$\mathcal{W}$-Algebras in Conformal Field Theory

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Abstract

Quantum $\mathcal{W}$-algebras are defined and their relevance for conformal field theories is outlined. We describe direct constructions of $\mathcal{W}$-algebras using associativity requirements. With this approach one explicitly obtains the first members of series of $\mathcal{W}$-algebras, including in particular 'Casimir algebras' (related to simple Lie algebras) and extended symmetry algebras corresponding to special Virasoro-minimal models. We also describe methods for the study of highest weight representations of $\mathcal{W}$-algebras. In some cases consistency of the corresponding quantum field theory already imposes severe restrictions on the admitted representations, i.e. allows to determine the field content. We conclude by reviewing known results on $\mathcal{W}$-algebras and RCFTs and show that most known rational conformal fields theories can be described in terms of Casimir algebras although on the level of $\mathcal{W}$-algebras exotic phenomena occur.

1. Introduction

The aim of this talk is to summarize briefly known facts about $\mathcal{W}$-algebras and the corresponding rational conformal field theories (RCFTs). Much more detailed reviews on this subject exist (see e.g. [1]) which we also recommend for further references. This talk is based on our efforts to find all $\mathcal{W}$-algebras with few fields of low conformal dimension and to fit them into the known patterns.

Virasoro minimal models have $c < 1$ [2]. But in statistical mechanics second order phase transitions are known which exhibit conformal invariance and lead to rational conformal field theories with $c > 1$. One class of examples are the $\mathbb{Z}_k$-parafermions [3]. Also in string theory one needs $c > 1$. Very important applications require $N = 2$ supersymmetric rational conformal field theories (RCFTs) with $c = 9$ [4].

Consequently, the classification of all RCFTs is a natural question. Since RCFTs with $c > 1$ can be constructed using $\mathcal{W}$-algebras one may expect that $\mathcal{W}$-algebras play a major rôle in this classification program.

Anyhow, one important motivation for the study of $\mathcal{W}$-algebras certainly is that $\mathcal{W}$-algebras have a very rich mathematical structure, their complete classification still being an open problem.

2. Basic Results on Virasoro Minimal Models

In this talk we restrict to the (left-)chiral part of the conformal field theory.
In our notations, the Virasoro algebra is given by

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{1}$$

where we adopt non-standard sign conventions in order to have the 'energy' L_0 bounded from below. Highest weight representations of the Virasoro algebra are defined by a cyclic vector $|h\rangle$ with the following properties:

$$L_0 \,|h\rangle = h\,|h\rangle \ , \qquad L_n\,|h\rangle = 0 \qquad \forall n < 0. \tag{2}$$

The irreducible representation modules are:

$$\mathcal{M}(c,h) := \frac{\mathrm{span}\{L_{n_k}\ldots L_{n_1}\,|h\rangle \mid n_k \geq n_{k-1} \geq \ldots n_1 > 0\}}{\{\text{maximal proper submodule}\}} \tag{3}$$

where the maximal proper submodule depends on h and c. One particular representation is the 'vacuum representation':

$$L_n|v\rangle = 0 \qquad \forall n < 2 \tag{4}$$

In the seminal work of BPZ in 1984 it was shown that in certain cases one can solve the field theory completely [2]. These so-called 'minimal models' are related to completely degenerate representations which are given by

$$c = 1 - 6\frac{(p-q)^2}{pq} \tag{5a}$$

$$h(p,q;r,s) = \frac{(pr-qs)^2-(p-q)^2}{4pq}\ , \quad 1 \leq r \leq q-1\ , \ \ 1 \leq s \leq p-1\,. \tag{5b}$$

with p, q coprime integers. For the irreducible representation modules $\mathcal{M}(c,h)$ one defines characters by:

$$\chi_h(\tau) := \mathrm{tr}_{\mathcal{M}(c,h)}\big(e^{2\pi i(L_0 - \frac{c}{24})\tau}\big). \tag{6}$$

One important tool is the modular group $PSL(2,\mathbb{Z})$ generated by:

$$T:\ \tau \mapsto \tau + 1\ , \qquad S:\ \tau \mapsto -\frac{1}{\tau}. \tag{7}$$

The characters (6) carry a natural representation of the modular group. The models (5) are rational in the sense that the representation of the modular group (7) on the characters (6) is finite dimensional. We will regard the property that characters form a finite dimensional representation of the modular group as a definition of RCFTs. In particular, the partition function

$$Z(\tau) := \sum_{i,j} N_{i,j} \chi^*_{h_i}(\tau) \chi_{h_j}(\tau) \tag{8}$$

of the complete theory is modular invariant if the integers are chosen appropriately. The 'diagonal' choice $N_{i,j} \sim \delta_{i,j}$ is always a solution. For a rational conformal field theory, the operator product expansion (OPE) of fields in the complete theory closes in finitely many families.

3. Definition of $\mathcal{W}$-Algebras

The Virasoro algebra has only rational models for $c < 1$ (see Eq. (5a)), even $c_{\text{eff}} = c - 24h_{\text{min}} = 1 - \frac{6}{pq} < 1$. As we mentioned in the introduction applications also need rational conformal field theories with $c > 1$. One class of examples are particular 2D k-states spin models, the so-called $\mathbb{Z}_k$ parafermions which were shown in 1985 by Fateev and Zamolodchikov to give rise to conformal field theories with $c = \frac{2k-2}{k+2}$ describing their second order phase transition [3]. Already in this construction, additional operators were introduced which mapped between different representation modules Eq. (3), or equivalently comprised several conformal families into a single one. If one can thus combine infinitely many conformal families into a single family one may hope to describe the field theory with finitely many of these bigger families. The original approach used chirally non-local fields. However, one would like to use better behaved chiral fields for such a mapping: The 'primary' local chiral fields. This leads to the definition of a $\mathcal{W}$-algebra. The first examples were constructed by Zamolodchikov in 1986 [5] which initiated a detailed study where also the $\mathbb{Z}_k$ parafermions found a new explanation as unitary minimal models of certain $\mathcal{W}$-algebras.

A local chiral primary field $\phi(z) = \sum_{n-d(\phi)\in\mathbb{Z}} \phi_n z^{n-d(\phi)}$ of conformal dimension $d(\phi) \in \frac{\mathbb{Z}_+}{2}$ is characterized by the commutator of its modes with the Virasoro algebra Eq. (1):

$$[L_m, \phi_n] = (n - (d(\phi) - 1)m)\phi_{n+m}. \tag{9}$$

$\mathcal{W}$-algebras encode properties of conformal field theory algebraically. A derivative ∂ is naturally defined in the space of fields. Furthermore, the singular part of the OPE of two fields gives rise to a Lie bracket of their modes whereas the regular part of the OPE leads to some standard normal ordering prescription $N(\phi, \psi)$ of two fields ϕ and ψ.

One of the first observations is that the commutator does not close in primary fields Eq. (9) only. Therefore, one is forced to introduce more general, so-called 'quasi-primary' fields. Iff a field ϕ satisfies Eq. (9) for $m \in \{-1,0,1\}$ this field is called quasi-primary.

Using conformal covariance one can deduce a general formula for the commutator of the modes of two quasi-primary local chiral fields [6]:

$$[\phi_m^{(i)}, \phi_n^{(j)}] = d_{ij}\, \delta_{n,-m} \binom{n + d(\phi^{(i)}) - 1}{2d(\phi^{(i)}) - 1} + \sum_{\substack{k\in I \\ d(\phi^{(k)})<d(\phi^{(i)})+d(\phi^{(j)})}} C_{ij}^k \; p_{ijk}(m,n)\, \phi_{m+n}^{(k)}. \tag{10}$$

The universal polynomials p_{ijk} depend only on the conformal dimensions $d(\phi^{(i)})$, $d(\phi^{(j)})$ and $d(\phi^{(k)})$ and are known explicitly [6–9]. The structure constants C_{ij}^k and d_{ij} are characteristic for the fields involved ($\phi^{(i)}$, $\phi^{(j)}$ and $\phi^{(k)}$). They are algebraic functions of the central charge c. If one requires the existence of a vacuum vector $|v\rangle$ satisfying

$$\phi_n|v\rangle = 0 \qquad \forall n < d(\phi) \tag{11}$$

the structure constants C_{ij}^k and d_{ij} can be expressed in terms of the three- and two-point functions $C_{ijk} = \langle v|\phi^{(k)}_{-d(\phi^{(k)})}\phi^{(i)}_{d(\phi^{(k)})-d(\phi^{(j)})}\phi^{(j)}_{d(\phi^{(j)})}|v\rangle$ and $d_{ij} = \langle v|\phi^{(i)}_{-d(\phi^{(i)})}$ $\phi^{(j)}_{d(\phi^{(j)})}|v\rangle$ with $\sum_k d_{lk}C_{ij}^k = C_{ijl}$. Thus, one is in principle able to calculate structure constants for composite fields.

The normal ordering prescription can be used to write down a convenient basis for the space of fields. However, the commutator formula is valid only for quasi-primary fields but the naive normal ordered product $N(\phi^{(i)}, \partial^n\phi^{(j)})$ is usually not quasi-primary. Therefore, we add correction terms and define a quasi-primary normal ordering prescription $\mathcal{N}$ [6]:

$$\begin{aligned}\mathcal{N}(\phi^{(i)}, \partial^n\phi^{(j)}) := N(\phi^{(i)}, \partial^n\phi^{(j)}) &- \sum_{r=1}^{n} \alpha_{ij}^r \; \partial^r N(\phi^{(j)}, \partial^{n-r}\phi^{(i)}) \\ &- \sum_k \beta_{ij}^k(n)\; C_{ij}^k \; \partial^{h(ijk)-n}\phi^{(k)}\end{aligned} \tag{12}$$

yielding a field of dimension $d(\phi^{(i)})+d(\phi^{(j)})+n$. Again the α_{ij}^r and $\beta_{ij}^k(n)$ are some universal, explicitly known polynomials in the dimensions of $\phi^{(i)}$ and $\phi^{(j)}$ as well as n, r and k [6,8]. The exponent $h(ijk)$ is fixed to give the correct scaling dimension.

The algebra generated by finitely many simple (i.e. non-composite) fields $\phi^{(1)},\ldots,\phi^{(k)}$ is called a '$\mathcal{W}(d(\phi^{(1)}),\ldots,d(\phi^{(k)}))$'.

Apart from the energy-momentum tensor L all simple fields are primary.

When performing explicit constructions one gives as input the dimensions of the simple fields. All structure constants can be expressed in terms of those connecting three simple fields and the central element c. The structure constants involving three simple fields and possibly c are fixed by checking the validity of the Jacobi identity for all simple fields.

It may turn out that for given simple fields there is no solution at all, there can be solutions for particular discrete values of the central charge c or the central charge is not restriced by the Jacobi identities. In some examples, one finds even several generically existing $\mathcal{W}$-algebras with the same dimensions of the simple fields. Thus, our notation of a $\mathcal{W}$-algebra is not always unique.

Example: $\mathcal{W}(2,3)$

In our notation Zamolodchikov's $\mathcal{W}(2,3)$ [5] is given by the following commutation relations of the simple fields L and W:

$$\begin{aligned}
[L_m, L_n] &= (n-m)\, L_{m+n} + \frac{c}{12}\,(n^3-n)\,\delta_{n,-m} \\
[L_m, W_n] &= (n-2m)\, W_{m+n} \\
[W_m, W_n] &= C_{WW}^{L}\, p_{332}(m,n)\, L_{m+n} + C_{WW}^{\Lambda}\, p_{334}(m,n)\, \Lambda_{m+n} + \frac{c}{3}\binom{n+2}{5}\,\delta_{n,-m}\ ,
\end{aligned} \tag{13}$$

where

$$\Lambda = \mathcal{N}(L,L) = N(L,L) - \frac{3}{10}\partial^2 L \tag{14a}$$

$$C_{WW}^{L} = 2, \qquad C_{WW}^{\Lambda} = \frac{32}{5c+22} \tag{14b}$$

$$p_{334}(m,n) = \frac{n-m}{2}\ , \qquad p_{332}(m,n) = \frac{n-m}{60}(2m^2 - mn + 2n^2 - 8)\ . \tag{14c}$$

Note that the structure constants Eq. (14b) and the polynomials Eq. (14c) are fixed by the conformal symmetry up to normalization. Validity of the Jacobi identity is then automatically ensured by the universal polynomials. This is a very special property of $\mathcal{W}(2,3)$ – in general Jacobi identities do yield non-trivial restrictions.

The appearance of the field Λ clearly shows the non-linear structure of this algebra. The commutator of this spin 4 field with itself will involve fields of higher dimension and so on.

3. Representations of $\mathcal{W}$-Algebras

Representations of $\mathcal{W}$-algebras can be defined along the same lines as those of the Virasoro algebra naturally generalizing Eqs. (2), (3) and (6). A highest weight vector $|\,h,w\rangle$ for a representation satisfies

$$\phi_n\,|\,h,w\rangle\ = 0 \qquad \forall n < 0 \tag{15}$$

for all fields ϕ. Furthermore, the representation of the zero modes ϕ_0 of all fields ϕ is required to be irreducible. Quite often, one can assume the L_0-eigenspace of minimal energy to be one-dimensional.

There are two approaches to the explicit study of HWRs of $\mathcal{W}$-algebras. Both are based on natural requirements on a field theory. First, fields that vanish in the vacuum representation –so-called 'null fields'– should vanish everywhere. Second, evaluation of any correlation function should not depend on the actual order of evaluation. It is surprising to note that the second approach yields any restrictions at all and thus in some cases completely fixes the field content of the theory [10]. However, this works only for isolated values of the central charge c, i.e. mainly for those $\mathcal{W}$-algebras existing only for c discrete. For algebras existing at c generic one usually has to rely on the study of null fields.

For RCFTs it turns out that an important quantity is the effective central charge

$$c_{\text{eff}} := c - 24h_{\text{min}} \tag{16}$$

where h_{min} is the smallest of all h-values. For unitary theories c_{eff} equals c because $h_{\text{min}} = 0$.

Example: $\mathcal{W}(2,3)$

For $\mathcal{W}(2,3)$ the highest weight vector $|h,w\rangle$ satisfies

$$L_0 \, |h,w\rangle = h \, |h,w\rangle \, , \qquad W_0 \, |h,w\rangle = w \, |h,w\rangle \, . \tag{17}$$

Here, the associativity requirement yields no restrictions on h, w and one has to study null fields. This can be done explicitly for a few values of the central charge.

For $c = \frac{4}{5}$ one obtains $h \in \{0, \frac{2}{3}, \frac{2}{5}, \frac{1}{15}\}$ with fixed corresponding w. This are precisely the $\mathbb{Z}_3$-parafermions. Of course, this model is already Virasoro-minimal.

At $c = -2$ one can explicitly examine null fields as well. Here, all conditions that have been studied are satisfied for all (h,w) which obey the following relation

$$w^2 = \frac{2}{27}(8h+1)h^2. \tag{18}$$

Thus, the CFT at $c = -2$ is most probably only degenerate but not rational with respect to $\mathcal{W}(2,3)$.

4. First Results on the Classification of RCFTs

A few very simple statements about RCFTs have been proven so far. All proofs strongly rely on modular invariance.

The first statement is that RCFTs necessarily involve *rational* central charge c and *rational* conformal dimensions h [11,12].

Another important result is that all RCFTs with $c_{\text{eff}} < 1$ can be understood in terms of characters of the Virasoro-algebra Eq. (6). Of course, this does not exclude that they might also have larger symmetry algebras. The proof uses the known representation theory of the Virasoro algebra contained in any $\mathcal{W}$-algebra in order to write down a lower bound for Fourier coefficients of the characters if $c < 1$. If the model is not Virasoro-minimal the asymptotic behavior of the characters implies $c_{\text{eff}} \geq 1$.

Also the RCFTs with $c_{\text{eff}} = 1$ are completely classified [13−15]. Here, one needs non-trivial extensions of the Virasoro algebra. This classification uses a theorem of Serre-Stark.

For $c_{\text{eff}} > 1$ one definitely needs *extended* symmetry algebras in order to obtain RCFTs. A more precise statement is [16,17]: All rational models of a bosonic $\mathcal{W}$-algebra with k simple fields satisfy $c_{\text{eff}} < k$. The proof is elementary and based on character asymptotics.

Another recent result is the classification of $c = c_{\text{eff}} = 24$ RCFTs with only 1 character [18]. Only for $c \equiv 0 \pmod 8$ non-trivial one character theories are possible. The classification of $c = 8$ and $c = 16$ theories is comparably simple. A.N. Schellekens showed under the assumption of unitarity that at most 71 such models can exist at $c = 24$ [18], not all of which have been constructed explicitly so far. The tedious though elementary proof again involves modular invariance. It uses Jacobi forms instead of modular forms which take into account the additional quantum numbers given by the zero modes of the currents.

For all RCFTs where the characters are known they are modular functions on some congruence subgroup Γ_N. The level N is the smallest integer such that T^N acts trivially on the characters. Probably, this is true for all RCFTs.

5. Known Series of $\mathcal{W}$-Algebras

In this section we present general construction principles for $\mathcal{W}$-algebras which predict complete 'series' with infinitely many members.

c Generic

- Kac-Moody algebras consist exclusively of currents (spin 1 fields). They are loop algebras over (simple) Lie algebras $\mathcal{G}$. For them the energy-momentum tensor L is composite and given by the Sugawara formula, i.e. as a quadratic form in terms of the currents.
- $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras obtained by Drinfeld-Sokolov reduction (for the classical case see [19] and for the quantization [20]). These algebras are constructed applying Hamiltonian reduction to Kac-Moody algebras associated to $\mathcal{G}$. The construction involves an embedding of $\mathcal{S} = SL(2, \mathbb{R})$ into $\mathcal{G}$ which determines the spin of the simple fields.

  These algebras have certain good properties; for example they do not contain null fields for generic values of c. It is probably possible to show that all algebras with these good properties can be obtained by the Drinfeld-Sokolov mechanism because to any such algebra a Lie algebra $\mathcal{G}$ and an embedding $\mathcal{S} \subset \mathcal{G}$ can be associated [21,9].

Historically, a subset of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ has been discovered first: The $\mathcal{W}\mathcal{G}$-algebras, or so-called 'Casimir'-algebras. In the Drinfeld-Sokolov framework such algebras are given by the principal embedding $\mathcal{S} \subset \mathcal{G}$ in a simple Lie algebra $\mathcal{G}$. These algebras can be realized in terms of free fields (or currents) using the Casimir invariants of some simple Lie algebra $\mathcal{G}$. For them, the dimension of the simple fields is given by the order of the Casimir invariants of $\mathcal{G}$. Examples include in particular $\mathcal{W}\mathcal{A}_{n-1} \cong \mathcal{W}(2,3,\ldots,n)$. These algebras were some of the first well understood ones [22−24]. Originally, Hamiltonian reduction was discovered for them in physical models like constrained WZNW models [25]. The Casimir algebras also arise as conserved currents of Toda field theories [26].

Kac, Wakimoto, Frenkel have used Drinfeld-Sokolov reductions for principal embeddings in order to derive the minimal series of Casimir algebras from representations of Kac-Moody algebras at fractional level [27].

- Orbifolds: If a $\mathcal{W}$-algebra has outer automorphisms [28] one can project onto the subspace invariant under the automorphism. For a finitely generated $\mathcal{W}$-algebra the orbifold is usually also generated finitely with more generators and generic nullfields. E.g. the $\mathbb{Z}_2$-orbifold of $\mathcal{W}(2,3)$ generically is a $\mathcal{W}(2,6,8,10,12)$ where the first generic null field occurs at dimension 16.
- Products: One can always take the tensor product of two RCFTs. On the level of algebras this is a complicated 'sum'. In particular, one has to take care of the energy-momentum tensor L and the central extension c. For example, a $\mathcal{W}(2^n)$ can always be obtained summing n copies of the Virasoro algebra.
- Commutants: If a $\mathcal{W}$-algebra has another $\mathcal{W}$-algebra as subalgebra (e.g. some residual currents) one can consider the commutant of this subalgebra giving rise to another $\mathcal{W}$-algebra. For particular situations, this is the so-called 'Coset-construction' [29] where one usually puts the emphasis on the inherited representations.

c Discrete

Most of the results for discrete c have been discovered by performing explicit constructions of $\mathcal{W}$-algebras. An exhaustive search was performed for $\mathcal{W}(2,\delta)$-algebras with $\delta \leq 11$ [8,30,31].

- Virasoro-minimal models sometimes also possess extended symmetries. For all non-diagonal partition functions in the ADE-classification of modular invariant partition functions [32] a $\mathcal{W}(2,\delta)$-algebra diagonalizing it exists [10,17].
- The classification of $c = 24$ modular invariant partition functions [18] also gives rise to $\mathcal{W}$-algebras. E.g. the symmetry algebra of the CFT invariant under the monster is a $\mathcal{W}(2^{196884}, 3^{21296876})$ – a surprisingly small algebra compared to the order of the Monster.

- Also the $c = 1$ classification of modular invariant partition functions [13,14] predicts $\mathcal{W}$-algebras: It predicts a $\mathcal{W}(2, 4, \delta)$ for arbitrary dimension δ and the algebras $\mathcal{W}(2, 16)$, $\mathcal{W}(2, 9, 16)$, $\mathcal{W}(2, 36)$ at $c = 1$.
- $\mathcal{W}(2, \delta)$-algebras existing for $c = 1 - 8\delta$ or $c = 1 - 3\delta$ complete the classification of $c_{\mathrm{eff}} = 1$ RCFTs [15].
- $\mathcal{W}(2, \delta^n)$ algebras with odd n can be constructed from Virasoro-degenerate models at $c = c_{1,q}$ [33]. The rationality of these models is still an open question. The models are certainly not rational but only degenerate for $n = 1$.

6. Known Exotic Constructions

All known exotic constructions have been found by explicit construction.

c Generic

One solution each to $\mathcal{W}(2, 4, 6)$ [30] and $\mathcal{W}(2, 3, 4, 5)$ [34] are still not understood. Both examples exist for c generic and generically have null fields. In the case of $\mathcal{W}(2, 4, 6)$ known constructions including orbifolds have been carefully ruled out [31]. It has been proposed recently that one could identify this solution for $\mathcal{W}(2, 4, 6)$ as $\mathcal{WD}_{-1}$ [35] but it is only a very formal continuation of the $\mathcal{WD}_n$-series at the moment. All minimal models of this $\mathcal{W}(2, 4, 6)$ are conjectured to be isomorphic to a minimal model of some $\mathcal{WB}_m$ or $\mathcal{WC}_m$ [36].

c Discrete

- $\mathcal{W}(2, \delta)$-algebras existing for irrational values of the central charge [8,30]. Here, no restrictions on the HWRs can be deduced [17]. Examples include $\mathcal{W}(2, 5)$ at $c = 134 \pm 60\sqrt{5}$ and $\mathcal{W}(2, 8)$ at $c = 350 \pm 252\sqrt{2}$.
- $\mathcal{W}$-algebras existing for rational values of the central charge [8,30] but giving rise only to degenerate models, not to rational ones [17]. Here, examples include $\mathcal{W}(2, 8)$ at $c = -\frac{1015}{2}$.
- $\mathcal{W}(2, 8)$ at $c = -\frac{712}{7}$ [8] belongs to the minimal series of $\mathcal{WE}_8$. At this particular value of the central charge $\mathcal{WE}_8$ presumably truncates to $\mathcal{W}(2, 8)$.
  For this rational model with 15 representations also an exotic construction using the $\mathbb{Z}_5$-parafermions and two copies of the Virasoro-minimal model at $c = c_{2,7}$ was found [17]. The identification works for the characters; for the partition functions it reads
$$Z = \frac{1}{2}\left(Z^P + (Z^{\mathrm{Vir}})^2\right). \tag{19}$$
- The rational model of $\mathcal{W}(2, 8)$ at $c = -\frac{3164}{23}$ occurs in the minimal series of $\mathcal{WE}_7$, the Casimir algebra probably again truncating to a $\mathcal{W}(2, 8)$.
  As this rational model has 22 representations and all h's have denominator 23 it was natural to assume also a direct connection to Γ_{23} [17]. However, it was a

major effort to write down the characters as modular functions on Γ_{23} [37]. The same construction also works for $\mathcal{W}(2,4)$ at $c = -\frac{444}{11}$ and $\mathcal{W}(2,6)$ at $c = -\frac{1420}{17}$. These examples cannot be continued to a series.

7. Outlook

$\mathcal{W}$-algebras including fermionic fields can be treated along the same lines. Then one needs straightforward generalizations like super Lie brackets. In particular, $N = 1$- and $N = 2$-super $\mathcal{W}$-algebras have been investigated [38–40]. For $N = 1$ [38,39] the picture is similar to $N = 0$ whereas for $N = 2$ [40] unitarity seems to play a much more fundamental rôle.

All known RCFTs fit into one of the series of section 5. Thus, the exotic examples we presented in section 6 could be irrelevant for the classification problem of RCFTs.

The mathematical structure of $\mathcal{W}$-algebras is very rich. Due to the known and not yet well understood exotic constructions the classification problem for them is still open. This situation is similar to many fields in mathematics before a classification had been achieved, e.g. simple finite groups.

For physical applications a major important open question remains a geometrical interpretation of $\mathcal{W}$-transformations.

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