are rather elementary, we believe that they show quite transparently how the relation between the $c = -2$ ghost system and the $c = 1$ complex fermion presented in [2] arises as a special case of an infinite set of identities.

Instead of viewing the b-c system as fundamental, one could have equally well chosen the complex fermion as the starting point. This would lead to formulae slightly different from, but equivalent to (2.16) and (2.17).

3. The bosonic ghost system

The same construction as for the fermionic ghost system can also be applied to a bosonic ghost system, also called β-γ system [1] (see also [31] for some remarks related to the following discussion). Here the only non-trivial basic OPE reads

$$\beta(z) \circ \gamma(w) = \frac{1}{(z-w)} + reg. \tag{3.1}$$

Like for the b-c system one can introduce a ghost-number current

$$J(z) = (\gamma\,\beta)(z)\,, \tag{3.2}$$

which up to a sign satisfies the same OPE as in (2.3), *i.e.* $J(z) \circ J(w) = -\frac{1}{(z-w)^2} + reg.$. The OPEs of the current (3.2) with the basic ghost fields read

$$J(z) \circ \beta(w) = -\frac{\beta(w)}{z-w} + reg.\,, \qquad J(z) \circ \gamma(w) = \frac{\gamma(w)}{z-w} + reg. \tag{3.3}$$

As before, these OPEs can be used to define a ghost number, which is $+1$ for the field $\gamma(z)$ and -1 for the field $\beta(z)$.

Also the β-γ systems admits a one-parameter family of energy-momentum tensors

$$T^\lambda(z) = (1-\lambda)(\beta\,\partial\gamma)(z) - \lambda(\gamma\,\partial\beta)(z) = T^0(z) - \lambda\partial J(z)\,, \tag{3.4}$$

with respect to which the fields β and γ are primary fields of dimension $\dim(\beta) = 1 - \lambda$ and $\dim(\gamma) = \lambda$. The Virasoro central charge associated to (3.4) is $c_\lambda = 2(6\lambda(\lambda-1)+1)$.

Like the field (2.2) the current (3.2) has conformal dimension one with respect to the corresponding energy-momentum tensor (3.4), but is not primary for $\lambda \neq \frac{1}{2}$.

The OPE (3.1) has the same $U(1)$-invariance as the OPE (2.1) which enables one to impose again twisted boundary conditions

$$\lim_{\theta\to 2\pi} \beta(\mathrm{e}^{i\theta}z) = \mathrm{e}^{-2\pi i\alpha}\beta(z)\,, \qquad \lim_{\theta\to 2\pi} \gamma(\mathrm{e}^{i\theta}z) = \mathrm{e}^{2\pi i\alpha}\gamma(z)\,. \tag{3.5}$$

One obtains the following commutation relations for the modes for $\beta_n^{(\alpha)}$ and $\gamma_k^{(\alpha)}$ which are introduced via the expansion (2.7):

$$[\beta_m^{(\alpha)}, \gamma_k^{(\alpha)}] = \delta_{k+m,-1}\,, \qquad [\beta_m^{(\alpha)}, \beta_n^{(\alpha)}] = [\gamma_k^{(\alpha)}, \gamma_l^{(\alpha)}] = 0\,. \tag{3.6}$$

The boundary conditions (3.5) translate into the choice $m, n \in \mathbb{Z}+\alpha$, $k, l \in \mathbb{Z}-\alpha$ in (3.6).

We introduce a Fock space $\mathcal{B}^{(\alpha)}$ for the β-γ system with a cyclic vector $|0\rangle$ that satisfies

$$\beta_n^{(\alpha)}|0\rangle = 0 = \gamma_k^{(\alpha)}|0\rangle\,, \qquad \forall\, n > -1+\alpha,\ k > -1-\alpha\,. \tag{3.7}$$

A basis for this space is given by

$$\beta_{n_1}^{(\alpha)} \cdots \beta_{n_r}^{(\alpha)} \gamma_{k_1}^{(\alpha)} \cdots \gamma_{k_s}^{(\alpha)} |0\rangle \qquad \text{with} \qquad \begin{cases} n_1 \leq n_2 \leq \ldots \leq n_r \leq -1+\alpha\,, \\ k_1 \leq k_2 \leq \ldots \leq k_s \leq -1-\alpha\,. \end{cases} \tag{3.8}$$

$\mathcal{B}^{(0)}$ carries the structure of a simple vertex algebra. It has a $\widehat{U(1)}$ vertex subalgebra corresponding to (3.2) which is generated by the state

$$|J\rangle = \gamma_{-1}\beta_{-1}|0\rangle\,. \tag{3.9}$$

The conformal structures given by (3.4) are now encoded in the one-parameter family of conformal vectors [3])

$$|T^\lambda\rangle = (1-\lambda)\beta_{-1}\gamma_{-2}|0\rangle - \lambda\gamma_{-1}\beta_{-2}|0\rangle\,. \tag{3.10}$$

For $\alpha \notin \mathbb{Z}$, $\mathcal{B}^{(\alpha)}$ can again be viewed as a twisted module of this vertex algebra [16].

Similar to the fermionic ghost system, one can use the twisted Borcherds identity (compare appendix A and (5.7) of [16]) to obtain

$$J_m^{(\alpha)} = \sum_{k<-\alpha} \gamma_k^{(\alpha)}\beta_{m-1-k}^{(\alpha)} + \sum_{k\geq-\alpha} \beta_{m-1-k}^{(\alpha)}\gamma_k^{(\alpha)} - \delta_{m,0}\alpha\,, \tag{3.11}$$

while for the modes of the family of energy-momentum tensors (3.4) one finds

$$\begin{aligned} L_n^{(\lambda,\alpha)} = (1-\lambda) &\left(\sum_{k<\alpha} (k-n)\beta_k^{(\alpha)}\gamma_{n-1-k}^{(\alpha)} + \sum_{k\geq\alpha} (k-n)\gamma_{n-1-k}^{(\alpha)}\beta_k^{(\alpha)} \right) \\ -\lambda &\left(\sum_{k<-\alpha} (k-n)\gamma_k^{(\alpha)}\beta_{n-1-k}^{(\alpha)} + \sum_{k\geq-\alpha} (k-n)\beta_{n-1-k}^{(\alpha)}\gamma_k^{(\alpha)} \right) \\ + &\left(\lambda\alpha - \frac{\alpha(\alpha-1)}{2} \right) \delta_{n,0}\,. \end{aligned} \tag{3.12}$$

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[3]) One can show that these are the most general conformal vectors with respect to which the fundamental ghost fields are primary using the same argument as for the b-c system.

Hence the formula (2.16) extends to this case which is also in agreement with (3.4). Also (2.17) remains valid for the β-γ system with the obvious modification that $\mathcal{F}^{(\alpha)}$ should be replaced with $\mathcal{B}^{(\alpha)}$ in the definition of the character. At $\lambda = 0$ one finds (*c.f.* eq. (2.7) in [2])

$$\chi^{(0,\alpha)}(q,z) = q^{-\frac{1}{12}} q^{-\frac{\alpha(\alpha-1)}{2}} z^{-\alpha} \prod_{n=1}^{\infty} (1 - zq^{n+\alpha-1})^{-1}(1 - z^{-1}q^{n-\alpha})^{-1} . \tag{3.13}$$

Therefore, the most general form of the character reads

$$\chi^{(\lambda,\alpha)}(q,z) = q^{-\frac{1}{12}} q^{-\frac{1}{2}(\lambda+\alpha)(\lambda+\alpha-1)} z^{-\alpha} \prod_{n=1}^{\infty} (1 - zq^{n+(\alpha+\lambda)-1})^{-1}(1 - z^{-1}q^{n-(\alpha+\lambda)})^{-1} . \tag{3.14}$$

Note that for $\lambda = \frac{1}{2}$ and $\alpha = 0$ we indeed obtain the character of the (untwisted) $c = -1$ bosonic ghost system.

Again, apart from the prefactor $z^{-\alpha}$, the character (3.14) depends only on $\lambda+\alpha$ which generalises the similarity between the characters for the cases $\lambda = 0$ and $\lambda = \frac{1}{2}$ that was observed in [2]. As was explained in detail for the b-c system, this can be interpreted in terms of a spectral flow. Of course, the choice of reference point $\lambda = 0$ is arbitrary here as well. However, at this point one can identify the bosonic ghost system with a subspace of a complex boson $\varphi(z)$, $\bar{\varphi}(z)$ via $\gamma(z) = \varphi(z)$, $\beta(z) = \partial\bar{\varphi}(z)$ (in [2] a more symmetric mapping was given which though involves non-local operations such as dividing by $\sqrt{\partial}$).

4. Symmetries and vertex subalgebras

This section focuses on vertex subalgebras of the ghost systems that are invariant under a certain symmetry. We will be interested only in subalgebras that can be interpreted within the context of conformal field theory. Therefore, we require the chosen energy-momentum tensor to be contained within the invariant subspace.

The ghost systems always have a $U(1)$ symmetry. At the symmetric point $\lambda = \frac{1}{2}$ (*i.e.* $c = 1$ and $c = -1$ for the fermionic and bosonic ghost system, respectively), this symmetry is enhanced to $Sp(2,\mathbb{C})$ rotations of the two fundamental ghost fields. The energy-momentum tensor and the ghost-number current are invariant under these transformations. In addition, if following [12] one considers the "small" algebras generated by b and ∂c or β and $\partial\gamma$ (*i.e.* states containing no c_0 or no γ_0), one has a further extension to $Sp(2,\mathbb{C})$ on this subspace for $\lambda = 0$, *i.e.* at $c = -2$ or $c = 2$ respectively. The energy-momentum tensor is invariant in this case, too, but the ghost-number current is not even contained in this small algebra.

A twisted representation of a vertex algebra can be decomposed into untwisted representations of the vertex subalgebra that is invariant under the group used for the twist (see e.g. [32] for a detailed explanation in the case of $\mathbb{Z}_2$ and section 4A of [27] for a general discussion). Because of this, the study of vertex subalgebras that are invariant under a given symmetry can be quite useful, apart from being interesting in its own right. A method for studying such subalgebras using invariant theory in the spirit of Weyl [33] has been established in [34].

In this section we will investigate some $\mathbb{Z}_N$- and $U(1)$-invariant vertex subalgebras of the bosonic or fermionic ghost system, respectively. We start by presenting the necessary elements of $\mathbb{Z}_N$-invariant theory (the cases $\mathbb{Z}_2$ and $U(1)$ are discussed in detail in [34]; the latter case can be considered as the limit $N \to \infty$ of $\mathbb{Z}_N$). Let ζ be a generator of $\mathbb{Z}_N$ that acts on two basic fields X^+ and X^- via

$$\zeta X^+ = \omega X^+ \qquad\qquad \zeta X^- = \omega^{-1} X^- , \tag{4.1}$$

where ω is a primitive Nth root of unity ($\omega^N = 1$). Then a polynomial generating set of all invariants is given by

$$U^{m,n} := (\partial^m X^+)(\partial^n X^-) , \tag{4.2a}$$
$$V^{k_1,\ldots,k_N} := (\partial^{k_1} X^+)(\partial^{k_2} X^+) \cdots (\partial^{k_N} X^+) , \tag{4.2b}$$
$$W^{k_1,\ldots,k_N} := (\partial^{k_1} X^-)(\partial^{k_2} X^-) \cdots (\partial^{k_N} X^-) . \tag{4.2c}$$

On the quantum level suitable normal-ordering is understood.

For classical, (anti-)commuting fields $X^\pm$, the ring of all relations satisfied by the $\mathbb{Z}_N$-invariants is generated by [4])

$$U^{k,l} U^{m,n} - \epsilon^3 U^{k,n} U^{m,l} = 0 , \tag{4.3a}$$
$$U^{m,n} V^{k_1,k_2\ldots,k_N} - \epsilon^3 U^{k_1,n} V^{m,k_2,\ldots,k_N} = 0 , \tag{4.3b}$$
$$U^{m,n} W^{k_1,k_2\ldots,k_N} - \epsilon U^{m,k_1} W^{n,k_2,\ldots,k_N} = 0 , \tag{4.3c}$$
$$V^{k_1,\ldots,k_N} W^{l_1,\ldots,l_N} - \epsilon^{(n-1)!} U^{k_1,l_1} \cdots U^{k_N,l_N} = 0 , \tag{4.3d}$$

where $\epsilon = -1$ if $X^\pm$ are fermions and $\epsilon = 1$ if they are bosons.

The proof that (4.2) and (4.3) generate the subrings of all invariants and all (classical) relations, respectively goes as follows. Since the $\mathbb{Z}_N$-action (4.1) is diagonal, it suffices to consider monomials in the basic fields $X^\pm$. Denote the exponent of the field X^+ by r and the exponent of X^- by s. Then, a monomial is invariant under (4.1) precisely if $r - s$ is a multiple of N. From this monomial we can factor out $\lfloor r/N \rfloor$ fields of type V and $\lfloor s/N \rfloor$ fields of type W. Now we must have $r - N\lfloor r/N \rfloor = s - N\lfloor s/N \rfloor$, *i.e.* the remaining fields must occur in pairs such that we can write them in terms of U's. This proves that the fields (4.2) generate all invariants.

On the classical level, a simple sorting argument along the following lines shows that (4.3) generate all relations among the fields (4.2): First one defines a basis in the space of monomials in U's, V's and W's (*i.e.* a standard form of monomials), e.g. by the requirement that at most $N-1$ factors of type U appear and ordering the fields as well as their indices lexicographically. Then it is not difficult to convince oneself that an arbitrary monomial in the U's, V's and W's can be brought into this standard form using the relations (4.3) (and (anti-)commutativity of any two fields as well as identities between V's and W's differing only in their order of indices).

It should be mentioned that so far we have regarded a field and its derivatives as independent. Implementing the action of ∂ yields the following additional relations:

$$\partial U^{m,n} = U^{m+1,n} + U^{m,n+1} , \tag{4.4a}$$
$$\partial V^{k_1,\ldots,k_N} = V^{k_1+1,\ldots,k_N} + \ldots + V^{k_1,\ldots,k_N+1} , \tag{4.4b}$$
$$\partial W^{k_1,\ldots,k_N} = W^{k_1+1,\ldots,k_N} + \ldots + W^{k_1,\ldots,k_N+1} . \tag{4.4c}$$

---

[4]) (Anti-)commutativity of the U, V and W as well as identities between fields that differ only in their order of indices are understood to hold as well.

For $N = 2$ we clearly recover the results of [34] concerning $\mathbb{Z}_2$-invariants. Considering $U(1)$ as $\lim_{N\to\infty} \mathbb{Z}_N$, the invariants W and V become infinite order and thus drop out. With this interpretation we also recover the result of [34] that the $U(1)$-invariants are generated by (4.2a), their relations by (4.3a) and that the derivative acts as (4.4a).

The above consideration describes the structure of the classical case completely, a prominent feature being an infinite set of generators satisfying infinitely many relations [34]. However, our main concern here are vertex algebras, *i.e.* quantum fields. Then, the relations (4.3) acquire normal-ordering corrections. These should be computed explicitly to understand the structure of invariant vertex subalgebras, since normal-ordered relations may be used to eliminate all but finitely many of the generators (4.2) [34]. Such computations will be performed in the sequel. It is convenient to perform these computations in the vacuum Fock space rather than in a field formulation – for some useful formulae we refer to appendix A.

First, let us look at the *fermionic ghost system*. We set $X^+ = b$ and $X^- = c$ and start with the fields U that are present for all $\mathbb{Z}_N$ and $U(1)$. We can choose $U^{0,m}$ as an independent set of generators due to the action of the derivative (4.4a). For them the classical relations (4.3a) turn into $U^{0,m}U^{0,n} = 0$, which can easily be understood since the fermionic field b satisfies a Pauli principle. We now compute the corrections arising when normal-ordering this relation. With some algebra (and a few formulae from appendix A) one finds that

$$\begin{aligned}(U^{0,n}\,U^{0,m})_{-1}|0\rangle =&U^{0,n}_{-1}U^{0,m}_{-1}|0\rangle \\ =&\left(\sum_{k<0} b_k(\partial^n c)_{-2-k} - \sum_{k\geq 0}(\partial^n c)_{-2-k}b_k\right) m!\; b_{-1}c_{-1-m}|0\rangle \\ =&m!\; n!\; (-1)^n b_{-2-n}c_{-1-m}|0\rangle + \frac{(m+n+1)!}{m+1}c_{-2-m-n}b_{-1}|0\rangle \\ =&-\left\{\frac{1}{n+1}+\frac{1}{m+1}\right\}(m+n+1)!\; b_{-1}c_{-2-m-n}|0\rangle + L_{-1}(\ldots)\,.\end{aligned} \tag{4.5}$$

From this we infer that

$$(U^{0,n}\,U^{0,m}) = \frac{(-1)^n}{n+1}U^{n+1,m} - \frac{1}{m+1}U^{0,m+n+1} = -\left\{\frac{1}{n+1}+\frac{1}{m+1}\right\}U^{0,m+n+1} + \partial(\ldots)\,. \tag{4.6}$$

With this information we have already complete control over the $U(1)$-invariant subspace: Eq. (4.6) shows that it is generated by $U^{0,0}$ only and that one can express the $U^{0,n}$ with $n > 0$ as normal-ordered products or derivatives thereof. Specialization of this result to $\lambda = \frac{1}{2}$ amounts to a rederivation of the simplest case ($L = 1$) of the truncation of $\mathcal{W}_{1+\infty}$ to a $\mathcal{W}(1, 2, \ldots, L)$ at $c = L$ [25]. Here we use (the $L = 1$ special case of) the fact that $\mathcal{W}_{1+\infty}$ at $c = L$ is isomorphic to the $U(L)$-invariants of L copies of the complex free fermion [16, 25, 27]. The original proof [25] of these truncations involved character considerations and quantized Drinfeld-Sokolov reduction. Our approach using invariant theory suggests a different proof in the case of an arbitrary integer L, too.

The statement that the $U(1)$-invariant vertex subalgebra is a $\widehat{U(1)} = \mathcal{W}(1)$ can also be proven using a counting argument and in this case (4.6) would not be really needed.

First note that the Jacobi triple product identity (see e.g. (7.27) in [35]) allows one to rewrite the character (2.19) at $\lambda = \frac{1}{2}$ and $\alpha = 0$ as

$$\chi^{(\frac{1}{2},0)}(q,z) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} z^n \,. \tag{4.7}$$

Using then (2.17) as $\chi^{(\lambda,\alpha)}(q,z) = q^{-\frac{c_\lambda - 1}{24}} \chi^{(\frac{1}{2},\alpha)}(q, q^{\lambda-\frac{1}{2}} z)$ one concludes further that

$$\chi^{(\lambda,0)}(q,z) = \frac{q^{-\frac{c_\lambda-1}{24}}}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}+n(\lambda-\frac{1}{2})} z^n \,. \tag{4.8}$$

From this one reads off that the coefficient of each z^n is proportional to $1/\eta(q)$. In particular, the z-independent part equals $q^{-\frac{c_\lambda-1}{24}}/\eta(q)$. This is precisely the character of an irreducible $\widehat{U(1)}$-representation. Furthermore, $U^{0,0}$ ($= -J$ in (2.2)) clearly generates a $\widehat{U(1)}$-subalgebra of the $U(1)$-invariant vertex subalgebra. Now equality of the characters suffices to show that this vertex subalgebra must be equal to its $\widehat{U(1)}$-subalgebra.

Another example over which we have already full control is the $U(1)$-invariant subspace generated by b and ∂c at $\lambda = 0$, *i.e.* $c = -2$. Since we exclude c itself here, the invariant subring is now generated by $U^{0,n}$ with $n \geq 1$ and using (4.6) one can further eliminate those with $n \geq 3$. Thus, the invariant vertex subalgebra is now actually generated by $U^{0,1} = -T^0$ and $U^{0,2}$ which has conformal dimension three, *i.e.* we recover Zamolodchikov's seminal $\mathcal{W}(2,3)$ [36] at $c = -2$. A realization of $\mathcal{W}(2,3)$ at $c = -2$ in terms of the b-c system has already been given some time ago [37], but the identification with the $U(1)$-invariant subspace is new.

Now we turn to $\mathbb{Z}_N$-invariant vertex subalgebras of the b-c system. Then we have to include the generators (4.2b) and (4.2c) and study the normal-ordered versions of the relations (4.3b) and (4.3c). The first non-vanishing new generators are $V^{0,\ldots,N-1}$ and $W^{0,\ldots,N-1}$. It is straightforward to check by elementary manipulations that

$$U^{0,0}_n V^{0,\ldots,N-1}_{-1}|0\rangle = U^{0,0}_n W^{0,\ldots,N-1}_{-1}|0\rangle = 0 \qquad\qquad \forall n > 0\,. \tag{4.9}$$

This means that both $V^{0,\ldots,N-1}_{-1}|0\rangle$ and $W^{0,\ldots,N-1}_{-1}|0\rangle$ are highest weight vectors for the $\widehat{U(1)}$-algebra generated by $U^{0,0}$ (the highest weight property can also be inferred from (4.8) since the character shows that there are no states whose conformal weight and ghost number are equal to those of the expressions in (4.9)).

Now we can again apply the counting argument that we already used before: The character of an irreducible $\widehat{U(1)}$-module equals $1/\eta(q)$ which is proportional to the coefficient of $z^{\pm N}$ in (4.8). So, we can obtain all fields with ghost number $-N$ and N by iterative normal-ordered products of $U^{0,0}$ and its derivatives with $V^{0,\ldots,N-1}$ and $W^{0,\ldots,N-1}$, respectively. This includes in particular all the fields (4.2b) and (4.2c). So, we have shown that the $\mathbb{Z}_N$-invariant vertex subalgebra is generated by $U^{0,0}$, $V^{0,\ldots,N-1}$ and $W^{0,\ldots,N-1}$ which is an algebra of type $\mathcal{W}(1, \frac{N^2}{2} + (\lambda - \frac{1}{2})N, \frac{N^2}{2} + (\frac{1}{2} - \lambda)N)$ for generic λ. At $c = 1$ we obtain a $\mathcal{W}(1, \frac{N^2}{2}, \frac{N^2}{2})$ which we believe to be the same as the algebra $\mathfrak{A}(N^2)$ constructed in Example 6.1 of [27] by other methods. Characters of this algebra are given in appendix B where we also establish a relation to a free boson compactified at radius $r = \frac{N}{2}$.

For the $\mathbb{Z}_N$-invariants at $c = -2$ ($\lambda = 0$) built out of b and ∂c we expect that the invariants are generated by $U^{0,1}$, $U^{0,2}$, $V^{0,\ldots,N-1}$ and $W^{1,\ldots,N}$ although we have in general no proof of this. In other words: we conjecture the $\mathbb{Z}_N$-invariant vertex subalgebra of b and ∂c to be of type $\mathcal{W}(2, 3, \frac{N(N+1)}{2}, \frac{N(N+1)}{2})$. We are now going to prove this at least for $N = 2$ where we find a $\mathcal{W}(2, 3^3)$ at $c = -2$ – an algebra which was constructed in [38]. Note that a construction of $\mathcal{W}(2, 3^3)$ in terms of the b-c system was already given in [12] though without a proof of being identical to any invariant vertex subalgebra. In this case we can select $V^{0,m}$ and $W^{1,m}$ as generators due to (4.4b) and (4.4c). A straightforward computation shows that

$$
\begin{aligned}
(U^{0,1}\, V^{0,m}) &= -\frac{m+4}{2(m+2)} V^{0,m+2} + \partial V^{0,m+1} - \frac{1}{2}\partial^2 V^{0,m}\,, \\
(U^{0,1}\, W^{1,m}) &= -\frac{m+3}{2(m+1)} W^{1,m+2} + \partial W^{1,m+1} - \frac{1}{2}\partial^2 W^{1,m}\,.
\end{aligned}
\tag{4.10}
$$

Using these relations one can eliminate $V^{0,m}$ for $m \geq 1$ and $W^{1,m}$ for $m \geq 2$. Therefore, the $\mathbb{Z}_2$-invariant vertex subalgebra of b and ∂c at $c = -2$ is generated by $U^{0,1}$, $U^{0,2}$, $V^{0,1}$ and $W^{1,2}$ and thus of type $\mathcal{W}(2, 3^3)$.

We conclude the discussion of the b-c system by defining an action of $Sp(2, \mathbb{C})$ on the subspace $\mathcal{W}$ generated by the two fields b and ∂c at $\lambda = 0$, *i.e.* $c = -2$ following [12]. For $G \in Sp(2, \mathbb{C})$ this action is simply given by $(b, \partial c)^t \mapsto G(b, \partial c)^t$. This action leaves the energy-momentum tensor (2.5) invariant. Special subgroups of this $Sp(2, \mathbb{C})$ are the groups $U(1)$ (which we have already studied in some detail) and $\mathbb{Z}_4$ acting by the following generators $X(a)$ and $\mathcal{J}$, respectively

$$
\begin{aligned}
X(a) b = a\, b\,, \quad & X(a)\partial c = a^{-1}\, \partial c\,, \\
\mathcal{J} b = \partial c\,, \quad & \mathcal{J}\partial c = -b\,.
\end{aligned}
\tag{4.11}
$$

The reason for explicitly writing down the action of the semidirect product $U(1) \times_s \mathbb{Z}_4$ on $\mathcal{W}$ is given by the first inclusion in the sequence

$$
\mathcal{W}(2, 10) \subset \mathcal{W}(2, 3) \subset \mathcal{W}(2, 3^3) \subset \mathcal{W}\,, \tag{4.12}
$$

which summarizes some of our results for $c = -2$. Indeed, we have seen above that the subalgebra of $\mathcal{W}$ which is invariant under $\mathbb{Z}_2 =< X(-1) >=< \mathcal{J}^2 >$ is the algebra of type $\mathcal{W}(2, 3^3)$ constructed in [38] by direct methods. Primary generators for it have been constructed in terms of the ghost system [12] and are explicitly given by eqs. (39)-(41) loc.cit. We have also seen that the $U(1)$-invariant subalgebra is Zamolodchikov's $\mathcal{W}(2, 3)$ [36] at $c = -2$. Its primary spin three generator is the same linear combination of $U^{0,2}$ and $\partial U^{0,1}$ as the generator with zero ghost number among the ones of $\mathcal{W}(2, 3^3)$ (eq. (40) of [12]). Now, let us look at the $U(1) \times_s \mathbb{Z}_4$ invariant subalgebra. One can first project to the $U(1)$-invariant subalgebra $\mathcal{W}(2, 3)$ and is then left with a $\mathbb{Z}_2$ whose generator is the generator $\mathcal{J}$ of the original $\mathbb{Z}_4$. $\mathcal{J}$ acts on the primary spin three generator of this $\mathcal{W}(2, 3)$ just by a flip of sign. Thus the $U(1) \times_s \mathbb{Z}_4$-invariant subspace is the $\mathbb{Z}_2$-orbifold of $\mathcal{W}(2, 3)$ at $c = -2$ which was discussed in section 2.2.2 of [17] and argued to be the $\mathcal{W}$-algebra of type $\mathcal{W}(2, 10)$ constructed by direct methods in [39]. So far, it has actually only been proven that the $U(1) \times_s \mathbb{Z}_4$-invariant algebra contains this $\mathcal{W}(2, 10)$ as its subalgebra, neither do we prove equality here. In any case, the inclusions indicated by (4.12) at $c = -2$ have been established rigorously.

We conclude this section with a look at the $U(1)$-invariant vertex subalgebra of the *bosonic ghost system*. We now set $X^+ = \beta$ and $X^- = \gamma$ and then can choose the $U^{0,n}$ as generators among (4.2a). Now one has to compute the effect of normal-ordering on the relations (4.3a). A computation along the same lines as (4.5) leads to

$$\begin{aligned}(U^{m,n}\,U^{i,j}) - (U^{m,j}\,U^{i,n}) = {}& \frac{(-1)^{n+1}}{i+n+1} U^{i+m+n+1,j} - \frac{(-1)^{j+1}}{i+j+1} U^{i+m+j+1,n} \\ & + (-1)^m \left\{ \frac{1}{m+j+1} - \frac{1}{m+n+1} \right\} U^{i,m+j+n+1} \\ ={}& (-1)^{i+m} \left\{ \frac{1}{n+i+1} + \frac{1}{m+j+1} - \frac{1}{i+j+1} - \frac{1}{m+n+1} \right\} U^{0,m+n+i+j+1} \\ & + \partial(\ldots)\,.\end{aligned} \tag{4.13}$$

This is analogous to (4.6) for the b-c system although here we have a two-term relation, since there is no Pauli principle.

The coefficient of $U^{0,m+n+i+j+1}$ in (4.13) vanishes if and only if $i = m$ or $n = j$. As a simple consequence we infer that all $U^{0,l}$ with $l \geq 3$ can be eliminated using (4.13) and thus the $U(1)$-invariant vertex subalgebra of the β-γ system is an algebra of type $\mathcal{W}(1,2,3)$ generated by $U^{0,0}$, $U^{0,1}$ and $U^{0,2}$. This is valid for all λ, thus also for $c = -1$ ($\lambda = \frac{1}{2}$). Using the construction of [16] we have thus rigorously proven the conjecture of [40] that $\mathcal{W}_{1+\infty}$ truncates to a $\mathcal{W}(1,2,3)$ at $c = -1$. We should mention that a proof of this fact which is also based on the construction of [16] has very recently been worked out independently in [41]. Actually, a slightly stronger statement was proven in [41], namely that $\mathcal{W}_{1+\infty}$ at $c = -1$ can be written as $\widehat{U(1)} \oplus \mathcal{W}(2,3)$ with the $\mathcal{W}(2,3)$ [36] at $c = -2$. Since this algebra is a $\mathcal{W}(1,2,3)$ at $c = -1$, the result of [41] implies ours. The converse, *i.e.* that a $\widehat{U(1)}$ can be factored out without affecting the field content of the remainder is a special property of $\mathcal{W}_{1+\infty}$ which is true for all c (see e.g. [17]).

More generally, it has been conjectured in [17] that $\mathcal{W}_{1+\infty}$ at $c = -L$ truncates to a $\mathcal{W}(1,2,\ldots,(L+1)^2-1)$. This conjecture is based on and supported by results of [26]. Since it was shown in [16] that $\mathcal{W}_{1+\infty}$ at $c = -L$ can be identified with the $U(L)$-invariant vertex subalgebra of L copies of the bosonic ghost system, we expect that this conjecture can be proven by generalizing the present computation to $U(L)$-invariants. More specifically, one would need to derive a generalization of (4.13) using the relations (VI) on p. 71 of [33] of which (4.3a) is the $L = 1$ special case.

At $c = 2$ ($\lambda = 0$) we can consider the $U(1)$-invariant vertex subalgebra generated by β and $\partial\gamma$. Now we can eliminate all $U^{0,l}$ with $l \geq 5$ using (4.13) and thus obtain a $\mathcal{W}(2,3,4,5)$ at $c = 2$ generated by $U^{0,n}$ with $n \in \{1,2,3,4\}$. This algebra is actually the $c = 2$ special case of a deformable $\mathcal{W}(2,3,4,5)$ that was first constructed by direct methods in [42], as can be understood as follows: The $\mathcal{W}(2,3,4,5)$ of [42] can be realized in terms of the coset $\widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ [34, 17] and $c = 2$ corresponds to the limit $k = \infty$. Using this coset realization and general arguments along the lines of [34] one can see that the limit $k \to \infty$ yields precisely the $U(1)$-invariants of J^- and J^+/k, which have finite OPEs in the limit $k \to \infty$ and can then be identified with β and $\partial\gamma$ using the Wakimoto realization of $\widehat{sl(2,\mathbb{R})}_k$ (see e.g. (15.279) in [9]). In this manner also the truncation of $\mathcal{W}_\infty$ at $c = 2$ to a $\mathcal{W}(2,3,4,5)$ [40] arises naturally, since our construction clearly gives rise to a linear algebra in the basis of the $U^{0,l}$. Actually, truncations of $\mathcal{W}_\infty$ are expected for $c \in 2\mathbb{Z}$ [40, 17] and one may speculate that our result on the $U(1)$-invariants generated by β and $\partial\gamma$ can be generalized in a way similar to $\mathcal{W}_{1+\infty}$.

Finally, recall that at the symmetric point $\lambda = \frac{1}{2}$ ($c = -1$), the symmetry is enlarged to an $Sp(2,\mathbb{C})$. The invariants of this group were studied in [34]. There it was shown by an explicit computation of normal-ordered relations that the $Sp(2,\mathbb{C})$-invariant vertex subalgebra of the β-γ system is of type $\mathcal{W}(2,4,6)$ [5]). The computation of [34] differs from the present one slightly in that it used rearrangement lemmas, but to the best of our knowledge this is the only non-trivial case where an invariant vertex subalgebra was rigorously shown to be of a certain type (viewed as a $\mathcal{W}$-algebra) prior to the present work.

5. Conclusion

The main results of this paper are summarized as follows:

In the first part we formulated fermionic and bosonic ghost systems as a single vertex algebra each. These vertex algebras admit the one-parameter families (2.12) and (3.10) of conformal vectors. We observed the simple relation (2.16) between the actions of the Virasoro algebra in each family. A corollary is the identity (2.17) between characters for different choices of conformal vectors. With this information one readily obtains the explicit formulae (2.19) and (3.14) for the characters of a general twisted module of a ghost system.

The second part was devoted to symmetries and vertex subalgebras invariant under them. First, we have discussed the invariant theory for the group $\mathbb{Z}_N$, including previous results [34] for $\mathbb{Z}_2$ and $U(1)$ as limiting cases. Then we have studied the $\mathbb{Z}_N$-invariant subalgebras of the fermionic ghost system in detail and shown that they are $\mathcal{W}$-algebras of type $\mathcal{W}(1, \frac{N^2}{2} + (\lambda - \frac{1}{2})N, \frac{N^2}{2} + (\frac{1}{2} - \lambda)N)$. At $c = -2$ the $\mathbb{Z}_N$-invariant subalgebras based on b and ∂c (*i.e.* omitting c) are conjectured to be of type $\mathcal{W}(2, 3, \frac{N(N+1)}{2}, \frac{N(N+1)}{2})$. However this has been shown rigorously only for $N = 2$ and $N = \infty$ where we recover a $\mathcal{W}(2,3^3)$ [38, 12] and a $\mathcal{W}(2,3)$ [36] at $c = -2$, respectively. Finally, we showed that the $U(1)$-invariant subalgebra of the generic bosonic ghost system is an algebra of type $\mathcal{W}(1,2,3)$. For $\lambda = 0$ the $U(1)$-invariant subalgebra of β and $\partial\gamma$ is a $\mathcal{W}(2,3,4,5)$ [42] with $c = 2$.

We should mention that the ghost-number currents (2.2) and (3.2) are primary only at the symmetric point $\lambda = \frac{1}{2}$. This means that if (as is sometimes done) we require the generators of a $\mathcal{W}$-algebra to be primary fields, we have to specialize the $\mathcal{W}(1, \frac{N^2}{2} + (\lambda - \frac{1}{2})N, \frac{N^2}{2} + (\frac{1}{2} - \lambda)N)$ and the $\mathcal{W}(1,2,3)$ obtained from the fermionic and bosonic ghost system, respectively to this symmetric point ($c = 1$ and $c = -1$, respectively).

The question of (generalized) rationality of the conformal field theories associated to the algebras studied in this paper remains to be investigated. This will involve logarithmic

[5]) This algebra is the $c = -1$ special case of a $\mathcal{W}(2,4,6)$ constructed in [43] by direct methods. The identification at $c = -1$ and its generalizations in [34] solved the problem [39] of identifying this $\mathcal{W}(2,4,6)$-algebra.

fields [44] since they are known to appear in certain rational conformal field theories at $c = -2$ [45, 46].

In the present paper we have always worked with a single ghost system. The discussion of the first part generalizes to several copies with arbitrarily chosen energy-momentum tensor and twist in each copy in the obvious way. A generalization of our study of invariant vertex subalgebras to several copies and other groups is more difficult, but would be very interesting. One application would be to check the conjecture of [17] on the structure of $\mathcal{W}_{1+\infty}$ at $c = -L$ using its construction in terms of the $U(L)$-invariants in L copies of the $c = -1$ bosonic ghost system [16]. More generally, in this way one may hope to gain further insight into the structure of conformal field theories at integer central charge.

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Appendix A. Twisted modules & useful formulae

In this appendix we summarize the main ingredients of the definition of twisted modules of vertex algebras [16] and add the necessary signs to include fermions. We also collect some useful formulae for the computations performed in this paper.

A twisted vertex operator Y assigns a twisted field $Y(|a\rangle, z)$ to any state $|a\rangle$ of a vertex algebra. When Y is the vacuum vertex operator the twist is trivial and this map becomes an isomorphism which also plays a central rôle in conformal field theory (see e.g. [47]). Throughout the remainder of this paper we use the notation

$$f(z) = Y(|f\rangle, z) = \sum_{n \in \phi+\mathbb{Z}} f_n^{(\alpha)} z^{-n-1}, \tag{A.1}$$

where $|f\rangle = f_{-1}|0\rangle$ is a state in the vacuum module of the vertex algebra and $f(z)$ is a ϕ-twisted field; only for the purposes of the appendix we keep the vertex map Y explicitly. Here the modes of a twisted field are denoted similarly to the bulk of the paper by an α as upper index characterizing the twisted module (which we omit for the vacuum module).

A twisted module is then characterized by the following three axioms [16]:

(M1) $Y(|0\rangle, z) = \mathbb{1}^{(\alpha)}$,

(M2) $Y(L_{-1}|a\rangle, z) = \partial Y(|a\rangle, z)$, (M3) (twisted Borcherds identity)

$$\sum_{j=0}^{\infty} \binom{m}{j} Y(a_{n+j}|b\rangle, z) z^{m-j} = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \Big\{ a^{(\alpha)}_{m+n-j} Y(|b\rangle, z) z^j \\ - (-1)^n \epsilon_{a,b} Y(|b\rangle, z) a^{(\alpha)}_{m+j} z^{n-j} \Big\} , \tag{A.2}$$

where $n \in \mathbb{Z}$, $m \in \mathbb{Z} + \alpha$ and α is the twist of the field $a(z)$. Here $\epsilon_{a,b} = -1$ if both a and b are fermions and $\epsilon_{a,b} = 1$ otherwise.

The $m = \alpha$, $n = -1$ special case of (A.2) yields the following formula for normal-ordered products in a twisted module [16]:

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} Y(a_{j-1}|b\rangle, z) z^{-j} = \, : Y(|a\rangle, z) Y(|b\rangle, z) : , \tag{A.3}$$

where the normal-ordered product of the two fields a and b is defined with the mode-expansion (2.7) by

$$: Y(|a\rangle, z) Y(|b\rangle, z) :_n = \sum_{k<-\alpha} a_k b_{n-1-k} + \epsilon_{a,b} \sum_{k \geq -\alpha} b_{n-1-k} a_k \, . \tag{A.4}$$

Here as in (M3) $\epsilon_{a,b} = 1$ unless both a and b are fermions, and α is the twist of a.

It should be noted that one usually also requires that for all $|v\rangle$ in the twisted module and all $|a\rangle$ in the vertex algebra $a_n^{(\alpha)}|v\rangle$ vanishes for n large enough (in the cases we consider this is indeed true as we are only dealing with highest weight representations). This ensures that the sums on the l.h.s. of (A.2) and also (A.3) are actually finite. In fact, for our purposes the sums are truncated in a manner that at most three terms in the vacuum representation of the ghost systems need to be considered.

We conclude this axiomatic part by mentioning that a vertex algebra is called simple if its vacuum representation is irreducible.

Finally, we collect a few useful formulae. Without a twist one has the highest-weight condition for the vacuum vector $|0\rangle$: $\phi_k|0\rangle = 0$ for all $k \geq 0$. Combining this with the specialization of (A.4) to a trivial twist implies $(a\,b)_{-1}|0\rangle = a_{-1}b_{-1}|0\rangle$.

From the mode expansion (2.7) one finds furthermore

$$(\partial^n \phi)_k = (-1)^n k(k-1) \cdots (k-n+1) \phi_{k-n} \, . \tag{A.5}$$

Finally, the Virasoro-generators $L_m^{(\alpha)}$ act as

$$[L_m^{(\alpha)}, \phi_n^{(\alpha)}] = \left((\dim(\phi) - 1)(m+1) - n \right) \phi^{(\alpha)}_{m+n} \, , \tag{A.6}$$

which is valid

- for all m if ϕ is a *primary* field of conformal dimension $\dim(\phi)$,
- for $m = 0$ and $m = \pm 1$ if ϕ is a *quasi-primary* field of conformal dimension $\dim(\phi)$,
- for $m = 0$ and $m = -1$ if ϕ has a definite conformal dimension $\dim(\phi)$, and
- for $m = -1$ always.

In particular, from combination of (A.5) with (A.6) one recovers the axiom (M2).

Appendix B. Orbifold characters

In this appendix we discuss the characters of the $\mathbb{Z}_N$-orbifolds of the fermionic ghost system at $c=1$ (*i.e.* the complex fermion).

For this end we need a suitable form of the characters of the fermionic ghost system with a twist α. One reads off from (2.19) that $\chi^{(\lambda,\alpha)}(q,z) = z^\alpha \chi^{(\lambda+\alpha,0)}(q,z)$. Inserting this into (4.8) one finds

$$\chi^{(\frac{1}{2},\alpha)}(q,z) = \frac{z^\alpha q^{\frac{\alpha^2}{2}}}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}+n\alpha} z^n \,. \tag{B.1}$$

Passing now to the $\mathbb{Z}_N$-orbifolds, each module of the ghost system with twist $N\alpha \in \mathbb{Z}$ will split into untwisted modules of the orbifold algebra. The characters of the modules of the orbifold are just the coefficient of ω^n in $\chi^{(\frac{1}{2},\alpha)}(q,\omega)/\omega^\alpha$ (where ω is the primitive Nth root of unity). There are N twisted modules, each splitting into N modules for the orbifold and therefore the orbifold algebra has a total of N^2 characters. One obtains from (B.1) and with a few manipulations the following orbifold-characters

$$\chi_s(q) = \frac{q^{\frac{s^2}{2N^2}}}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{\frac{n^2N^2}{2}+ns} = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{\frac{(s+nN^2)^2}{2N^2}} \,, \tag{B.2}$$

where $0 \le s \le N^2-1$. This yields the partition function

$$Z(q) = \sum_{s=0}^{N^2-1} \chi_s(\bar{q})\chi_s(q) = \frac{1}{\eta(\bar{q})\eta(q)} \sum_{\substack{\bar{k},k\in\mathbb{Z} \\ \bar{k}-k\in N^2\mathbb{Z}}} \bar{q}^{\frac{\bar{k}^2}{2N^2}} q^{\frac{k^2}{2N^2}} \,. \tag{B.3}$$

From this one immediately recognizes the partition function of a free boson compactified at radius $r=\frac{N}{2}$ if N is even (in the notation of [13]). This identification is also valid for odd N, with the only further subtlety that the Ramond sector of the orbifold algebra must be added to (B.3) to recover the full modular invariant partition function of the free boson at radius $r=\frac{N}{2}$.

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