

# A Perturbative Approach to Spectrum and Correlation Functions of the Chiral Potts Model

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## Abstract

The massive high-temperature phase of the chiral Potts quantum chain is studied using perturbative methods. For the  $\mathbb{Z}_3$ -chain we present high-temperature expansions for the groundstate energy and the dispersion relations of the two single-particle states as well as two-particle states at general values of the parameters. We also present a perturbative argument showing that a large class of massive  $\mathbb{Z}_n$ -spin quantum chains have quasiparticle spectra with  $n - 1$  fundamental particles. It is known from earlier investigations that—at special values of the parameters—some of the fundamental particles exist only for limited ranges of the momentum. In these regimes our argument is not rigorous as one can conclude from a discussion of the radius of convergence of the perturbation series.

We also derive correlation functions from a perturbative evaluation of the groundstate for the  $\mathbb{Z}_3$ -chain. In addition to an exponential decay we observe an oscillating contribution. The oscillation length seems to be related to the asymmetry of the dispersion relations. We show that this relation is exact at special values of the parameters for general  $\mathbb{Z}_n$  using a form factor expansion.

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## 1. Introduction

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In this paper we discuss the chiral Potts model in its spin quantum chain formulation. The first chiral Potts model that was introduced in 1981 by Ostlund in order to describe incommensurate phases of physisorbed systems [1] was a classical 2D spin model. The associated quantum chain Hamiltonians were obtained in 1981-82 by Rittenberg et al. [2][3]. Because this chain was not self-dual the location of the critical manifold was difficult. In 1983, Howes, Kadanoff and denNijs introduced a self-dual  $\mathbb{Z}_3$ -symmetric chiral quantum chain [4], which however, does not correspond to a two-dimensional model with positive Boltzmann weights. Soon afterwards, von Gehlen and Rittenberg noticed that the remarkable property of the first gap of this model being linear in the inverse temperature also applies to the second gap and can be generalized to arbitrary  $\mathbb{Z}_n$  [5]. Furthermore, the authors of [5] showed that the Ising-like form of the eigenvalues is related to this  $\mathbb{Z}_n$ -Hamiltonian satisfying the Dolan-Grady integrability condition [6] – or equivalently [7] Onsager’s algebra [8]. It was then shown by Au-Yang, Baxter, McCoy, Perk et al. that this integrability property – nowadays called ‘superintegrability’ – can be implemented in a 2D classical model with Boltzmann weights defined on higher genus Riemann surfaces that satisfy a generalized Yang-Baxter relation. In the sequel the chiral Potts model attracted much attention because of these mathematical aspects, i.e. on the one hand the generalized Yang-Baxter relations [9 – 17] and on the other hand because of Onsager’s algebra [7][18 – 21]. In this paper we present new results showing that the model is also ‘physically’ very interesting although it is not directly related to a realistic 2D physisorbed system.

Our observations will apply to general  $\mathbb{Z}_n$ -spin quantum chains: The superintegrable  $\mathbb{Z}_n$ -chiral Potts quantum chains can be generalized (not necessarily demanding integrability) to include further known integrable models, in particular the conformally invariant models of Fateev and Zamolodchikov with  $\mathcal{WA}_{n-1}$ -symmetry [22 – 25]. Recently, Cardy introduced an integrable chiral perturbation of these models [26]. The  $\mathbb{Z}_n$ -spin quantum chains describe both this perturbation as well as the integrable thermal perturbations of the conformal field theories (see e.g. [27 – 31]).

In previous papers we provided numerical evidence that the low-lying excitations in the zero momentum sectors can be explained in terms of  $n - 1$  fundamental particles for  $n = 3, 4$  at general values of the parameters [32][33] and checked for  $n = 3$  that this quasiparticle picture extends to general momenta [34]. For the superintegrable  $\mathbb{Z}_3$ -chiral Potts model McCoy et al. have derived a quasiparticle picture of the complete spectrum using Bethe ansatz techniques [35]. Recently, they argued that this quasiparticle picture should in general be valid for the integrable  $\mathbb{Z}_3$ -chiral Potts quantum chain [36]. In this paper we will show that both results can be combined into the general statement that the massive high-temperature phases of general chiral Potts quantum chains have quasiparticle spectra. In fact, this quasiparticle picture will in certain cases give small corrections to the additivity of energy in the momentum zero sectors observed in [32].

The massive low-temperature phases of the  $\mathbb{Z}_n$ -spin quantum chains exhibit spectra that are dual to those in the high-temperature phases, the main difference being that the rôle of charge and boundary conditions is interchanged [37]. Therefore, our results about the massive high-temperature phase can be transferred to the massive low-temperature phase using duality.

In this paper we also use perturbation series in order to continue the systematic study of correlation functions which has been started in [33][34][37]. We pay special attention to the oscillatory behaviour which is also present in the massive phases and show how it can be related to the parity violation of the excitation spectrum via a form factor decomposition.

The outline of this paper is as follows. In section 2 we recall some well-known facts about the chiral Potts quantum chain and introduce basic notions. Section 3 presents a short summary of perturbation theory which is applied in section 4 to the dispersion relations of the lowest excitations of the  $\mathbb{Z}_3$ -chain. In section 5 we derive the main statement of our paper: The quasiparticle structure of the massive high-temperature phase. Details of the proof are shifted to an appendix. This argument can also be used in order to obtain some control on the finite-size effects. In section 6 we apply perturbation expansions and form factor decompositions to the correlation functions, our main interest being the oscillatory contribution. Then we specialize to vanishing chiral angles and discuss some of the results obtained previously in more detail. The final section 8 where we discuss the radius of convergence of the perturbation series completes our investigation.

## 2. The chiral Potts quantum chain

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This section summarizes well-known basic facts about  $\mathbb{Z}_n$ -spin quantum chains. We also introduce some notions that will be useful later on. For more details see e.g. the review [38].

A general  $\mathbb{Z}_n$ -spin quantum chain with  $N$  sites is defined by the Hamiltonian:

$$H_N^{(n)} = - \sum_{j=1}^N \sum_{k=1}^{n-1} \bar{\alpha}_k \sigma_j^k + \lambda \alpha_k \Gamma_j^k \Gamma_{j+1}^{n-k} + \hat{\lambda} \hat{\alpha}_k \Xi_j^k \Xi_{j+1}^{n-k}. \quad (2.1)$$

For reasons to be explained below we will in all subsequent sections set  $\hat{\lambda} = 0$ , i.e. we will neglect the extra term in (2.1) introduced in ref. [21] and will consider

$$H_N^{(n)} = - \sum_{j=1}^N \sum_{k=1}^{n-1} \bar{\alpha}_k \sigma_j^k + \lambda \alpha_k \Gamma_j^k \Gamma_{j+1}^{n-k} \quad (2.2)$$

instead.  $\sigma_j$ ,  $\Gamma_j$  and  $\Xi_j$  freely generate a finite dimensional associative algebra with involution by the following relations ( $1 \leq j, l \leq N$ ):

$$\begin{aligned} \sigma_j \sigma_l &= \sigma_l \sigma_j, & \sigma_j \Gamma_l &= \Gamma_l \sigma_j \omega^{\delta_{j,l}}, \\ \Gamma_j \Gamma_l &= \Gamma_l \Gamma_j, & \sigma_j^n &= \Gamma_j^n = \Xi_j^n = (\Xi_j \Gamma_j)^n = \mathbf{1}, \\ \Xi_j \Xi_l &= \Xi_l \Xi_j, & \sigma_j \Xi_l &= \Xi_l \sigma_j \omega^{\delta_{j,l}}, & \underbrace{\Xi_j \Gamma_j \Xi_j \Gamma_j \dots}_{n \text{ operators}} &\neq \mathbf{1}, \\ \sigma_j^+ &= \sigma_n^{n-1}, & \Gamma_j^+ &= \Gamma_j^{n-1}, & \Xi_j^+ &= \Xi_j^{n-1} \end{aligned} \quad (2.3)$$

where  $\omega$  is the primitive  $n$ th root of unity  $\omega = e^{\frac{2\pi i}{n}}$ . In the following we will consider only periodic boundary conditions for  $H_N^{(n)}$ , i.e.  $\Gamma_{N+1} = \Gamma_1$ .

The Hamiltonian (2.1) contains  $3n - 1$  parameters: The temperature-like parameters  $\lambda$  and  $\hat{\lambda}$  which we choose to be real and the complex constants  $\bar{\alpha}_k$ ,  $\alpha_k$  and  $\hat{\alpha}_k$ .  $H_N^{(n)}$  is hermitean iff  $\bar{\alpha}_k = \bar{\alpha}_{n-k}^*$ ,  $\alpha_k = \alpha_{n-k}^*$  and  $\hat{\alpha}_k = \hat{\alpha}_{n-k}^*$ .

The algebra (2.3) is conveniently represented in

$$\mathcal{H}_N := \underbrace{\mathbb{C}^n \otimes \mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n}_{N \text{ times}} \quad (2.4)$$

labeling the standard basis of  $\mathbb{C}^n$  by  $\{e_0, \dots, e_{n-1}\}$ . Then a basis for (2.4) is given by:

$$|i_1 \dots i_N\rangle := e_{i_1} \otimes \dots \otimes e_{i_N}, \quad 0 \leq i_j \leq n-1. \quad (2.5)$$

Now the following operation in the space (2.4) is a faithful irreducible representation  $r$  of the algebra (2.3):

$$\begin{aligned} r(\sigma_j) |i_1 \dots i_j \dots i_N\rangle &= \omega^{i_j} |i_1 \dots i_j \dots i_N\rangle, \\ r(\Gamma_j) |i_1 \dots i_j \dots i_N\rangle &= |i_1 \dots (i_j + 1 \bmod n) \dots i_N\rangle, \\ r(\Xi_j) |i_1 \dots i_j \dots i_N\rangle &= \begin{cases} -|i_1 \dots (i_j + 1) \dots i_N\rangle, & \text{if } i_j < n-1; \\ |i_1 \dots 1 \dots i_N\rangle, & \text{if } i_j = n-1. \end{cases} \end{aligned} \quad (2.6)$$

The involution is the adjoint operation with respect to the standard scalar product in the tensor product of  $\mathbb{C}^n$ .

The Hamiltonian (2.1) commutes with the  $\mathbb{Z}_n$  charge operator  $\hat{Q} := \prod_{j=1}^N \sigma_j$  acting on the vectors (2.5) as

$$r(\hat{Q}) |i_1 \dots i_N\rangle = \omega^{(\sum_{j=1}^N i_j)} |i_1 \dots i_N\rangle \quad (2.7)$$

which shows that the eigenvalues of  $\hat{Q}$  have the form  $\omega^Q$  with  $Q$  integer. Thus,  $H_N^{(n)}$  has  $n$  charge sectors which we shall refer to by  $Q = 0, \dots, n-1$ .

$H_N^{(n)}$  also commutes with the translation operator  $T_N$  that acts on the basis vectors (2.5) in the following way:

$$r(T_N) |i_1 i_2 \dots i_N\rangle = |i_2 \dots i_N i_1\rangle. \quad (2.8)$$

The eigenvalues of  $T_N$  are  $N$ th roots of unity. We label them by  $e^{iP}$  and call  $P$  the ‘momentum’. We choose  $0 \leq P < 2\pi$  corresponding to the first Brillouin zone and have  $P \in \{0, \frac{2\pi}{N}, \dots, \frac{2\pi(N-1)}{N}\}$ . Note that the states

$$\begin{aligned} \||i_1 i_2 \dots i_{N-1} i_N\rangle\rangle_P := & \frac{1}{\sqrt{\mathcal{N}}} \left( |i_1 i_2 \dots i_{N-1} i_N\rangle + e^{iP} |i_N i_1 i_2 \dots i_{N-1}\rangle + \dots \right. \\ & \left. + e^{iP(N-1)} |i_2 \dots i_{N-1} i_N i_1\rangle \right) \end{aligned} \quad (2.9)$$

are eigenstates of  $T_N$  with eigenvalue  $e^{iP}$ .  $\mathcal{N}$  is a suitable normalization constant. If the state  $|i_1 \dots i_N\rangle$  has no symmetry (i.e.  $T_N^k |i_1 \dots i_N\rangle \neq |i_1 \dots i_N\rangle$  for all  $0 < k < N$ ), one has  $\mathcal{N} = N$ . This will apply to most cases below where we need (2.9).

In this paper we will use the following parametrization of the constants  $\alpha_k$  and  $\bar{\alpha}_k$ , fixing their dependence on  $k$ :

$$\alpha_k = \frac{e^{i\phi(\frac{2k}{n}-1)}}{\sin \frac{\pi k}{n}}, \quad \bar{\alpha}_k = \frac{e^{i\varphi(\frac{2k}{n}-1)}}{\sin \frac{\pi k}{n}}. \quad (2.10)$$

This is a suitable choice because it includes a large class of interesting models.

For  $\phi = \varphi = 0$  one obtains real  $\alpha_k = \bar{\alpha}_k = \frac{1}{\sin \frac{\pi k}{n}}$ . This leads to models with a second order phase transition at  $\lambda = 1$  which can be described by a parafermionic conformal field theory in the limit  $N \rightarrow \infty$  at criticality [22][23]. These so-called Fateev-Zamolodchikov-models [24] lead to extended conformal algebras  $\mathcal{WA}_{n-1}$  where the simple fields have conformal dimension  $2, \dots, n$  for generic values of the central charge  $c$  [25]. The spectrum of the Hamiltonian (2.2) can be described by the first unitary minimal model of the algebra  $\mathcal{WA}_{n-1}$ . For  $n = 3$  the symmetry algebra is Zamolodchikov's well-known spin-three extended conformal algebra [39] at  $c = \frac{4}{5}$ .

Choosing  $\phi = \varphi = \frac{\pi}{2}$  in (2.10) for the Hamiltonian (2.2) yields the superintegrable chiral Potts model. For  $n = 3$  such complex parameters in a spin chain Hamiltonian were first investigated in detail by Howes, Kadanoff and denNijs [4]. The integrability of this chain was then recognized by von Gehlen and Rittenberg who also generalized it to higher  $\mathbb{Z}_n$  [5]. More precisely, the authors of [5] showed that the  $\mathbb{Z}_n$ -Hamiltonian (2.2) with (2.10) at  $\phi = \varphi = \frac{\pi}{2}$  is integrable for all values of the inverse temperature  $\lambda$  using the Dolan-Grady integrability condition [6]. This particular kind of integrability is called ‘superintegrability’ (note that this terminology is not used entirely consistent in the literature – in contrast to us, some authors include the generalized Yang-Baxter relations in the notion of superintegrability). Ahn et al. [21] showed that the Hamiltonian (2.1) is still integrable at  $\phi = \varphi = \frac{\pi}{2}$  for  $\hat{\alpha}_k = \alpha_k = \bar{\alpha}_k = 1 - i \cot \frac{\pi k}{n}$  and any  $\lambda, \hat{\lambda}$ . Their argument used Onsager's algebra in order to construct an infinite set of commuting conserved charges. Note that the Hamiltonian (2.1) subject to the above constraints is *not superintegrable* for general values of the parameters. Anyway, one can introduce a further parameter  $\hat{\lambda}$  into (2.2) without spoiling integrability <sup>1)</sup>.

The parametrization (2.10) also includes the family of integrable models discovered in [9–14] which interpolates between the integrable cases at  $\phi = \varphi = 0, \lambda = 1$  and  $\phi = \varphi = \frac{\pi}{2}$ . The Hamiltonian (2.2) is integrable if one imposes the additional constraint

$$\cos \varphi = \lambda \cos \phi \quad (2.11)$$

on the parametrization (2.10). For  $\phi = \varphi = 0$  this yields  $\lambda = 1$  – the conformally invariant critical points. At  $\phi = \varphi = 0$ , the Hamiltonian is self-dual, i.e. it is invariant under a duality-transformation such that  $H_N^{(n)}(\lambda) \cong \lambda H_N^{(n)}(\lambda^{-1})$ . The Hamiltonian is also self-dual on the superintegrable line  $\phi = \varphi = \frac{\pi}{2}$ .  $H_N^{(n)}$  with the choices (2.10), (2.11) is in

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<sup>1)</sup> For  $n = 2$  and  $\lambda = \hat{\lambda}$  this gives rise to extra symmetries of the Hamiltonian – one obtains an XY quantum chain [21] that is invariant under an additional global  $U(1)$  symmetry group. However, one can easily check that for  $n > 2$  the Hamiltonian (2.1) is not invariant under any non-trivial change of bases  $\Gamma_j \rightarrow a\Gamma_j + b\Xi_j, \Xi_j \rightarrow c\Gamma_j + d\Xi_j$ .

general not self-dual any more whereas particular choices yield a self-dual Hamiltonian. If we choose for (2.10)  $\phi = \varphi$  and neglect (2.11)  $H_N^{(n)}$  will be self-dual again. Therefore we choose to refer to (2.2) with (2.10) as the general ‘chiral Potts model’. We will not consider the integrable case where the additional constraint (2.11) is satisfied in detail.

We will now explain why we are going to focus on the Hamiltonian (2.2) rather than considering the more general case (2.1). For  $\lambda = 0$  (2.1) is just a different representation of (2.2). Thus, although we will certainly obtain different numerical results, the main structures are unchanged by the extra term in (2.1). In this paper we will use for example perturbation theory. The free part of the Hamiltonian  $H_0$  is the same in (2.2) and in (2.1):  $H_0 = -\sum_{j,k} \bar{\alpha}_k \sigma_j^k$ . Only the potential  $V$  is changed. For (2.2) we have  $V = -\sum_{j,k} \alpha_k \Gamma_j^k \Gamma_{j+1}^{n-k}$  whereas for (2.1) we have an extra term  $\hat{V} = -\sum_{j,k} \alpha_k \Gamma_j^k \Gamma_{j+1}^{n-k} + h \hat{\alpha}_k \Xi_j^k \Xi_{j+1}^{n-k}$  with  $h := \lambda^{-1} \hat{\lambda}$ . Obviously, the action of  $V$  and  $\hat{V}$  on the eigenstates (2.9) of charge and momentum is the same apart from different numerical constants. Furthermore, the extra term in (2.1) spoils duality. Thus, we will not consider the Hamiltonian (2.1) explicitly any more. It is always understood that our results apply to it with only minor modifications. In particular, the quasiparticle picture we will derive for the Hamiltonian (2.2) will hold for (2.1) as well.

Our main interest is the spectrum in the limit  $N \rightarrow \infty$  of  $H_N^{(n)}$ . Of course, we have to specify how the limit is to be taken. In order to be able to study the spectrum in this limit we concentrate on the  $N$ -dependence of the Hilbert spaces  $\mathcal{H}_N = \mathcal{D}(H_N^{(n)})$ . Consider the following embedding of Hilbert spaces:

$$\begin{aligned} \mathcal{H}_N &\rightarrow \mathcal{H}_M & N < M \\ \|i_1 \dots i_N\rangle\rangle_P &\mapsto \|i_1 \dots i_N \underbrace{0 \dots 0}_{M-N \text{ times}}\rangle\rangle_P. \end{aligned} \quad (2.12)$$

This definition is motivated by the well-known fact that matrix elements of (2.2) in momentum space are almost independent of  $N$ . We will see in the following sections that this definition is indeed useful.

Using the inclusion map (2.12) we can define the Hilbert space  $\mathcal{H}$  as the closure of an inductive limit

$$\mathcal{H} := \{|x\rangle \mid \exists N : |x\rangle \in \mathcal{H}_N\}^{\sim}. \quad (2.13)$$

Furthermore, we shall not consider the limit of  $H_N^{(n)}$  directly. Instead, we shall subtract the groundstate energy  $E_N^0$  first and then consider the weak limit of the operator

$$\Delta H_N^{(n)} := H_N^{(n)} - E_N^0 \mathbb{1}. \quad (2.14)$$

Similarly, we define  $T$  to be the weak limit of  $T_N$ . For each finite  $N$  eqs. (2.7) and (2.9) imply that the Hilbert space  $\mathcal{H}_N$  is graded into charge and momentum eigenspaces:

$$\mathcal{H}_N = \bigoplus_P \bigoplus_{Q=0}^{n-1} \mathcal{H}_N^{P,Q}. \quad (2.15)$$

In the limit  $N \rightarrow \infty$  the grading (2.15) translates into the fact that  $\Delta H^{(n)}$  and  $T$  can be written in terms of the same projection-valued measure  $\{\Pi_\mu^Q\}$ :

$$T = \sum_{Q=0}^{n-1} \int e^{iP(\mu)} d\Pi_\mu^Q, \quad \Delta H^{(n)} = \sum_{Q=0}^{n-1} \int \Delta E(\mu) d\Pi_\mu^Q \quad (2.16)$$

with  $0 \leq P(\mu) < 2\pi$ . The  $\{\Pi_\mu^Q\}$  can be thought of as infinite dimensional generalizations of projection operators onto eigenspaces of charge  $Q$  and momentum  $P(\mu)$ . Thus, (2.16) is just the proper formulation of (2.15) in the infinite dimensional case. The existence of the limits and projection valued measures in (2.16) is not at all obvious. However, this is guaranteed by the quasiparticle picture whereof a proof is presented in appendix A.

The definition in (2.14) is motivated by the fact that the smallest eigenvalue of (2.2) has a leading term proportional to  $N$  and the excitation spectrum usually is defined with respect to this reference eigenvalue. With the definition (2.14) the Hamiltonian  $\Delta H_N^{(n)}$  is bounded from below. This automatically yields an operator  $\Delta H^{(n)} = \lim_{N \rightarrow \infty} \Delta H_N^{(n)}$  with positive spectrum and an eigenvector for eigenvalue 0. Note that this definition of the limit implies that any point where at finite  $N$  eigenvalues exist that are arbitrarily close to it belongs to the spectrum. In particular, the spectrum forms a closed set.

Before proceeding let us make a few further comments on our definition of the limit. First note that  $H_N^{(n)}$  is defined only on  $\mathcal{D}(H_N^{(n)}) = \mathcal{H}_N \subset \mathcal{H}$ . Of course, we could extend it linearly (e.g. by 0) to the complete Hilbert space  $\mathcal{H}$ . However, it is easy to show that the limit  $\Delta H^{(n)}$  does not depend on the particular extension chosen as long as it is uniformly bounded for all  $N$ . We will therefore not make use of any particular extension.

Secondly, it is convenient to let  $H_N^{(n)}$  act on vectors in  $\mathcal{D}(H_N^{(n)}) = \mathcal{H}_N$  which corresponds to choosing a particular representative for a vector in the Hilbert space  $\mathcal{H}$ . This is useful because  $H_N^{(n)}$  naturally acts on chains of length  $N$ . However, such a vector always has to be thought of as lying in  $\mathcal{H}$  and, in particular, in all  $\mathcal{H}_M$  with  $M \geq N$ . Although the notation might propose this, a limit in the chain length never has to be applied to momentum eigenstates. Of course, other states than (2.9) are not naturally embedded into  $\mathcal{H}$  and therefore have to be expanded in terms of them. This might lead to  $N$ -dependent coefficients and a limit might have to be applied to the *coefficients*.

Finally, it is noteworthy that the Hilbert space  $\mathcal{H}$  can be thought of as a kind of universal tensor product. Any tensor product of spaces  $\mathcal{H}_N$  and  $\mathcal{H}_M$  can be naturally identified with  $\mathcal{H}_{N+M}$ :  $\mathcal{H}_N \otimes \mathcal{H}_M \cong \mathcal{H}_{N+M}$ . Therefore the definition (2.13) yields an object that is closed with respect to taking tensor products. Note that we have chosen a particular topology on  $\mathcal{H}$  which is not the one usually chosen on the tensor algebra of a vector space. Still, this observation is useful to guarantee the completeness of the construction to be presented in section 5.

### **3. Generalities about Perturbation Theory**

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In this section we review the general outline for perturbation theory to all orders as presented in [40] which directly applies to the degenerate case as well.

The Hamiltonian (2.2) can be written as

$$H = H_0 + \lambda V \quad (3.1)$$

with  $H_0 = -\sum_{j,k} \bar{\alpha}_k \sigma_j^k$ ,  $V = -\sum_{j,k} \alpha_k \Gamma_j^k \Gamma_{j+1}^{n-k}$ . The eigenstates for  $H_0$  are obvious, thus we have solved:

$$H_0 |a\rangle = E_{|a\rangle}^{(0)} |a\rangle. \quad (3.2)$$

Now one can solve

$$H |a(\lambda)\rangle = E_{|a\rangle} |a(\lambda)\rangle \quad (3.3)$$

for small  $\lambda$  as follows: Let  $q_{|a\rangle}$  be the projector onto the eigenspace of  $H_0$  with eigenvalue  $E_{|a\rangle}^{(0)}$ . We can treat non-degenerate and degenerate perturbation theory alike if we choose  $|a\rangle$  such that

$$q_{|a\rangle} V |a\rangle = E_{|a\rangle}^{(1)} |a\rangle \quad (3.4)$$

with a constant  $E_{|a\rangle}^{(1)}$ , i.e.  $q_{|a\rangle} V q_{|a\rangle}$  is to be chosen diagonal. One also needs a regularized resolvent  $g(z)$  of  $H_0$ :

$$g(z) := (\mathbf{1} - q_{|a\rangle}) (z - H_0)^{-1}. \quad (3.5)$$

Then, the Wigner-Brillouin perturbation series

$$E_{|a\rangle} = \sum_{\nu=0}^{\infty} \lambda^{\nu} E_{|a\rangle}^{(\nu)}, \quad |a(\lambda)\rangle = \sum_{\nu=0}^{\infty} \lambda^{\nu} |a, \nu\rangle \quad (3.6)$$

is given by the following recurrence relations [40]:

$$\begin{aligned} |a, 0\rangle &= |a\rangle \\ |a, \nu\rangle &= g(E_{|a\rangle}^{(0)}) \left\{ V |a, \nu - 1\rangle - \sum_{\mu=1}^{\nu-1} |a, \nu - \mu\rangle E_{|a\rangle}^{(\mu)} \right\}, \\ E_{|a\rangle}^{(\nu+1)} &= \langle a | V |a, \nu\rangle. \end{aligned} \quad (3.7)$$

Note that neither  $|a(\lambda)\rangle$  nor  $|a, \nu\rangle$  are in general normalized although  $|a\rangle$  must be normalized to one. Observe that the derivation of (3.7) does not rely on  $H$  being hermitean. Therefore, (3.7) may also be applied to diagonalizable but non-hermitean  $H$ .

The radius of convergence of the series (3.6) can be more easily discussed in a different framework. Therefore, we postpone such a discussion to section 8.

There is one observation that makes explicit evaluation of high orders for the  $\mathbb{Z}_n$ -Hamiltonian (2.2) possible. The energy-eigenvalues  $E_{|a\rangle}$  of  $H_N^{(n)}$  do depend on the chain length  $N$ . However, for the low lying gaps  $\Delta E_{|a\rangle}$ <sup>2)</sup> of  $\Delta H_N^{(n)}$  (see (2.14)) the coefficients for powers of  $\lambda$  become independent of  $N$  up to order  $\lambda^{N-2}$  (see e.g. [41]). Intuitively, this can be inferred from the fact that (2.2) shows only nearest neighbour interaction and thus

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<sup>2)</sup> This will apply precisely to the fundamental quasiparticle states to be discussed below.



we need  $N - 1$  powers in  $V$  to bring us around a chain of length  $N$ . Smaller powers in  $V$  (or  $\lambda$ ) do not feel that the length of the chain is finite.

#### 4. High-temperature expansions

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In this section we study the low lying levels in the spectrum of the  $\mathbb{Z}_3$ -chiral Potts model perturbatively. In particular, we calculate the dispersion relations of the lowest excitations in the charge sectors  $Q = 1$  and  $Q = 2$ . Some first results in this direction have been presented in [32] for the self-dual version of this model. In this section we derive higher orders and admit general  $\phi \neq \varphi$ <sup>3)</sup>. We also present some explicit results on the next excitations.

Perturbation expansions had already been used in [4], and were again used e.g. in [42] and [41] in order to obtain some results for spectra and order parameters on the superintegrable line. Recently, low-temperature expansions have been applied in [37] to spectra and correlation functions for general values of the parameters. Here, we focus on the high-temperature regime.

For arbitrary  $n, N$  the groundstate of the Hamiltonian (2.2) in the limit  $\lambda \rightarrow 0$  is given by:

$$|\text{GS}\rangle := |0 \dots 0\rangle \quad (4.1)$$

provided that  $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ . For  $n = 3$  (4.1) will be the groundstate for the larger range  $-\pi \leq \varphi \leq \pi$  and for  $n = 4$  (4.1) is the groundstate for  $-\frac{5\pi}{6} \leq \varphi \leq \frac{5\pi}{6}$ .

The first excited states at  $\lambda = 0$  for  $Q > 0$  and arbitrary  $P$  are the states

$$\|s^Q\rangle\rangle_P := \|Q0 \dots 0\rangle\rangle_P \quad (4.2)$$

in the range  $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ . According to our definition of the space  $\mathcal{H}$  in section 2, the states (4.2) give rise to proper eigenstates in the limit of  $\Delta H^{(n)}$ . Thus, the corresponding gaps  $\Delta E_{Q,0}(P)$  belong to the point spectrum of  $\Delta H^{(n)}$ .

More generally, we wish to argue later on that the complete spectrum can be explained in terms of quasiparticles. At  $\lambda = 0$ , a single-particle excitation corresponds to just one flipped spin (4.2). Due to the absence of interactions  $k$ -particle states have  $k$  flipped spins at  $\lambda = 0$ . For  $\lambda > 0$  one would have to take the interactions into account using perturbation theory. Although we are in general not able to perform such a computation directly, it may still be suggestive to think in terms of such states. In fact, such a picture is quite traditional (see e.g. [43]).

In the following we will use the abbreviations:

$$\mathcal{C} := \cos\left(\frac{\varphi}{3}\right), \quad \hat{\mathcal{C}} := \cos\left(\frac{\pi - \varphi}{3}\right), \quad \mathcal{R} := 1 - 4\mathcal{C}^2, \quad \bar{\mathcal{C}}_r := \cos\left(\frac{r\phi}{3}\right). \quad (4.3)$$

---

<sup>3)</sup> Note that the main limitation of the length of most of the series to be presented in this section is that we explicitly keep the dependence on the parameters.

For  $n = 3$  we can calculate the groundstate energy per site  $e_0$  which is defined by  $E_N^0 = N e_0$  perturbatively:

$$e_0 = -\frac{4}{\sqrt{3}}\mathcal{C} - \frac{2\lambda^2}{3\sqrt{3}\mathcal{C}} - \frac{\bar{\mathcal{C}}_3\lambda^3}{9\sqrt{3}\mathcal{C}^2} + \frac{\sqrt{3}}{81\mathcal{C}} \left\{ \frac{1}{2\mathcal{C}^2} + \frac{4}{\mathcal{R}} \right\} \lambda^4 + \frac{\sqrt{3}\bar{\mathcal{C}}_3}{81\mathcal{C}^2} \left\{ \frac{3}{4\mathcal{C}^2} + \frac{4}{\mathcal{R}} \right\} \lambda^5 + \mathcal{O}(\lambda^6). \quad (4.4)$$

Eq. (4.4) is independent of the chain length  $N$  if  $N > 5$ . In order to convey some idea of the quality of such an expansion we mention that for  $\phi = \varphi = \frac{\pi}{2}$  and  $\lambda = \frac{1}{2}$  the difference between (4.4) and the result of a numerical diagonalization of the Hamiltonian (2.2) performed with 12 sites is of magnitude  $10^{-4}$ . Further comments on the accuracy of (4.4), in particular at the boundary of the phase, can be found in [37] where the same expansion has been calculated for the massive low-temperature phase.

Furthermore, for  $n = 3$  we obtain for the lowest  $Q = 1$  gap and  $P = 0$  using the states (4.2):

$$\begin{aligned} \Delta E_{1,0}(\phi, \varphi) &= 4 \sin\left(\frac{\pi - \varphi}{3}\right) - \lambda \frac{4}{\sqrt{3}}\bar{\mathcal{C}}_1 - \lambda^2 \frac{2}{3\sqrt{3}} \left\{ \frac{\bar{\mathcal{C}}_2 - 2}{\mathcal{C}} + \frac{\bar{\mathcal{C}}_2 + 1}{\hat{\mathcal{C}}} \right\} \\ &+ \lambda^3 \frac{1}{9\sqrt{3}} \left\{ -\frac{4\bar{\mathcal{C}}_1}{\mathcal{C}\hat{\mathcal{C}}} + \frac{\bar{\mathcal{C}}_1}{\mathcal{C}^2} + \frac{3\bar{\mathcal{C}}_1 + \bar{\mathcal{C}}_3}{\hat{\mathcal{C}}^2} \right\} \\ &- \lambda^4 \frac{1}{27} \left\{ \frac{\bar{\mathcal{C}}_4 + 1}{\sqrt{3}\hat{\mathcal{C}}\mathcal{C}^2} + \frac{\bar{\mathcal{C}}_4 - 1}{2\sqrt{3}\hat{\mathcal{C}}^2\mathcal{C}} + \frac{\bar{\mathcal{C}}_4 + 4\bar{\mathcal{C}}_2 + 3}{2\sqrt{3}\hat{\mathcal{C}}^3} + \frac{4\bar{\mathcal{C}}_4 - 12\bar{\mathcal{C}}_2 + 9}{2\sqrt{3}\mathcal{C}^3} \right. \\ &\quad \left. - \frac{1}{3 \sin\left(\frac{\pi - \varphi}{3}\right)\mathcal{C}^2} - \frac{\bar{\mathcal{C}}_2 + 1}{3 \sin\left(\frac{\pi + \varphi}{3}\right)\hat{\mathcal{C}}^2} + \frac{\bar{\mathcal{C}}_2 - 2}{3 \sin\left(\frac{\pi + \varphi}{3}\right)\mathcal{C}^2} \right\} \\ &+ \mathcal{O}(\lambda^5). \end{aligned} \quad (4.5)$$

$\Delta E_{2,0}(\phi, \varphi)$  is given by  $\Delta E_{2,0}(\phi, \varphi) = \Delta E_{1,0}(-\phi, -\varphi)$ .

For  $n = 3$  and general  $P$  we obtain from the states (4.2) the following perturbation expansion for the dispersion relation of the lowest  $Q = 1$  excitation:

$$\begin{aligned} E_1(P) &:= \Delta E_{1,0}(P, \phi, \varphi) = 4 \sin\left(\frac{\pi - \varphi}{3}\right) - \lambda \frac{4}{\sqrt{3}} \cos\left(P - \frac{\phi}{3}\right) \\ &- \lambda^2 \frac{2}{3\sqrt{3}} \left\{ \frac{\cos\left(P + \frac{2\phi}{3}\right) + 1}{\hat{\mathcal{C}}} + \frac{\cos\left(2P - \frac{2\phi}{3}\right) - 2}{\mathcal{C}} \right\} \\ &+ \lambda^3 \frac{1}{9\sqrt{3}} \left\{ -\frac{2 \cos\left(2P + \frac{\phi}{3}\right) - 3 \cos\left(P - \frac{\phi}{3}\right) + 2 \cos(3P - \phi) - 2\bar{\mathcal{C}}_3}{\mathcal{C}^2} \right. \\ &\quad \left. - \frac{2 \cos\left(2P + \frac{\phi}{3}\right) + 2 \cos\left(P - \frac{\phi}{3}\right)}{\mathcal{C}\hat{\mathcal{C}}} + \frac{\cos\left(2P + \frac{\phi}{3}\right) + 2 \cos\left(P - \frac{\phi}{3}\right) + \bar{\mathcal{C}}_3}{\hat{\mathcal{C}}^2} \right\} \\ &+ \mathcal{O}(\lambda^4) \end{aligned} \quad (4.6)$$

and the lowest  $Q = 2$  excitation is given by:

$$E_2(P) := \Delta E_{2,0}(P, \phi, \varphi) = \Delta E_{1,0}(P, -\phi, -\varphi). \quad (4.7)$$

Eqs. (4.6) and (4.7) have already been presented in [34] in a different form. Note that the agreement of (4.6) and (4.7) with the results of a numerical diagonalization is usually good as was discussed in detail in [34].

In the previous section we mentioned that the  $k$ th orders of (4.5) – (4.7) are independent of  $N$  if  $N \geq k + 2$ . In particular, this implies the existence of the limits  $N \rightarrow \infty$  of (4.5) and (4.6) if the perturbative series converge at all.

In the derivation of (4.6) we have not assumed that the Hamiltonian (2.2) is hermitean. Thus, we may admit  $\phi \in \mathbb{C}$ . We have checked in a few cases that results of a numerical diagonalization at  $N = 12$  sites are still in good agreement with (4.6) also for complex  $\phi$ .

We would like to mention that it is no problem to compute further orders of the series (4.4), (4.5) and (4.6). In fact, we have indeed done so (see eq. (4.2.6) of [44] for the 7th order result of (4.4) and eq. (4.2.7) of [44] for the 4th order contribution to (4.6)) but refrain from presenting the results because the next orders are very complicated and not relevant for our purposes.

Obviously, for  $\phi = \varphi = \frac{\pi}{2}$  we have to perform degenerate perturbation theory. The correct perturbative excited state for  $Q = 2$  and  $P = 0$  is for odd  $N$ :

$$\sqrt{\frac{2}{N+1}} \left( \|20 \dots 0\|_0 + \|110 \dots 0\|_0 + \dots \|1 \underbrace{0 \dots 0}_{\frac{N-3}{2}} 10 \dots 0\|_0 \right) \quad (4.8a)$$

and for even  $N$ :

$$\sqrt{\frac{2}{N+1}} \left( \|20 \dots 0\|_0 + \|110 \dots 0\|_0 + \dots \|1 \underbrace{0 \dots 0}_{\frac{N}{2}-2} 10 \dots 0\|_0 + \frac{1}{\sqrt{2}} \|1 \underbrace{0 \dots 0}_{\frac{N}{2}-1} 10 \dots 0\|_0 \right). \quad (4.8b)$$

With this state we obtain for  $N > 9$ :

$$\Delta E_{2,0} \left( \frac{\pi}{2}, \frac{\pi}{2} \right) = 4(1 - \lambda) + \mathcal{O}(\lambda^9) \quad (4.9)$$

as expected. In fact (4.9) has been proven exactly [11] using different methods but previous perturbative calculations were restricted to the non-degenerate case  $\Delta E_{1,0}$  at  $\phi = \varphi = \frac{\pi}{2}$ . This demonstrates the universality of the approach to perturbation expansions outlined in section 3.

Also for the higher excitations we must apply degenerate perturbation theory. The next simplest case are the states where two spins are different from zero. For general  $P$ ,  $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$  the space of the excitation with one spin flipped into charge sector  $Q_1$  and another one flipped into charge sector  $Q_2$  is spanned by the states

$$\|t_j^{Q_1, Q_2}\|_P := \|Q_1 \underbrace{0 \dots 0}_{j-1 \text{ times}} Q_2 0 \dots 0\|_P, \quad 1 \leq j \leq \begin{cases} N-1, & \text{if } Q_1 \neq Q_2; \\ \lfloor \frac{N}{2} \rfloor, & \text{if } Q_1 = Q_2. \end{cases} \quad (4.10)$$

Obviously, we will have to consider two cases:  $Q_1 \neq Q_2$  and  $Q_1 = Q_2$ .

Let us first consider  $Q_1 \neq Q_2$ . For  $n = 3$  we can choose  $Q_1 = 1$ ,  $Q_2 = 2$ . Then we may omit the upper indices of  $t$  because they are uniquely fixed:  $\|t_j\rangle\rangle_P := \|t_j^{1,2}\rangle\rangle_P$ . Now, the potential  $V$  acts in the space (4.10) as:

$$\begin{aligned}
qr(V)\|t_1\rangle\rangle_P &= -\frac{2}{\sqrt{3}} \left( e^{-i(\frac{\phi}{3}-P)} \|t_{N-1}\rangle\rangle_P + 2 \cos\left(\frac{P}{2}\right) e^{-i(\frac{\phi}{3}+\frac{P}{2})} \|t_2\rangle\rangle_P \right) \\
qr(V)\|t_j\rangle\rangle_P &= -\frac{2}{\sqrt{3}} \left( 2 \cos\left(\frac{P}{2}\right) e^{i(\frac{\phi}{3}+\frac{P}{2})} \|t_{j-1}\rangle\rangle_P + 2 \cos\left(\frac{P}{2}\right) e^{-i(\frac{\phi}{3}+\frac{P}{2})} \|t_{j+1}\rangle\rangle_P \right) \\
&\qquad\qquad\qquad 1 < j < N-1 \\
qr(V)\|t_{N-1}\rangle\rangle_P &= -\frac{2}{\sqrt{3}} \left( 2 \cos\left(\frac{P}{2}\right) e^{i(\frac{\phi}{3}+\frac{P}{2})} \|t_{N-2}\rangle\rangle_P + e^{i(\frac{\phi}{3}-P)} \|t_1\rangle\rangle_P \right)
\end{aligned} \tag{4.11}$$

where  $q$  is the projector onto the space (4.10). Although it is not difficult to diagonalize (4.11) numerically for comparably long chains (e.g.  $N = 100$ ), we did not succeed in obtaining a closed expression for the eigenvalues or eigenvectors.

In the second case, i.e.  $Q := Q_1 = Q_2$  introduce the abbreviation  $W$  by:

$$-\frac{2}{\sin\left(\frac{\pi Q}{n}\right)} \cos\left(\frac{P}{2} - \left(1 - \frac{2Q}{n}\right)\phi\right) W\|t_j^{Q,Q}\rangle\rangle_P := qr(V)\|t_j^{Q,Q}\rangle\rangle_P. \tag{4.12}$$

In the case of two identical excitations we will also have to distinguish between even and odd momenta in terms of lattice sites. It is therefore convenient to introduce a further abbreviation  $\delta_P^N$  encoding this distinction:

$$\delta_P^N := 0, \quad \text{if } \frac{PN}{2\pi} \text{ odd}; \qquad \delta_P^N := 1, \quad \text{if } \frac{PN}{2\pi} \text{ even}. \tag{4.13}$$

The action of the potential  $V$  now is

$$\begin{aligned}
W\|t_1^{Q,Q}\rangle\rangle_P &= \left( e^{-i\frac{P}{2}} \|t_2^{Q,Q}\rangle\rangle_P \right) \\
W\|t_j^{Q,Q}\rangle\rangle_P &= \left( e^{i\frac{P}{2}} \|t_{j-1}^{Q,Q}\rangle\rangle_P + e^{-i\frac{P}{2}} \|t_{j+1}^{Q,Q}\rangle\rangle_P \right) \quad 1 < j < \left[\frac{N}{2}\right] - 1 \\
W\|t_{\left[\frac{N}{2}\right]-1}^{Q,Q}\rangle\rangle_P &= \begin{cases} \left( e^{i\frac{P}{2}} \|t_{\frac{N-5}{2}}^{Q,Q}\rangle\rangle_P + e^{-i\frac{P}{2}} \|t_{\frac{N-1}{2}}^{Q,Q}\rangle\rangle_P \right), & \text{if } N \text{ odd;} \\ \left( e^{i\frac{P}{2}} \|t_{\frac{N-2}{2}}^{Q,Q}\rangle\rangle_P + \delta_P^N \sqrt{2} e^{-i\frac{P}{2}} \|t_{\frac{N}{2}}^{Q,Q}\rangle\rangle_P \right), & \text{if } N \text{ even} \end{cases} \\
W\|t_{\left[\frac{N}{2}\right]}^{Q,Q}\rangle\rangle_P &= \begin{cases} \left( e^{i\frac{P}{2}} \|t_{\frac{N-3}{2}}^{Q,Q}\rangle\rangle_P - (-1)^{\delta_P^N} \|t_{\frac{N-1}{2}}^{Q,Q}\rangle\rangle_P \right), & \text{if } N \text{ odd;} \\ \delta_P^N \sqrt{2} e^{i\frac{P}{2}} \|t_{\frac{N}{2}-1}^{Q,Q}\rangle\rangle_P, & \text{if } N \text{ even.} \end{cases}
\end{aligned} \tag{4.14}$$

At first sight (4.14) looks much more complicated than (4.11). This is however misleading and the matrix  $W$  can be diagonalized explicitly. In order to do so, we exploit a connection to graph theory (see e.g. [45]). In this section, we restrict to a graphical representation of (4.11) and (4.14) – the calculation is spelled out in detail in appendix A of [44].

Each vector  $\|t_j^{Q_1, Q_2}\rangle\rangle_P$  will be symbolized as a ‘ $\bullet$ ’ with the index written above. The action of the potential  $V$  is symbolized by lines, with the square of the matrix elements

(up to an isomorphism presented in appendix A of [44]) attached to them. Assume first that we could distinguish the two flips we make. Then the graphical representation for the action of the potential  $V$  (or  $W$ ) would be

$$\begin{array}{c} 1 \quad 2 \quad \dots \quad N-2 \quad N-1 \\ \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \quad 1 \qquad \qquad \qquad \quad 1 \end{array} = (\mathcal{A}_{N-1}). \quad (4.15)$$

Here ‘ $(\mathcal{L}_k)$ ’ denotes the incidence matrix derived from the Cartan matrix of a Lie algebra  $\mathcal{L}_k$ . However, the states  $\|t_j^{Q,Q}\rangle_P$  and  $\|t_{N-j}^{Q,Q}\rangle_P$  are proportional to each other and must therefore be identified. Furthermore, it turns out that for  $N$  even and  $\frac{NP}{2\pi}$  odd  $\|t_{\frac{N}{2}}^{Q,Q}\rangle_P = 0$  vanishes identically. This already splits the graph (4.15) into two disjoint parts. Therefore, a graphical representation of (4.14) is given by:

$$\begin{aligned} W &\cong \begin{array}{c} 1 \quad 2 \quad \dots \quad \frac{N}{2}-2 \quad \frac{N}{2}-1 \\ \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \quad 1 \qquad \qquad \qquad \quad 1 \end{array} = \left(\mathcal{A}_{\frac{N}{2}-1}\right) & \text{for } N \text{ even, } \frac{NP}{2\pi} \text{ odd} \\ W &\cong \begin{array}{c} 1 \quad 2 \quad \dots \quad \frac{N-3}{2} \quad \frac{N-1}{2} \\ \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \quad 1 \qquad \qquad \qquad \quad 1 \end{array} \text{---} \text{---} \text{---} 1 = \left(\mathcal{T}_{\frac{N-1}{2}}\right) & \text{for } N \text{ odd} \\ W &\cong \begin{array}{c} 1 \quad 2 \quad \dots \quad \frac{N}{2}-2 \quad \frac{N}{2}-1 \quad \frac{N}{2} \\ \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \quad 1 \qquad \qquad \qquad \quad 1 \quad 2 \end{array} = \left(\mathcal{B}_{\frac{N}{2}}\right) & \text{for } N \text{ even, } \frac{NP}{2\pi} \text{ even.} \end{aligned} \quad (4.16)$$

Fortunately, all the graphs (4.16) have norm less or equal to  $2^4$ ). Because the eigenvalues of such graphs are classified [45] we can derive the first order explicitly.

In the case of (4.11) the situation is different. In order to simplify the discussion consider the case  $P = \phi = 0$ . Then one can represent (4.11) as

$$V \approx \dots \dots \dots \begin{array}{c} N-1 \quad 1 \quad 2 \quad \dots \quad N-2 \quad N-1 \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \quad 1 \quad 4 \quad 2 \quad \dots \quad 4 \quad 1 \end{array} \dots \dots \dots \quad (4.17)$$

Note that instead of drawing a closed diagram we have represented part of it twice. It is easy to see that the norm of (4.17) is larger than 3 (it tends to 4 for  $N \rightarrow \infty$ ). The absence of explicit expressions for the eigenvalues of such graphs prevented us from deriving an explicit expression for the first order of two-particle states in the  $Q = 0$  sector.

The result of the calculation in appendix A of [44] for the eigenvectors of the matrix  $W$  as given by (4.14) is:

$$\begin{aligned} \|\tau_k^{Q,Q}\rangle_P &:= \frac{2}{\sqrt{N}} \left\{ \sum_{j=1}^{[\frac{N}{2}]-1} \sin\left(\frac{(2k - \delta_P^N)j\pi}{N}\right) e^{-i\frac{P}{2}(j-1)} \|t_j^{Q,Q}\rangle_P \right. \\ &\quad \left. + \frac{\sqrt{2}}{\sqrt{3 + (-1)^N}} \sin\left(\frac{(2k - \delta_P^N)[\frac{N}{2}]\pi}{N}\right) e^{-i\frac{P}{2}([\frac{N}{2}]-1)} \|t_{[\frac{N}{2}]}^{Q,Q}\rangle_P \right\}. \end{aligned} \quad (4.18)$$

<sup>4)</sup>  $(\mathcal{T}_k)$  is the Tadpole graph.

The final result for the first order expansion of the energy for these excitations is for  $N \geq 3$ :

$$\begin{aligned} \Delta E_{2Q,k}(P, \phi, \varphi) = & 2 \left( \sum_{k=1}^{n-1} \bar{\alpha}_k (1 - \omega^{Qk}) \right) - 4\lambda \frac{\cos \left( \frac{P}{2} - \left( 1 - \frac{2Q}{n} \right) \phi \right) \cos \left( \frac{(2k - \delta_P^N) \pi}{N} \right)}{\sin \left( \frac{\pi Q}{n} \right)} \\ & + \mathcal{O}(\lambda^2), \quad 1 \leq k \leq \left\lfloor \frac{N + \delta_P^N - 1}{2} \right\rfloor. \end{aligned} \quad (4.19)$$

For remarks on the second order see appendix A of [44].

## 5. Evidence for quasiparticle spectrum

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In this section we present an argument using perturbation theory that the spectrum of the  $\mathbb{Z}_n$ -Hamiltonian (2.2) can be interpreted in terms of quasiparticles for a wide range of parameters. In the case of  $\mathbb{Z}_3$ , the dispersion relations of the two fundamental particles with  $Q = 1$  and  $Q = 2$  are given by (4.6) and (4.7).

The results in [32 – 36] suggest that we may expect a quasiparticle spectrum. More precisely, all excitation energies  $\Delta E_{Q,r}(P, \phi, \varphi)$  should satisfy

$$\Delta E_{Q,r}(P, \phi, \varphi) = \sum_{k=1}^{m_r} E_{Q^{(k)}}(P^{(k)}), \quad P = \sum_{k=1}^{m_r} P^{(k)} \pmod{2\pi}, \quad Q = \sum_{k=1}^{m_r} Q^{(k)} \pmod{n} \quad (5.1)$$

where  $E_1(P), \dots, E_{n-1}(P)$  are the energies of the  $n - 1$  fundamental quasiparticles. Additionally, the fundamental quasiparticles seem to satisfy a Pauli principle, i.e.  $Q^{(i)} = Q^{(j)}$  implies  $P^{(i)} \neq P^{(j)}$ . In particular, for  $n = 3$ , (4.6) and (4.7) are the dispersion relations of the fundamental  $Q = 1$  and  $Q = 2$  quasiparticles and all other states can be obtained by composition under the assumption that energy, momentum and charge are additive.

It should be clear to the reader that a particle interpretation is not directly incorporated into the Hamiltonian (2.2) neither is it related to any particular integrability properties of the Hamiltonian. Even on the superintegrable line the derivation of (5.1) given in [35] for  $n = 3$  only yields the quasiparticle spectrum at the very end. This is due to the fact that standard Bethe ansatz methods (which would automatically ensure a quasiparticle spectrum) have failed for the integrable chiral Potts model such that functional relations for the transfer matrix had to be used in order to determine the excitation spectrum (see [35] and references therein). Instead of specializing to the integrable chiral Potts model and using particular integrability properties we will argue below that the quasiparticle interpretation (5.1) follows from the basic physical properties of finite correlation length and absence of long-range order which are not related to integrability at all. Although this is not a surprising observation, a rigorous derivation of (5.1) using this argument is to the best of our knowledge not contained in the literature. This motivated us to spell this argument out in more detail below as well as in appendix A.

Before presenting a proof of (5.1) we would like to add a remark on Figs. 2–4 of [34]: In the limit  $N \rightarrow \infty$  the eigenvalues seem to become dense such that we may expect to interpret the energy bands as continuous spectrum in the weak limit of the Hamiltonian. Note that according to our definition, the single particle excitations (4.2) lead to point

spectrum. One also observes that the energy bands are filled from the interior such that their boundaries do not belong to the spectrum for any finite  $N$ . However, we have pointed out in section 2 that the spectrum is closed in the infinite chain limit. Thus, the boundaries of the energy bands will belong to the spectrum in this infinite  $N$  limit. It is worthwhile noting that the normalization factors  $\frac{2}{\sqrt{N}}$  for the two-particle states in (4.18) demonstrate that these states tend to zero for  $N \rightarrow \infty$  and will therefore not give rise to proper eigenvectors. This confirms that with our definition of the limit composite particle states belong to the continuous spectrum.

Before proceeding with the general discussion let us first look a little closer at the two-particle states. Comparing (4.19) with the first order expansion for the single-particle states eq. (39) of [32] one observes that this first order expansion of the two-particle excitations is in agreement with the quasiparticle rule (5.1). Up to first order the composite particle states satisfy either  $2\Delta E_{Q,0}(P, \phi, \varphi) < \Delta E_{2Q,k}(2P, \phi, \varphi) < 2\Delta E_{Q,0}(P + 2\pi, \phi, \varphi)$  or  $2\Delta E_{Q,0}(P, \phi, \varphi) > \Delta E_{2Q,k}(2P, \phi, \varphi) > 2\Delta E_{Q,0}(P + 2\pi, \phi, \varphi)$  depending on which one of the single particle energies is larger. Thus, the two-particle states do indeed lie inside the energy band of two single-particle states and the boundaries are not included. Even more, we can see from (4.19) that the two-particle states become dense in this energy band for  $N \rightarrow \infty$ .

Let us now present a more abstract argument which ensures the validity of (5.1). The interaction in the Hamiltonian (2.1) is very short ranged – in fact, only among nearest neighbours. In the massive high-temperature phase there is no spontaneous order and the correlation length is finite. Thus, if one puts two excitations of ‘short’ chains with a sufficient separation on a longer chain, the interaction will be negligible. For example, putting one single-particle excitation on the left half of the chain and another on the right half will approximate a two-particle excitation.

We make this derivation of the quasiparticle interpretation of the spectrum more precise using perturbative arguments. According to the remarks at the beginning of section 4 the quasiparticle spectrum with flat dispersion curves is easily verified for  $\lambda = 0$ . In this section we sketch a proof that the quasiparticle picture remains valid for  $\lambda > 0$  – we just present the main ideas. A modified rigorous version of the proof is spelled out in appendix A.

First, we notice that

$$\begin{aligned} \Delta H_{N+M}^{(n)} &= \Delta H_N^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes \Delta H_M^{(n)} + \mathcal{O}(\Delta H_{N,M}) , \\ T_{N+M} &= \{\mathbf{1} + \mathcal{O}(T_{N,M})\} T_N \otimes T_M \end{aligned} \tag{5.2}$$

where ‘ $\mathcal{O}(\Xi_{N,M})$ ’ denotes an operator acting only at sites  $0, N-1, N$  and  $N+M-1$ . One of the main steps of the proof is to show that these boundary operators vanish in the limit  $N, M \rightarrow \infty$ . It should be clear to the reader that the coproduct rule (5.2) is going to be crucial for the derivation of the quasiparticle picture (5.1). In particular, our proof cannot be easily modified to accommodate more complicated selection rules and will therefore be specific for  $\mathbb{Z}_n$ -spin quantum chains

If we can build a composite state of any two states we have to show that energy, charge and momentum behave additive under this composition and that we can construct all states. Then, the quasiparticle structure follows by induction.

Composite particle states are expected to give rise to continuous spectrum. This is a technical complication in the argument we are going to give because it is not possible to use eigenstates but we must show that the resolvent is unbounded. However, for each finite  $N$  the Hamiltonian has a complete set of eigenstates. We have already argued in section 2 that the resolvent becomes unbounded for any energy if it can be approximated by eigenvalues of  $\Delta H_N^{(n)}$ . This in turn can be ensured by providing a sequence of vectors  $\|k; E\rangle\rangle_P$  that approximate eigenvectors of  $\Delta H_N^{(n)}$  to eigenvalue  $E$  for  $N$  large. Thus, we would have to take two limits simultaneously. However, a standard argument shows that it is no loss of generality to restrict to the diagonal sequence  $k = N$ .

The assumption in the induction is that we can choose two sequences of states  $\|N; E_1\rangle\rangle_{P_1} \in \mathcal{H}_N$  and  $\|M; E_2\rangle\rangle_{P_2} \in \mathcal{H}_M$  such that in the weak limits of  $\Delta H^{(n)}$  and  $T$  they give rise to unbounded resolvents at  $E_k, e^{iP_k}$ :

$$\begin{aligned} \lim_{N \rightarrow \infty} (\Delta H_N^{(n)} - E_1) \|N; E_1\rangle\rangle_{P_1} &= 0, & \lim_{N \rightarrow \infty} (T_N - e^{iP_1}) \|N; E_1\rangle\rangle_{P_1} &= 0, \\ \lim_{M \rightarrow \infty} (\Delta H_M^{(n)} - E_2) \|M; E_2\rangle\rangle_{P_2} &= 0, & \lim_{M \rightarrow \infty} (T_M - e^{iP_2}) \|M; E_2\rangle\rangle_{P_2} &= 0. \end{aligned} \quad (5.3)$$

We know that such sequences of states exist at least for the single-particle states – the perturbative series for  $\|s^Q\rangle\rangle_P$ .

The second major step in the proof is to consider now the state  $\|N; E_1\rangle\rangle_{P_1} \otimes \|M; E_2\rangle\rangle_{P_2} \in \mathcal{H}_{N+M}$ . From (5.2) one has

$$\begin{aligned} \Delta H_{N+M}^{(n)} (\|N; E_1\rangle\rangle_{P_1} \otimes \|M; E_2\rangle\rangle_{P_2}) &= (\Delta H_N^{(n)} \|N; E_1\rangle\rangle_{P_1}) \otimes \|M; E_2\rangle\rangle_{P_2} \\ &\quad + \|N; E_1\rangle\rangle_{P_1} \otimes (\Delta H_M^{(n)} \|M; E_2\rangle\rangle_{P_2}) \\ &\quad + \mathcal{O}(\Delta H_{N,M}) (\|N; E_1\rangle\rangle_{P_1} \otimes \|M; E_2\rangle\rangle_{P_2}), \\ T_{N+M} (\|N; E_1\rangle\rangle_{P_1} \otimes \|M; E_2\rangle\rangle_{P_2}) &= (T_N \|N; E_1\rangle\rangle_{P_1}) \otimes (T_M \|M; E_2\rangle\rangle_{P_2}) \\ &\quad + \mathcal{O}(T_{N,M}) (T_N \|N; E_1\rangle\rangle_{P_1}) \otimes (T_M \|M; E_2\rangle\rangle_{P_2}). \end{aligned} \quad (5.4)$$

The vanishing of the boundary terms in (5.4) can be shown using e.g. perturbative arguments. The crucial point in the argumentation is that the momentum eigenstates have normalization factors  $N^{-\frac{1}{2}}, M^{-\frac{1}{2}}$ . Any operator acting only at boundaries yields only a finite part of these states in contrast to the operators  $T_N$  and  $\Delta H_N^{(n)}$  which act on the complete chain and yield complete momentum eigenstates. The finite pieces of momentum eigenstates are suppressed by the normalization factors  $N^{-\frac{1}{2}}$  in the infinite chain length limit. For example, for the translation operator  $T_N$  it is easy to verify explicitly that the boundary terms tend to zero at  $\lambda = 0$  using precisely this argument. The argumentation for the Hamiltonian is analogous but slightly more complicated. A similar perturbative argument has already been presented in [37] in order to show the vanishing of the  $Q$ -dependence in the low-temperature regime.

These rather technical details are spelled out in appendix A.

We have shown that the boundary operators  $\mathcal{O}(\Delta H_{N,M})$  and  $\mathcal{O}(T_{N,M})$  vanish as  $N, M$  go to infinity. Thus, in this limit

$$\begin{aligned} \lim_{N, M \rightarrow \infty} (\Delta H_{N+M}^{(n)} - (E_1 + E_2)) (\|N; E_1\rangle\rangle_{P_1} \otimes \|M; E_2\rangle\rangle_{P_2}) &= 0, \\ \lim_{N, M \rightarrow \infty} (T_{N+M} - e^{i(P_1+P_2)}) (\|N; E_1\rangle\rangle_{P_1} \otimes \|M; E_2\rangle\rangle_{P_2}) &= 0 \end{aligned} \quad (5.5)$$



holds. This shows that energy  $E$  and momentum  $P$  are additive – the additivity of the charge  $Q$  is obvious. One can always build a basis for the space  $\mathcal{H}_K = \otimes^K \mathcal{H}_1$  by considering tensor products of basis vectors in  $\mathcal{H}_N$  and  $\mathcal{H}_M$  with  $N + M = K$ . This is precisely what we have done. Thus, the above procedure does indeed yield the complete spectrum.

One should be careful about the requirements that enter in our proof of the quasiparticle picture in order not to mistake it for more general than it is. Note that the vanishing of boundary terms is a crucial part of the proof. However, boundary terms are substantial for conformally invariant systems with long ranged correlations. Also in the low-temperature phase boundary terms play an important rôle because the free part of the Hamiltonian depends on the difference of neighbouring spins (see also [37]). Thus, our proof applies neither to critical points where one might have conformal invariance nor to the low-temperature phase. Furthermore, we have used the explicit form (2.1) of the Hamiltonian (for example for the selection rules in (5.1)).

It should be clear to the reader that our argument relies on a perturbation series for the single-particle states and is valid only if this series is convergent. We will discuss the radius of convergence for the  $\mathbb{Z}_3$ -chain in more detail in section 8. At this place we would just like to mention that this perturbative argument cannot be applied to massless incommensurate phases because the main limitations on the convergence come from level crossings which are characteristic for massless incommensurate phases.

Note that we have not assumed the Hamiltonian to be hermitean. In particular, the quasiparticle picture should also be valid for  $\phi \in \mathbb{C}$  as long as the single-particle excitations exist and converge. This is indeed supported by numerical calculations [46].

The argument proving the quasiparticle structure can be refined in order to give an upper estimate for the rate of convergence in  $N$  of the energy of a  $k$ -particle state. As an approximation to a  $k$ -particle state for  $kN$  sites, total energy  $E_{\text{tot}}$  and total momentum  $P$  we may take the  $k$ -fold tensor product of single-particle states

$$\|kN; E_{\text{tot}}\rangle_P := \|N; E_1\rangle_{P_1} \otimes \dots \otimes \|N; E_k\rangle_{P_k} \quad (5.6)$$

with  $E_{\text{tot}} = \sum_{l=1}^k E_l$ ,  $P = \sum_{l=1}^k P_l$ . Now, the deviation from the limit  $N \rightarrow \infty$  is given by:

$$\begin{aligned} P \langle\langle kN; E_{\text{tot}} | \Delta H_{kN}^{(n)} | kN; E_{\text{tot}} \rangle\rangle_P - E_{\text{tot}} &= \prod_{l=1}^k P_l \langle\langle N; E_l | \mathcal{O}(\Delta H_N) | N; E_l \rangle\rangle_{P_l} \\ &= \mathcal{O}(N^{-k}). \end{aligned} \quad (5.7)$$

$\mathcal{O}(\Delta H_N)$  is some operator that acts only at sites 1 and  $N$ . The first equality simply uses the definition of the scalar product in tensor products. The last equality is more profound and due to the fact that operators acting only at boundaries of the chain are suppressed by  $N^{-1}$  due to the normalization factor in the finite Fourier transformation for momentum eigenstates. This shows that the deviation of the energy of a  $k$ -particle state ( $k > 1$ ) from the limit is at most of order  $N^{-k}$  for  $N \rightarrow \infty$ . Of course, one might find better approximations for the eigenstates and the convergence could be faster. Thus, the  $N$ -dependence of some energy eigenvalue gives only a lower bound on the number  $k$  of particles involved.

This general argument is confirmed by our results for the two-particle states. Expanding  $\cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}(x^4)$  we can read off from (4.19) that the first order correction of the  $k$ th two-particle state with respect to the boundary of the energy band behaves as  $N^{-2}$ . This is precisely what we expect from the general considerations.

This argument shows in particular that in a finite-size system the *energy* of any state remains unchanged to order  $\frac{1}{N}$ . According to the argument presented at the end of section 3 the energies of the fundamental particle states have to converge exponentially in  $N$  and the energies of composite particle states have corrections at most of order  $\frac{1}{N^2}$ . Thus, the only modification in (5.1) at order  $\frac{1}{N}$  in the massive high temperature phase is a discretization of the momentum (and possible minor modifications of the Brillouin zones and selection rules [36]).

Note that the proof of the vanishing of boundary terms as sketched above and presented in detail in appendix A also directly applies to the Hamiltonian (2.2) itself. So far, we have restricted ourselves to periodic boundary conditions  $\Gamma_{N+1} = \Gamma_1$ . However, one could also impose toroidal boundary conditions: ‘Cyclic’ boundary conditions  $\Gamma_{N+1} = \omega^{-R}\Gamma_1$  or ‘twisted’ boundary conditions  $\Gamma_{N+1} = \omega^{-R}\Gamma_1^+$ . Even ‘free’ boundary conditions  $\Gamma_{N+1} = 0$  are well-known in the literature. Our argument shows that all these different choices lead to the same spectrum in the limit  $N \rightarrow \infty$ . In particular, our results are valid for all choices of boundary conditions and one is free to choose those which seem most appropriate, e.g. one can leave the ends of the chain open instead of the unnatural end-identification for a realistic physical system.

Again, this observation for the massive high-temperature phase is to be contrasted with other situations. In particular, at the second order phase transition  $\phi = \varphi = 0$ ,  $\lambda = 1$  the correlation length becomes infinite and the boundary terms are very important [47 – 49]. Even in the massive low-temperature phase one observes long range order and boundary terms cannot be neglected [37].

So far, we have not addressed the question of whether the fundamental particles satisfy a Pauli principle or not – note that the above discussion is intrinsically insensitive to a Pauli principle because the limit was defined such that the spectrum forms a closed set. Nevertheless, for the special case  $n = 3$  and  $\phi = \varphi = \frac{\pi}{2}$ , eq. (5.1) was obtained in [35] supplemented with the Pauli principle mentioned below (5.1). Fortunately, due to (4.19), we have some control over the finite-size dependence of the scattering states of two identical particles in the general case. Up to first order in  $\lambda$  these finite-size effects do essentially neither depend on the charge  $Q$  nor on the number of states  $n$ . Therefore, the nature of the fundamental excitations can be determined by looking at one particular choice of  $Q$  and  $n$ . However, for  $n = 2$  one obtains the Ising model where it is well-known that the excitation spectrum can be explained in terms of one fundamental *fermion* (see e.g. [50]). This indicates that the fundamental excitations for general  $n$  should be regarded as fermions. In particular, for a scattering state of two identical excitations  $i$  and  $j$  the momenta must satisfy  $P_i \neq P_j$ . In a scattering state of two *different* fundamental particles these two fundamental particles can easily be distinguished because they carry different  $\mathbb{Z}_n$ -charges. Therefore, two different particles should not be subject to a Pauli principle (like it is the case for two different non-interacting fermions).

## 6. Correlation functions

In recent papers a systematic investigation of the correlation functions of the  $\mathbb{Z}_3$ -chiral Potts model in the massive phases has been started. First, a non-vanishing wave vector has been predicted in [33][51] for the massive high-temperature phase and its critical exponent was calculated from level crossings. Next, perturbative calculations for the massive *low-temperature phase* analogous to those to be presented below have been reported in [37]. We also studied the correlation function for the operator  $\Gamma$  in the massive high-temperature phase numerically in [34] and were able to demonstrate an oscillation. In [34] some of the results to be presented below we already cited without derivation. Note also that for the massless phases around  $\lambda \sim 1$  of the  $\mathbb{Z}_3$ -chain correlation functions have been derived in [52] borrowing results from conformal field theory.

In this section we study correlation functions for the  $\mathbb{Z}_3$ -chiral Potts model perturbatively. Before defining correlation functions, we first note that the two-point functions are translationally invariant because the groundstate  $|v\rangle$  is translationally invariant:

$$\begin{aligned} \langle v | \Gamma_{x+r}^+ \Gamma_r | v \rangle &= \langle v | \Gamma_{x+1}^+ \Gamma_1 | v \rangle , \\ \langle v | \sigma_{x+r}^+ \sigma_r | v \rangle &= \langle v | \sigma_{x+1}^+ \sigma_1 | v \rangle . \end{aligned} \quad (6.1)$$

Thus, it makes sense to define the correlation function for an operator  $\Xi$  by the following expression:

$$C_{\Xi}(x) := \frac{\langle v | \Xi_{x+1}^+ \Xi_1 | v \rangle}{\langle v | v \rangle} - \frac{\langle v | \Xi_{x+1}^+ | v \rangle \langle v | \Xi_1 | v \rangle}{\langle v | v \rangle^2} \quad 0 \leq x < \frac{N}{2} \quad (6.2)$$

where  $|v\rangle$  is the eigenvector of the Hamiltonian to lowest energy. Here, we do not assume that  $|v\rangle$  is normalized to one and have therefore included the proper normalization factors in (6.2). The correlation functions of the operators  $\Gamma_x$  and  $\sigma_x$  have the property

$$\begin{aligned} C_{\Gamma}(-x) &= C_{\Gamma}(x)^* , \\ C_{\sigma}(-x) &= C_{\sigma}(x)^* = C_{\sigma}(x) \end{aligned} \quad (6.3)$$

such that it makes sense to restrict to positive  $x$ . Note that (6.3) follows by complex conjugation using (6.1). Explicit calculations show the validity of (6.1) and (6.3) as well.

For simplicity we will first neglect the correction term for the uncorrelated part as well as the normalization in (6.2) and consider the following expression:

$$c_{\Xi}(x) := \langle v | \Xi_{x+1}^+ \Xi_1 | v \rangle \quad 0 \leq x < \frac{N}{2}. \quad (6.4)$$

The operator  $\Xi$  for the  $\mathbb{Z}_3$ -chiral Potts model can be either  $\Gamma$  or  $\sigma$ . For  $n > 3$  also different powers of these operators may be interesting.

One can use the quasiparticle picture which we have already derived in order to rewrite a correlation function  $C_{\Xi}(x)$  as follows:

$$\begin{aligned} C_{\Xi}(x) &= \frac{\sum_{n=0}^{\infty} \int_0^{2\pi} \left( \prod_{i=1}^n dp_i \right) \langle v | \Xi_{x+1}^+ | p_1, \dots, p_n \rangle \langle p_1, \dots, p_n | \Xi_1 | v \rangle}{\langle v | v \rangle} - \frac{|\langle v | \Xi_1 | v \rangle|^2}{\langle v | v \rangle^2} \\ &= \sum_{n=1}^{\infty} \int_0^{2\pi} \left( \prod_{i=1}^n dp_i \right) e^{ix((\sum_{j=1}^n p_j) - P_v)} \frac{|\langle p_1, \dots, p_n | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} \end{aligned} \quad (6.5)$$

where we have inserted a complete set of normalized  $n$ -particle states  $|p_1, \dots, p_n\rangle$ . Representations similar to (6.5) have been used in quantum field theory for a long time (see e.g. [53]) and are well-known to be useful for the evaluation of correlation functions of statistical models (see e.g. [31]). According to (6.5) one could compute the correlation function  $C_{\Xi}(x)$  by computing its ‘form factors’  $\langle p_1, \dots, p_n | \Xi_1 | v \rangle$ , but one can even derive interesting results without doing so. Clearly, if the groundstate  $|v\rangle$  has non-zero momentum  $P_{|v\rangle} \neq 0$  we expect an oscillatory contribution to the correlation function. However, one can read off from (6.5) that an oscillatory contribution is also to be expected if  $P_{|v\rangle} = 0$  but the model breaks parity which precisely applies to the massive high-temperature phase of the chiral Potts model. The correlation functions of massive models in general have an exponential decay, i.e.  $C_{\Xi}(x) = e^{-\frac{x}{\xi}} f_{\Xi}(x)$  where  $f_{\Xi}(x)$  is some bounded function. According to (6.5) we also expect an oscillatory contribution of the form  $e^{i\frac{2\pi x}{L}}$ . In summary, we expect correlation functions of the approximate form

$$C_{\Xi}(x) \sim e^{-\frac{x}{\xi} + i\frac{2\pi x}{L}}. \quad (6.6)$$

$\xi$  is called ‘correlation length’ and  $L$  is the ‘oscillation length’ ( $L^{-1}$  is the ‘wave vector’).

More precisely, for the  $\mathbb{Z}_3$ -chiral Potts model the operator  $\Gamma_1$  creates  $Q = 1$ -single-particle excitations from the groundstate. The dispersion relations of these particles clearly violate parity. Therefore we expect that  $C_{\Gamma}(x)$  is of the form (6.6). The action of the operator  $\sigma_1$  is much less spectacular. In particular, it leaves the charge sector  $Q = 0$  invariant and thus it need not necessarily have an oscillatory contribution. In fact, from (6.3) we see that  $C_{\sigma}(x)$  should be real which in view of (6.6) implies the absence of oscillations.

Symmetries of the Hamiltonian translate into symmetries of the form factors. In certain cases these symmetries are already sufficient to compute the oscillation length  $L$ . In appendix B we demonstrate this in a few cases for the correlation function  $C_{\Gamma Q}(x)$  of the  $\mathbb{Z}_n$ -chiral Potts model. For  $\text{Re}(\phi) = \pi$  one observes a shifted parity symmetry [46] that can be derived e.g. along the lines of [14]. Using this symmetry one can show (see appendix B) that

$$C_{\Gamma Q}(x) = e^{\frac{2\pi i x}{L}} f_{Q,r}(x) \quad (6.7a)$$

with

$$\begin{aligned} f_{Q,r}(x) &\in \mathbb{R} && \forall x, \\ L &= \infty && \text{for } \phi = r\pi, r \in \mathbb{Z} \text{ or } \varphi \in \mathbb{R}, \text{Re}(\phi) = 0, \\ L &= \frac{2n}{n - 2Q} && \text{for } \varphi \in \mathbb{R}, \text{Re}(\phi) = \pi \text{ and } 0 < Q < n. \end{aligned} \quad (6.7b)$$

Let us now turn to the explicit computation of correlation functions for the  $\mathbb{Z}_3$ -chain. In order to be able to calculate the correlation functions we need to know the groundstate  $|v\rangle$ . We will calculate it from the free ground state  $|\text{GS}\rangle$  using the perturbation expansion (3.7). We should stress again that although we assume the free groundstate  $|\text{GS}\rangle$  to be normalized to 1, this is not necessarily true for the complete state  $|v\rangle$ . The expansion of the groundstate  $|v\rangle$  provides us with an expansion for the correlation functions in powers of  $\lambda$

$$c_{\Xi}(x) = \sum_{\nu=0}^{\infty} \lambda^{\nu} c_{\Xi}^{(\nu)}(x) \quad (6.8)$$

where we again neglect an irrelevant overall normalization factor which depends on  $\lambda$ . Note that according to (3.7) a  $k$ th order expansion of the groundstate yields a  $k + 1$ th order expansion of the groundstate energy as a byproduct.

Using the state (4.1) one can calculate for the  $\mathbb{Z}_3$ -chiral Potts model in the high-temperature phase the first orders in  $\lambda$  for  $c_\Gamma(x)$ :

$$\begin{aligned} c_\Gamma^{(0)}(x) &= \delta_{x,0} , & c_\Gamma^{(1)}(x) &= \delta_{x,1} \frac{e^{i\frac{\phi}{3}}}{3\mathcal{C}} , \\ c_\Gamma^{(2)}(x) &= \frac{1}{6\mathcal{C}^2} \left\{ \delta_{x,0} \frac{N}{3} + \delta_{x,1} \frac{e^{-i\frac{2\phi}{3}}}{2} + \delta_{x,2} e^{i\frac{2\phi}{3}} \right\} . \end{aligned} \quad (6.9)$$

In order to save place we present higher orders only in the final, properly normalized form (6.15).

For the first orders of  $c_\sigma(x)$  we obtain

$$\begin{aligned} c_\sigma^{(0)}(x) &= 1 , & c_\sigma^{(1)}(x) &= 0 , \\ c_\sigma^{(2)}(x) &= \frac{1}{3\mathcal{C}^2} \left\{ \delta_{x,0} + \frac{\delta_{x,1}}{4} + \frac{N-6}{6} \right\} . \end{aligned} \quad (6.10)$$

Again, we have postponed presentation of higher orders to the final, properly normalized result (6.14).

Let us now discuss the correction terms in (6.2). The operator  $\Gamma_x$  creates charge such that charge conservation implies  $\langle v | \Gamma_x^\dagger | v \rangle = \langle v | \Gamma_x | v \rangle = 0$  for all  $x$ . Thus

$$C_\Gamma(x) = \frac{c_\Gamma(x)}{\langle v | v \rangle} . \quad (6.11)$$

The corrections for the operator  $\sigma$  are more complicated. Using the expansion (3.7) for the groundstate one obtains independent of  $x$

$$\begin{aligned} \langle v | \sigma_x^\dagger | v \rangle &= \langle v | \sigma_x | v \rangle^* , \\ \langle v | \sigma_x | v \rangle &= 1 + \lambda^2 \frac{N-3}{18\mathcal{C}^2} + \lambda^3 \frac{(N-3)\bar{\mathcal{C}}_3}{54\mathcal{C}^3} \\ &+ \lambda^4 \frac{1}{81\mathcal{C}^2} \left\{ \frac{9i \sin\left(\frac{2\varphi}{3}\right) + 9 - 4N + (28N - 90)\mathcal{C}^2}{3\mathcal{R}^2} + \frac{(N-3)^2}{8\mathcal{C}^2} \right\} \\ &+ \mathcal{O}(\lambda^5) \quad \forall x. \end{aligned} \quad (6.12)$$

In order to be able to evaluate (6.2) we have to divide (6.12) by the norm squared of  $|v\rangle$  before we subtract it. We apply  $(1 + \sum_{\nu=1}^{\infty} a_\nu \lambda^\nu)^{-1} = \sum_{\mu=0}^{\infty} (-\sum_{\nu=1}^{\infty} a_\nu \lambda^\nu)^\mu$  to the norm of  $|v\rangle$

$$\langle v | v \rangle = 1 + \lambda^2 \frac{N}{18\mathcal{C}^2} + \lambda^3 \frac{N\bar{\mathcal{C}}_3}{54\mathcal{C}^3} + \lambda^4 \frac{N}{81\mathcal{C}^2} \left\{ \frac{4(7\mathcal{C}^2 - 1)}{3\mathcal{R}^2} + \frac{N}{8\mathcal{C}^2} \right\} + \mathcal{O}(\lambda^5) \quad (6.13)$$

and obtain a normalized expression for the one-point function (6.12). It is not surprising that up to the order calculated one has the equality  $|\frac{\langle v | \sigma_x | v \rangle}{\langle v | v \rangle}| = |C_\Gamma^{\text{lt}}(1)|$  at the dual point

in the low-temperature phase. In fact, this is to be expected from the proof of duality presented in the appendix of [37] (eq. (A.5)).

Inserting (6.12) and (6.13) into (6.10) leads to

$$\begin{aligned}
C_\sigma^{(0)}(x) = C_\sigma^{(1)}(x) = 0, \quad C_\sigma^{(2)}(x) = \frac{1}{3\mathcal{C}^2} \left\{ \delta_{x,0} + \frac{\delta_{x,1}}{4} \right\}, \quad C_\sigma^{(3)}(x) = \frac{\bar{\mathcal{C}}_3}{9\mathcal{C}^3} \left\{ \delta_{x,0} + \frac{\delta_{x,1}}{4} \right\}, \\
C_\sigma^{(4)}(x) = \frac{1}{27\mathcal{C}^2} \left\{ -\delta_{x,0} \left( \frac{2(1-10\mathcal{C}^2)}{\mathcal{R}^2} + \frac{3}{2\mathcal{C}^2} \right) + \delta_{x,1} \left( \frac{1+20\mathcal{C}^2}{3\mathcal{R}^2} - \frac{1}{\mathcal{C}^2} \right) \right. \\
\left. + \delta_{x,2} \left( \frac{2(1+2\mathcal{C}^2)}{3\mathcal{R}^2} + \frac{1}{16\mathcal{C}^2} \right) \right\}.
\end{aligned} \tag{6.14}$$

Note that also the  $N$ -dependence in (6.9) is due to the  $N$ -dependence (6.13) of the norm of  $|v\rangle$ . If we normalize  $|v\rangle$  properly to 1 we have

$$\begin{aligned}
C_\Gamma^{(0)}(x) = \delta_{x,0}, \quad C_\Gamma^{(1)}(x) = \delta_{x,1} \frac{e^{i\frac{\phi}{3}}}{3\mathcal{C}}, \\
C_\Gamma^{(2)}(x) = \frac{1}{6\mathcal{C}^2} \left\{ \delta_{x,1} \frac{e^{-i\frac{2\phi}{3}}}{2} + \delta_{x,2} e^{i\frac{2\phi}{3}} \right\}, \\
C_\Gamma^{(3)}(x) = \frac{1}{54\mathcal{C}} \left\{ -\delta_{x,1} e^{i\frac{\phi}{3}} \left( \frac{1}{\mathcal{C}^2} + \frac{8}{\mathcal{R}} \right) + \delta_{x,2} e^{-i\frac{\phi}{3}} \left( \frac{2}{\mathcal{C}^2} - \frac{8}{\mathcal{R}} \right) + \delta_{x,3} \frac{5e^{i\phi}}{\mathcal{C}^2} \right\}
\end{aligned} \tag{6.15a}$$

which obviously is  $N$ -independent. Finally, in this case we obtain for the fourth order

$$\begin{aligned}
C_\Gamma^{(4)}(x) = \frac{1}{81\mathcal{C}^2} \left\{ -\delta_{x,1} \left( e^{i\frac{4\phi}{3}} + 4e^{-i\frac{2\phi}{3}} \right) \left( \frac{9}{16\mathcal{C}^2} + \frac{3}{\mathcal{R}} \right) \right. \\
+ \delta_{x,2} \left( \frac{8e^{i\frac{2\phi}{3}}(19\mathcal{C}^2 - 4)}{3\mathcal{R}^2} + \frac{3e^{-i\frac{4\phi}{3}} - 20e^{i\frac{2\phi}{3}}}{8\mathcal{C}^2} - \frac{3e^{-i\frac{4\phi}{3}}}{\mathcal{R}} \right) \\
\left. + \delta_{x,3} \left( \frac{40\mathcal{C}^2 - 7}{\mathcal{R}^2} + \frac{9}{4\mathcal{C}^2} \right) + \delta_{x,4} \frac{35e^{i\frac{4\phi}{3}}}{8\mathcal{C}^2} \right\}
\end{aligned} \tag{6.15b}$$

which is also  $N$ -independent. More precisely,  $C_\Gamma^{(k)}(x)$  and  $C_\sigma^{(k)}(x)$  are independent of  $N$  if  $N > 2k$  and  $x \leq k$ .

$C_\sigma(x)$  is real and positive for all values of  $\phi$  and  $\varphi$  up to the order calculated. However, it is not easy to read off from (6.14) what might be the form for large  $x$ . Thus, we specialize to  $\phi = \frac{\pi}{2}$  and calculate two further orders for  $C_\sigma(x)$ :

$$\begin{aligned}
C_\sigma^{(0)}(x) = C_\sigma^{(1)}(x) = C_\sigma^{(3)}(x) = C_\sigma^{(5)}(x) = 0, \\
C_\sigma^{(2)}(x) = \frac{1}{9} \{4\delta_{x,0} + \delta_{x,1}\}, \\
C_\sigma^{(4)}(x) = \frac{1}{81} \{5\delta_{x,0} + 2\delta_{x,2}\}, \\
C_\sigma^{(6)}(x) = \frac{1}{6561} \{190\delta_{x,0} - 13\delta_{x,1} + 38\delta_{x,2} + 60\delta_{x,3}\}.
\end{aligned} \tag{6.16}$$

As a byproduct we verified in this case two further orders of the equality  $|\frac{\langle v | \sigma_x | v \rangle}{\langle v | v \rangle}| = |C_\Gamma^{tt}(1)|$  at the dual point in the low-temperature phase.

$C_\Gamma(x)$  in general has a non-vanishing imaginary part and therefore is worth while being considered in more detail. Thus, we specialize again to the superintegrable case  $\phi = \varphi = \frac{\pi}{2}$  and obtain after calculating two further orders

$$\begin{aligned}
C_\Gamma^{(0)}(x) &= \delta_{x,0} , & C_\Gamma^{(1)}(x) &= \delta_{x,1} \left( \frac{1}{3} + i \frac{\sqrt{3}}{9} \right) , \\
C_\Gamma^{(2)}(x) &= \frac{1}{18} \{ \delta_{x,1} + 2\delta_{x,2} \} + i \frac{\sqrt{3}}{18} \{ -\delta_{x,1} + 2\delta_{x,2} \} , \\
C_\Gamma^{(3)}(x) &= \frac{1}{81} \{ 4\delta_{x,1} + 10\delta_{x,2} \} + i \frac{\sqrt{3}}{243} \{ 4\delta_{x,1} - 10\delta_{x,2} + 20\delta_{x,3} \} , \\
C_\Gamma^{(4)}(x) &= \frac{1}{1458} \{ 27\delta_{x,1} + 18\delta_{x,2} + 210\delta_{x,3} - 70\delta_{x,4} \} + i \frac{\sqrt{3}}{1458} \{ -27\delta_{x,1} + 18\delta_{x,2} + 70\delta_{x,4} \} , \\
C_\Gamma^{(5)}(x) &= \frac{1}{2187} \{ 45\delta_{x,1} + 108\delta_{x,2} + 252\delta_{x,4} - 126\delta_{x,5} \} \\
&\quad + i \frac{\sqrt{3}}{2187} \{ 15\delta_{x,1} - 36\delta_{x,2} - 14\delta_{x,3} + 84\delta_{x,4} + 42\delta_{x,5} \} , \\
C_\Gamma^{(6)}(x) &= \frac{1}{39366} \{ 381\delta_{x,1} + 214\delta_{x,2} + 2314\delta_{x,3} + 784\delta_{x,4} + 2310\delta_{x,5} - 1848\delta_{x,6} \} \\
&\quad + i \frac{\sqrt{3}}{39366} \{ -381\delta_{x,1} + 214\delta_{x,2} - 784\delta_{x,4} + 2310\delta_{x,5} \} .
\end{aligned} \tag{6.17}$$

Of course, we still have to calculate the sum (6.8). Thus, changes of signs in individual orders need not necessarily turn up in the final result. In fact, it turns out that the imaginary part of  $C_\Gamma(x)$  is always positive up to order 6 because the smallest orders are positive and they dominate the others. However, for sufficiently small  $\lambda$  the real part does indeed change signs around  $x = 4$ . Although we are not able to verify if it becomes positive again around  $x = 12$  (which would need more than the double of the orders which we have calculated) it is in good agreement with the expected form (6.6). Therefore we fit (6.17) by a complex exponential function. In summary, (6.17) indicates that

$$C_\Gamma(x) = a e^{\left(\frac{2\pi i}{L} - \frac{1}{\xi_\Gamma}\right)x} + (1 - a)\delta_{x,0} , \tag{6.18a}$$

$$C_\sigma(x) = p e^{-\frac{x}{\xi_\sigma}} + q\delta_{x,0} \tag{6.18b}$$

such that  $C_\Gamma(x)$  is of the form (6.6) for  $x > 0$ . In (6.18) we have also taken into account that from (6.16)  $\frac{C_\sigma(0)}{C_\sigma(1)} \approx 4$  independent of the correlation length  $\xi_\sigma$ .

If (6.18a) is the correct form for  $C_\Gamma(x)$  we infer from (6.17) that  $L$  is about 14 for small  $\lambda$ . We can also see from the higher orders that  $L$  increases with increasing  $\lambda$  such that it might well be singular at  $\lambda = 1$ . The correlation length  $\xi$  tends to zero as  $\lambda \rightarrow 0$ . This implies that – after proper re-normalization of the Hamiltonian – the mass gap becomes infinite at  $\lambda = 0$ . It has already been observed in [32] that there are physical reasons to divide (2.2) by  $\sqrt{\lambda}$  which would have exactly the effect of infinite mass at  $\lambda = 0$ . Fits to (6.18) for  $\lambda \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$  in the superintegrable case are given by the values in the following table:

$\lambda$	$\xi_\Gamma$	$a$	$L$	$\xi_\sigma$	$p$	$q$	$P_{\min}$	$\frac{LP_{\min}}{2\pi}$
0.25	0.55(3)	0.55(5)	14.3(2)	0.25(2)	0.35(4)	0.32(4)	0.471	1.07(2)
0.50	0.9(1)	0.59(3)	16.5(8)	0.38(4)	0.35(3)	0.24(3)	0.401	1.05(5)
0.75	1.5(6)	0.64(3)	18.3(8)	0.55(6)	0.36(2)	0.09(2)	0.308	0.90(4)

Table 1: Parameters for the correlation functions (6.18) at  $\phi = \varphi = \frac{\pi}{2}$

The estimates in table 1 have been obtained as follows. First,  $\xi_\Gamma$  has been estimated by calculating  $\text{Re}(\ln(\frac{C_\Gamma(x)}{C_\Gamma(x+1)}))^{-1}$  and averaging over  $x$ . Next, the zero of  $\text{Re}(e^{\frac{x}{\xi_\Gamma}} C_\Gamma(x))$  has been estimated by linear interpolation for two neighbouring values and  $\frac{L}{4}$  was obtained by averaging. Finally,  $a$  was estimated such that the difference

$$\text{Re}(C_\Gamma(x)) - ae^{-\frac{x}{\xi_\Gamma}} \cos\left(\frac{2\pi x}{L}\right) \quad (6.19)$$

is minimal for  $x = 1, 2$ . That this procedure yields reasonable fits is demonstrated by Fig. 1 which shows the stretched correlation function  $e^{\frac{x}{\xi_\Gamma}} C_\Gamma(x)$  in comparison to the fits. The ‘error bars’ are not really error bars but given by  $ae^{-\frac{x-6}{\xi_\Gamma}}$  which gives an idea how much the values have actually been stretched and what might be the contribution of the next orders in the perturbation expansion. The agreement for all  $x$  not only in the real part but also in the imaginary part is convincing.

Table 2 shows the values  $C_\Gamma^{\text{pert.}}(x)$  corresponding to Fig. 1. This table also contains the numerical results for the correlation function  $C_\Gamma^{\text{num.}}(x)$  which were obtained in [34] for  $N = 12$  sites at  $\phi = \varphi = \frac{\pi}{2}$ ,  $\lambda = \frac{1}{2}$ .

$x$	0	1	2	3	4	5	6
$C_\Gamma^{\text{pert.}}(x)$	1	.18868+.07384 <i>i</i>	.04561+.03980 <i>i</i>	.00992+.01747 <i>i</i>	.00091+.00674 <i>i</i>	-.00088+.00263 <i>i</i>	-.00074
$C_\Gamma^{\text{num.}}(x)$	1	.18881+.07385 <i>i</i>	.04587+.03967 <i>i</i>	.01004+.01737 <i>i</i>	.00126+.00679 <i>i</i>	-.00056+.00224 <i>i</i>	-.00080

Table 2: Perturbative results (6.17) and numerical results at  $N = 12$  sites for the correlation function  $C_\Gamma(x)$  at  $\phi = \varphi = \frac{\pi}{2}$ ,  $\lambda = \frac{1}{2}$

The agreement between the results of both methods is good. This shows that on the one hand higher orders are indeed negligible in (6.17) for  $x < 7$  and on the other hand that the finite chain length does not considerably affect the correlation function  $C_\Gamma(x)$ .

Let us now discuss the implications of (6.15) under the assumption that (6.18a) is the correct form for general values of the chiral angles. From the leading orders in (6.15a) we read off the following identity for the ratio of  $C_\Gamma(1)$  and  $C_\Gamma(2)$ :

$$\frac{C_\Gamma(2)}{C_\Gamma(1)} = \frac{\frac{e^{i\frac{2\phi}{3}}}{6\mathcal{C}^2}\lambda^2 + \mathcal{O}(\lambda^3)}{\frac{e^{i\frac{\phi}{3}}}{3\mathcal{C}}\lambda + \mathcal{O}(\lambda^2)} = \frac{e^{i\frac{\phi}{3}}}{2\mathcal{C}}\lambda + \mathcal{O}(\lambda^2). \quad (6.20a)$$

On the other hand we immediately obtain from (6.18a)

$$\frac{C_\Gamma(2)}{C_\Gamma(1)} = e^{-\frac{1}{\xi_\Gamma}} e^{\frac{2\pi i}{L}}. \quad (6.20b)$$



Comparison of (6.20a) and (6.20b) leads to

$$L = \frac{6\pi}{\text{Re}(\phi)}, \quad \xi_\Gamma = -\frac{1}{\ln\left(\frac{\lambda}{2\cos\left(\frac{\varphi}{3}\right)}\right) - \frac{\text{Im}(\phi)}{3}} \quad (6.21)$$

for small values of  $\lambda$ . It is noteworthy that we obtain the same result for the oscillation length  $L$  if we apply a similar argument to  $\frac{C_\Gamma(x_1)}{C_\Gamma(x_2)}$  in lowest non-vanishing order with  $x_1, x_2 \in \{1, 2, 3, 4\}$ . At  $\phi = \frac{\pi}{2}$  (6.21) yields the approximations  $L = 12, \xi_\Gamma = 0.52, 0.80, 1.2$  for  $\lambda = 0.25, 0.50, 0.75$ . The agreement with the numbers of table 1 is very good. Thus, for very high temperatures the oscillation length  $L$  is proportional to the inverse chiral angle  $\phi^{-1}$ . In particular, the oscillation vanishes smoothly for  $\phi \rightarrow 0$ . In [32] it was shown that for very high temperatures the minimum of the dispersion relation of the fundamental particles is also proportional to  $\phi$ . More precisely, we read off from (4.6) that the minimum of the dispersion relation is in first order perturbation theory at  $P_{\min} = \frac{\text{Re}(\phi)}{3}$ . Thus, we obtain from (6.21) for very high temperatures

$$P_{\min}L \big|_{\lambda \rightarrow 0} = 2\pi \quad \forall \phi, \varphi. \quad (6.22)$$

Furthermore, the second order in (4.6) shows that the minimal momentum  $P_{\min}$  decreases with increasing  $\lambda$  (compare also [34]). Similarly, we read off from (6.15) that the inverse oscillation length  $L^{-1}$  also decreases with increasing inverse temperature  $\lambda$ . Thus, (6.22) has a chance to be valid for all values of  $\lambda$  in the massive high-temperature phase. Indeed, using the values of  $P_{\min}$  given in table 8 of [32] we see that  $P_{\min}L = 2\pi$  holds quite accurately for  $\lambda = 0.25, 0.5, 0.75$  at  $\phi = \varphi = \frac{\pi}{2}$  (compare table 1). Using numerical methods we have checked in [34] that  $P_{\min}L = 2\pi$  is indeed valid within the numerical accuracy for general values of the parameters. The identity  $P_{\min}L = 2\pi$  can e.g. be derived from the form factor expansion (6.5) if the Hamiltonian has suitable symmetries as is demonstrated in appendix B for certain special cases. However, it may well be that in general this relation is not exact but an excellent approximation.

Note that even at  $\phi = \varphi = \frac{\pi}{2}$  the correlation lengths  $\xi_\Gamma$  and  $\xi_\sigma$  are clearly different. Furthermore,  $\xi_\sigma$  coincides with its dual in the low-temperature phase whereas  $\xi_\Gamma$  does not (see [37]). Recall that for the correlation function  $C_\sigma(x)$  only the spectrum in the charge sector  $Q = 0$  is relevant but  $C_\Gamma(x)$  comes from the  $Q = 1$  sector. Using (6.5) this explains the agreement of  $\xi_\sigma$  with  $\xi$  in the low-temperature phase; in this phase all charge sectors have a spectrum that is identical with the spectrum in the  $Q = 0$ -sector at the dual point in the high-temperature phase [37].

## 7. The parity conserving Potts case

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So far, we have studied correlation functions for general values of the parameters and for the superintegrable case. In this section we discuss the standard parity-conserving  $\mathbb{Z}_3$  case in more detail and compare the correlation length to the inverse mass gap. We also examine the dispersion relation of the particle-/anti-particle pair closer for this special case and show that, for general  $\lambda$ , there is no simple relation between the square of the energy and the momentum like the Klein-Gordon equation.

First, we note that for  $\phi = \varphi = 0$  eq. (4.5) simplifies considerably and we can calculate even higher orders:

$$m(\lambda) := \Delta E_{1,0}(0,0) = \Delta E_{2,0}(0,0) = \frac{1}{\sqrt{3}} \left( 6 - 4\lambda - 2\lambda^2 + \lambda^3 - \frac{179}{162} \lambda^4 + \frac{1099}{1458} \lambda^5 - \frac{15865}{26244} \lambda^6 + \frac{163717}{629856} \lambda^7 - \frac{4564375}{68024448} \lambda^8 \right) + \mathcal{O}(\lambda^9). \quad (7.1)$$

In this case the mass gap is located at zero momentum. Therefore, we defined (7.1) as ‘ $m(\lambda)$ ’. With respect to the quality of the approximation (7.1) we would just like to mention that e.g. comparison with numerical values shows that the accuracy of the series (7.1) is good for all  $\lambda \in [0, 1]$ . However, (7.1) is an alternating sum and gives only good approximations if an even number of orders is used. In particular, close to the phase transition  $\lambda = 1$  higher order contributions do not always improve the approximation which reflects that close to  $\lambda = 1$  this perturbation series is slowly convergent.

It is well-known that the critical exponent for  $m(\lambda)$  at  $\lambda = 1$  equals  $\frac{5}{6}$ . The series eq. (7.1) can be used to verify this critical exponent with a DLog-*Padé* analysis. In fact, this check has already been performed in [4]. One can also use (7.1) or a numerical evaluation of the mass gap  $m(\lambda)$  in order to test the critical behaviour throughout the massive high-temperature phase. One finds that  $(1 - \lambda)^{\frac{5}{6}} m(\lambda)$  is a very slowly varying function (see also [32][54]). This means that the normalization of the Hamiltonian (2.2) is indeed meaningful even far away from the critical region.

Next we will discuss the correlation function  $C_\Gamma(x)$  for  $\phi = \varphi = 0$ . If a statistical system has an isotropic field theory as limit, the correlation length is related in this limit to the inverse of the smallest gap between the ground state and the first excitation [50]. Therefore, one expects a relation  $\xi \sim m(\lambda)^{-1}$  [50]. Note that for small values of  $\lambda$  we expect a different behaviour according to (6.21):  $\xi_\Gamma \sim \ln(\frac{\lambda}{2})^{-1}$ . We will now study these two relations more closely by considering the correlation function  $C_\Gamma(x)$ . First, we specialize (6.15) to  $\phi = \varphi = 0$  and calculate two further orders. This leads to:

$$\begin{aligned} C_\Gamma^{(0)}(x) &= \delta_{x,0} , & C_\Gamma^{(1)}(x) &= \frac{\delta_{x,1}}{3} , \\ C_\Gamma^{(2)}(x) &= \frac{1}{12} \{ \delta_{x,1} + 2\delta_{x,2} \} , & C_\Gamma^{(3)}(x) &= \frac{1}{162} \{ 5\delta_{x,1} + 14\delta_{x,2} + 15\delta_{x,3} \} , \\ C_\Gamma^{(4)}(x) &= \frac{1}{11664} \{ 315\delta_{x,1} + 478\delta_{x,2} + 852\delta_{x,3} + 630\delta_{x,4} \} , & & (7.2) \\ C_\Gamma^{(5)}(x) &= \frac{1}{209952} \{ 3525\delta_{x,1} + 5870\delta_{x,2} + 9007\delta_{x,3} + 12016\delta_{x,4} + 6804\delta_{x,5} \} , \\ C_\Gamma^{(6)}(x) &= \frac{1}{3779136} \{ 44744\delta_{x,1} + 77659\delta_{x,2} + 111952\delta_{x,3} + 153196\delta_{x,4} \\ &\quad + 162940\delta_{x,5} + 74844\delta_{x,6} \} . \end{aligned}$$

Using (7.2) one can calculate the correlation length  $\xi_\Gamma$  by the procedure described in the previous section. We just mention a few pairs  $[\lambda, \xi_\Gamma]$ :

$[0, 0]$ ,  $[0.00005, 0.094(2)]$ ,  $[0.0005, 0.121(3)]$ ,  $[0.005, 0.167(6)]$ ,  $[0.05, 0.27(2)]$ ,  $[0.25, 0.50(5)]$ ,  $[0.5, 0.8(1)]$ ,  $[0.75, 1.2(2)]$ .

Fig. 2 shows a plot including more estimates for the correlation length. At  $\lambda = 0$  there is no correlation between different sites, i.e. the correlation length is zero. One observes that it increases drastically for  $\lambda > 0$ . It is clearly different from zero even for very small values of  $\lambda$ . We have also plotted the estimate (6.21) in Fig. 2. This crude estimate fits the numerical results surprisingly well for all values of  $\lambda$  accessible to us. In particular, it nicely reproduces the behaviour for small  $\lambda$  as it is expected from our derivation of the estimate. In Fig. 2 we also plotted the properly normalized inverse mass  $m(\lambda)^{-1}$ . The agreement is good for  $\lambda > 0.3$ . For  $\lambda < 0.1$  one observes a clear disagreement. Note that in this region  $\frac{C_\Gamma(x)}{C_\Gamma(x+1)} \approx e^{\frac{1}{5\Gamma}} < 10$  and one should therefore expect that at least in this region the finite lattice spacing is important.

It has been observed in [32] that for  $\phi = \varphi = 0$  the dispersion relation (4.6) agrees with a Klein-Gordon dispersion relation up to order  $\lambda^2$ . Furthermore, it was shown in [55] that at the second order phase transition  $\phi = \varphi = 0$ ,  $\lambda = 1$  the dispersion relation is of Klein-Gordon type with mass  $m(1) = 0$ . Using the abbreviation

$$K := 2 \sin\left(\frac{P}{2}\right) \quad (7.3)$$

for the lattice analogue of the momentum we can specialize (4.6) to  $\phi = \varphi = 0$  and calculate two further orders. This yields the dispersion relation

$$\begin{aligned} \mathcal{E}(K) := \Delta E_{1,0}(P, 0, 0) = \Delta E_{2,0}(P, 0, 0) = \\ = \frac{1}{\sqrt{3}} \left\{ 6 + 2(K^2 - 2)\lambda - \frac{\lambda^2}{3}(K^4 - 6K^2 + 6) + \frac{\lambda^3}{18}(2K^6 - 14K^4 + 19K^2 + 18) \right. \\ \left. - \frac{\lambda^4}{324}(15K^8 - 152K^6 + 531K^4 - 738K^2 + 358) \right. \\ \left. + \frac{\lambda^5}{2916}(63K^{10} - 764K^8 + 3214K^6 - 5087K^4 + 1121K^2 + 2198) \right\} + \mathcal{O}(\lambda^6) \end{aligned} \quad (7.4)$$

for the two fundamental quasiparticles. The Klein-Gordon dispersion relation predicts

$$\begin{aligned} \mathcal{E}(K) = \sqrt{m(\lambda) + a(\lambda)K^2} = m(\lambda) + a(\lambda)\frac{K^2}{2m(\lambda)} - a(\lambda)^2\frac{K^4}{8m(\lambda)^3} + a(\lambda)^3\frac{K^6}{16m(\lambda)^5} \\ - a(\lambda)^4\frac{5K^8}{128m(\lambda)^7} + a(\lambda)^5\frac{7K^{10}}{256m(\lambda)^9} + \mathcal{O}(K^{12}) \end{aligned} \quad (7.5)$$

where we have included a free normalization constant  $a(\lambda)$  which corresponds to the velocity of light. Rewriting (7.4) in the form (7.5) leads to

$$\begin{aligned} \mathcal{E}(K) = m(\lambda) + \frac{\lambda}{729}(-4435\lambda^4 + 3618\lambda^3 - 2754\lambda^2 + 1944\lambda + 5832)\frac{K^2}{2m(\lambda)} \\ - \frac{16\lambda^2}{243}(-2221\lambda^3 + 567\lambda^2 + 324\lambda + 972)\frac{K^4}{8m(\lambda)^3} \\ + \frac{256\lambda^3}{81}(-223\lambda^2 + 144\lambda + 162)\frac{K^6}{16m(\lambda)^5} - \frac{4096\lambda^4}{135}(134\lambda + 135)\frac{5K^8}{128m(\lambda)^7} \\ + 32768\lambda^5\frac{7K^{10}}{256m(\lambda)^9} + \mathcal{O}(\lambda^6). \end{aligned} \quad (7.6)$$

In (7.6) the coefficients of  $K^n$  with lowest order in  $\lambda$  agree with the Klein-Gordon dispersion relation (7.5)<sup>5</sup>). However, using  $m(\lambda)$  as in (7.1) and fitting  $a(\lambda)$  from the coefficient of  $K^2$ , all but these leading orders disagree with (7.5). Thus, although a Klein-Gordon dispersion relation is certainly a good approximation to (7.4) it is unfortunately not the exact form.

It happens quite frequently in two-dimensional quantum field theories that the dispersion relation is sin-Gordon or sinh-Gordon. However, even these possibilities can be ruled out because the first five orders of these dispersion relations agree with Klein-Gordon ( $\sin(x^2) + \mathcal{O}(x^6) = x^2 = \sinh(x^2) + \mathcal{O}(x^6)$ ) but the deviation from Klein-Gordon occurs already at order  $K^4$ . Therefore we consider an even more general dispersion relation of the form

$$\mathcal{E}(K) = \sqrt{m(\lambda) + \frac{g(b(\lambda)a(\lambda)K^2)}{b(\lambda)}}, \quad (7.7)$$

$$g(x) = x + x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \mathcal{O}(x^6)$$

which contains the Klein-Gordon relation (7.5) for  $b(\lambda) \rightarrow 0$ . In particular, (7.7) is a good approximation to the Klein-Gordon dispersion relation for small  $b(\lambda)$ . (7.7) would also include both the sin-Gordon and sinh-Gordon relations for  $c_2 = 0$  but since this has already been ruled out  $c_2$  has been absorbed in  $b(\lambda)$ . Determining from the first orders of the Taylor expansion of (7.7) with respect to  $K^2$  first  $m(\lambda)$ , then  $a(\lambda)$ ,  $b(\lambda)$  and  $c_3$  it turns out that  $c_3$  depends on  $\lambda$ . Thus (7.7) can be ruled out if the function  $g$  is required to be universal for all  $\lambda$ .

## 8. Convergence of single-particle excitations

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As far as the proof of the quasiparticle picture is concerned the main open question is the convergence of the single-particle states, or equivalently the existence of the limits  $N \rightarrow \infty$  of the corresponding eigenvalues of the Hamiltonian. We have argued in section 4 that convergence of the perturbation expansions is sufficient to guarantee the existence of the limits  $N \rightarrow \infty$ . Therefore we will discuss the radius of convergence for the perturbation expansion of the single-particle excitations in this section.

For bounded operators—in particular finite dimensional ones—one could use criteria involving operator norms similar to those for v. Neumann series. Unfortunately, the potential for  $\Delta H_N^{(n)}$  as defined in (2.2) and (2.14) is unbounded if  $N$  is not fixed. Thus, we have to apply the slightly more complicated Kato-Rellich theory of regular perturbations. Reviews of this subject can be found e.g. in the monographs [56][57]. The main results we are going to use were originally published in [58][59]. The theory of Kato and Rellich applies in particular to operators of the form (3.1), i.e.  $H(\lambda) = H_0 + \lambda V$ .

Suppose that the single-particle eigenvalues  $\Delta E$  have a non-zero distance from the scattering eigenvalues (the continuous spectrum) at  $\lambda = 0$ . Then it is clear from the discussion in the previous sections that these eigenvalues are non-degenerate and isolated. In particular, the resolvent  $(\Delta H_N^{(n)}(\lambda) - z)^{-1}$  is bounded for  $|\Delta E - z| > 0$ . Restricting

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<sup>5</sup>) According to [55] one should have  $\mathcal{E}(K) = 3|K|$  at  $\lambda = 1$ . At  $\lambda = 1$  the series (7.4) does not really converge any more. Nevertheless, it seems that (7.4) is compatible with  $\mathcal{E}(K)^2 = 9K^2$  at  $\lambda = 1$ .

to the hermitean case, this is sufficient to guarantee that the  $\Delta H_N^{(n)}(\lambda)$  are an analytic family in the sense of Kato. In this case, the Kato-Rellich theorem ([56] Theorem XII.8) may be used to guarantee a non-zero radius of convergence  $r_0 > 0$  for the single-particle eigenvalues of  $\Delta H_N^{(n)}(\lambda)$ .

In order to obtain explicit estimates of the radius of convergence one needs the inequality

$$\|V |a\rangle\| \leq \mathcal{V} \|H_0 |a\rangle\| + \mathcal{K} \| |a\rangle\| \quad (8.1)$$

on  $\mathcal{D}(H_0)$  which in our case is dense in the complete Hilbert space  $\mathcal{H}$ . Then, the isolated point eigenvalues of  $H(\lambda)$  are convergent at least for

$$\lambda < r_1 := \mathcal{V}^{-1} \quad (8.2)$$

as long as these eigenvalues *do not come in contact with continuous spectrum* [59]. On the one hand this criterion is very simple, on the other hand one must estimate not only the constant  $\mathcal{V}$  but also examine the level crossings between single-particle excitations and scattering states. There is another estimate  $r_2$  that guarantees the separation of eigenvalues as well but gives smaller radii of convergence. For self-adjoint  $H_0$  with isolated eigenvalue  $E_0^{(0)}$  where the nearest eigenvalue  $E_1^{(0)}$  has distance  $\epsilon := |E_1^{(0)} - E_0^{(0)}|$  ( $\epsilon^{-1} = \|g(E_0^{(0)})\|$ ) the perturbation expansion of  $E_0(\lambda)$  is convergent for

$$\lambda < r_2 := \frac{\epsilon}{2(\mathcal{K} + \mathcal{V}(|E_0^{(0)}| + \epsilon))} \quad (8.3)$$

and there are no crossings with neighbouring levels. In order to compare the estimates (8.2) and (8.3) let us assume  $\mathcal{K} = 0$  and  $|E_0^{(0)}| = \epsilon$ . For this almost optimal case one has  $r_1 = 4r_2$  showing that the criterion (8.3) is much more restrictive.

Let us now apply these general results to the present case of  $\mathbb{Z}_n$ -spin quantum chains. For non-degenerate single-particle eigenvalues the Kato-Rellich theorem can be applied to guarantee a positive radius of convergence  $r_0$ . Then we know from section 5 and appendix A that the spectrum of  $\Delta H_N^{(n)}(\lambda)$  is a quasiparticle spectrum for  $\lambda < r_0$ . This fact can be used to calculate the constant  $\mathcal{V}$  and obtain explicit estimates  $r_1$  (where level crossings still have to be discussed) or  $r_2$ . One can obtain the estimate (8.1) with  $\mathcal{K} = 0$  using Schwarz' inequality:

$$\mathcal{V} := \sup_{|a\rangle \in \mathcal{H}} \frac{\langle a | \Delta V | a \rangle}{\|\Delta H_{N,0}^{(n)} | a \rangle\|}. \quad (8.4)$$

In general, this supremum need not be finite but then it is very difficult to ensure convergence at all. In our case, the important observation is that due to the quasiparticle picture we can evaluate (8.4) exclusively from the single-particle excitations. To see this one performs a first order expansion in  $\lambda$  for any composite particle state, compares coefficients and uses the quasiparticle property to expand the expectation values of  $\Delta H_{N,0}^{(n)}$  and  $\Delta V$  in single-particle excitations. Thus,  $\mathcal{V}$  can be calculated as

$$\mathcal{V} = \max_{Q,P} \frac{P \langle\langle s^Q | \Delta V | s^Q \rangle\rangle_P}{\|\Delta H_{N,0}^{(n)} | s^Q \rangle\|_P}. \quad (8.5)$$

In order to implement this program explicitly we specialize to the case of  $\mathbb{Z}_3$  with the parametrization (2.10). At  $\lambda = 0$  both single-particle eigenvalues are isolated for  $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ . This guarantees a non-zero radius of convergence  $r_0$ .

The simplest case is the parity conserving case  $\phi = \varphi = 0$ . Here, the maxima are located at zero momentum  $P = 0$  and both charge sectors are degenerate. Furthermore, we have  $\|\Delta H_{N,0}^{(3)}\|s^Q\|_0 = \epsilon = E_0^{(0)}$ . From (7.1) we can therefore read off  $\mathcal{V} = \frac{2}{3}$ , or in terms of radii of convergence

$$r_1 = \frac{3}{2}, \quad r_2 = \frac{3}{8}, \quad \text{for } n = 3, \phi = \varphi = 0. \quad (8.6)$$

$r_2 = 0.375$  is certainly too small which can easily be seen applying a naive ratio test to (7.1). Extrapolating  $\frac{m^{(\nu)}}{m^{(\nu+1)}}$  to  $\nu = \infty$  one obtains an estimate for the radius of absolute convergence of about 1.3. Thus, for  $\phi = \varphi = 0$  the radius of convergence is expected to be close to the boundary of the phase  $\lambda = 1$  which is also supported by the calculations in section 7.

For general angles  $0 \leq \varphi < \frac{\pi}{2}$ , the free part of the Hamiltonian  $\|\Delta H_{N,0}^{(3)}\|s^Q\|_0$  is minimized for  $Q = 1$  and the potential  $P\langle\langle s^Q\|\Delta V\|s^Q\rangle\rangle_P$  is maximal for  $P = \frac{\phi}{3}$ . Thus, we read off from (4.6)  $\mathcal{V} = (\sqrt{3} \sin(\frac{\pi-\varphi}{3}))^{-1}$ . Furthermore, one has  $\epsilon = 8 \sin(\frac{\pi-\varphi}{3}) - 4 \sin(\frac{\pi+\varphi}{3})$  and  $E_0^{(0)} = 4 \sin(\frac{\pi-\varphi}{3})$ . This amounts to the following radii

$$r_1 = \sqrt{3} \sin\left(\frac{\pi-\varphi}{3}\right), \quad r_2 = \frac{\sqrt{3} \sin\left(\frac{\pi-\varphi}{3}\right) (2 \sin\left(\frac{\pi-\varphi}{3}\right) - \sin\left(\frac{\pi+\varphi}{3}\right))}{2 (3 \sin\left(\frac{\pi-\varphi}{3}\right) - \sin\left(\frac{\pi+\varphi}{3}\right))} \quad (8.7)$$

for  $n = 3$ ,  $0 \leq \varphi < \frac{\pi}{2}$ . For  $\varphi \rightarrow \frac{\pi}{2}$  the situation is contrary to that at  $\phi = \varphi = 0$ . The  $Q = 2$  particle state becomes degenerate with two  $Q = 1$  scattering states at  $\varphi = \frac{\pi}{2}$  such that the radius of convergence must tend to zero for  $\varphi \rightarrow \frac{\pi}{2}$ . Whereas  $r_2$  has precisely this property,  $r_1$  tends to 0.866... which is certainly too large.

Because for small  $\varphi$  we would prefer the large radius of convergence  $r_1$  but at  $\varphi \approx \frac{\pi}{2}$  this is much too large and  $r_2$  seems more appropriate we have to enhance the estimate given by  $r_1$  by a discussion of level crossings between single-particle states and scattering states. For  $0 \leq \varphi < \frac{\pi}{2}$  the first level crossing of this kind will take place between the  $Q = 2$  single-particle excitation and a two  $Q = 1$  particles scattering state.

It is very difficult to determine those values of  $\lambda$  explicitly and precisely where they take place. Therefore, we will use the first order approximation of the perturbation expansion. We are looking for those values of  $\lambda$  where a single point  $P$  exists such that  $x := 2\Delta E_{1,0}(\frac{P}{2}, \phi, \varphi) - \Delta E_{2,0}(P, \phi, \varphi)$  vanishes. The fact that we are looking for no real crossings but  $x = 0$  implies  $\frac{dx}{dP} = 0$ . Inserting (4.6) and (4.7) up to first order leads to the condition

$$\sin\left(\frac{P}{2} - \frac{\phi}{3}\right) = \sin\left(P + \frac{\phi}{3}\right) \quad (8.8)$$

Eq. (8.8) has a solution

$$P = \frac{2\pi}{3} \quad (8.9)$$

that does not depend on  $\phi$ . Now we can solve the linear equation  $x|_{\lambda_0} = 0$  for the value  $\lambda_0$ . One obtains

$$\lambda_0 = \frac{\cos\left(\frac{\varphi}{3}\right) - \sqrt{3}\sin\left(\frac{\varphi}{3}\right)}{\cos\left(\frac{\phi}{3}\right) + \sqrt{3}\sin\left(\frac{\phi}{3}\right)}. \quad (8.10)$$

Fig. 3 shows a plot of the estimates (8.7) and (8.10) for the self-dual case  $\varphi = \phi$ . Note that  $r_1$  and  $r_2$  are independent of  $\phi$ . However, we have assumed that the Hamiltonian is hermitean and therefore  $\phi$  must be real. For  $\varphi$  close to  $\frac{\pi}{2}$ ,  $\lambda_0$  is smaller than  $r_2$  which is an apparent contradiction because there are no level crossings for  $\lambda < r_2$ . Remember, however, that  $\lambda_0$  has been calculated approximately such that this difference is not significant. At  $\phi = \varphi = 0$  we find  $\lambda_0 \approx 1$ . This is reassuring because perturbation expansions should not be valid beyond the second order phase transition at  $\lambda = 1$ . Although  $\lambda_0$  was estimated by looking at non-zero momenta there are also level crossings in the zero momentum sectors at  $\lambda = 1$ . Thus, the radius of convergence is indeed smaller than  $r_1$ . Still, our results agree in magnitude with the intuitive expectations from the ‘ratio test’. The dots in Fig. 3 indicate the two models whereof the spectra have been presented in Fig. 2 and Fig. 3 of [34]. For the left dot one expects a converging perturbation expansion whereas in the other case it should not converge (compare [34]). Indeed, both estimates  $r_2$  and  $\lambda_0$  make this distinction.

For completeness we have also included an estimate for the boundary of the massive high-temperature phase in Fig. 3. At this boundary, levels of the  $Q = 1$  particle with generically non-zero momentum cross with the ground state. Its explicit location has been obtained estimating the minimum of the dispersion relation (4.6) with  $P = \frac{\phi}{3}$  and solving the second order approximation  $\Delta E_{1,0}(\frac{\phi}{3}, \phi, \phi) = 0$  for  $\lambda$ . At  $\phi = 0$  the agreement with the exact value  $\lambda = 1$  is excellent. For small non-zero angles the true value is smaller than 1 but our approximation gives values that are slightly larger than 1. Also at  $\phi = \frac{\pi}{2}$  we observe a small deviation from the exact value  $\lambda = 0.901292\dots$  [14]: Our estimate yields  $\lambda = 0.866\dots$  (the agreement with  $r_1$  is a coincidence).

The level crossings transition  $\lambda_0$  divides the massive high-temperature phase of the  $\mathbb{Z}_3$ -chiral Potts model into two parts which we label I and II. In part I the derivation of the quasiparticle picture as presented in section 5 and appendix A is rigorous. Thus, in regime I the spectrum is a quasiparticle spectrum with two fundamental particles existing for all momenta. In [34] we have presented numerical evidence that regime II probably also exhibits a quasiparticle spectrum with two fundamental particles where the  $Q = 2$  particle has the unusual property that it exists only in a limited range of the momentum  $P$ . At  $\varphi = \phi = \frac{\pi}{2}$  this statement has been proven rigorously in [35]. We expect that the idea to approximate multi-particle states by putting single-particle states of ‘small’ chains with a sufficient separation on a longer chain and to use the finite correlation length in order to ensure vanishing of boundary terms (which we cannot show directly like in appendix A any more) will apply also in regime II for general angles  $\phi, \varphi$ . However, in contrast to section 5 we loose control over the fundamental  $Q = 2$  excitation because the perturbation series does not converge any more and there is no guarantee for the completeness of this construction. At least it is plausible to still expect a quasiparticle spectrum in regime II with two fundamental particles of which the  $Q = 2$  particle may have a Brillouin zone that is smaller than the interval  $[0, 2\pi]$ .

## 9. Conclusion and outlook

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In this paper we have presented an argument using perturbation theory proving that the massive high-temperature phases of all  $\mathbb{Z}_n$ -spin quantum chains exhibit quasiparticle spectra with  $n - 1$  fundamental particles. Since the argument relies on perturbation theory it applies rigorously only to very high temperatures. Due to the perturbative nature of the details we were not able to give it any predictive power for those case where some of the fundamental particles cross with scattering states. For these cases one needs different methods, e.g. Bethe ansatz techniques [35][36] or numerical methods [34]. Nevertheless, the basic idea of approximating a multi-particle state by single-particle states sitting on subparts of the chain might be applicable in the entire massive high-temperature phase. One could even speculate that a similar argument can be applied to  $\mathbb{Z}_n$ -spin models in higher dimensions as well.

A refined (but less rigorous) version of this argument can be used to control the finite-size effects of  $k$ -particle states showing in particular that the *energy* of the excitations does not pick up any corrections at order  $\frac{1}{N}$ .

Furthermore, our derivation of the quasiparticle picture involving  $n - 1$  fundamental particles applies to the scaling region near the conformal point  $\lambda = 1$ ,  $\phi = \varphi = 0$ , the only non-rigorous part of the proof being the radius of convergence. This region ( $\lambda < 1$ ,  $\phi$ ,  $\varphi$  small) corresponds to perturbations of conformal field theories with the thermal operator [27 – 31] and a small additional perturbation of the type presented in [26] that breaks parity.

Using duality [37] our results about the quasiparticle spectra can be pulled over to the massive low-temperature phase of  $\mathbb{Z}_n$ -spin quantum chains.

Having derived such a quasiparticle picture the main open problem is to find the corresponding massive field theory and to obtain the associated scattering matrix.

We also studied the correlation functions using a perturbation expansion for the ground state of the  $\mathbb{Z}_3$ -model. Although this approach is limited to short ranges, we were not only able to estimate correlation lengths in the massive high-temperature phase but it also turned out that the correlation functions have oscillatory contributions. For very high temperatures the oscillation length is proportional to the inverse of one of the chiral angles  $L \sim \phi^{-1}$ . We further observed that the oscillation length is closely related to the minimum of the dispersion relations for general values of the parameters. The relation  $LP_{\min} = 2\pi$  is valid on the lattice with a much better accuracy than the well-known relation  $\xi \sim m^{-1}$ . For special values of the parameters we were able to derive the relation  $LP_{\min} = 2\pi$  from a form factor decomposition but one should certainly understand it better in the general case.

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## Appendix A: Proof of quasiparticle picture

In this appendix we present a modified version of the proof in section 5 that the spectrum of the Hamiltonian (2.2) can be explained in terms of  $n - 1$  fundamental quasiparticles.

The main steps of the proof will be as presented in section 5. However, instead of considering general multi-particle states we will go directly back to the single-particle excitations. The corresponding perturbative eigenstates are given by (4.2). One also has to be careful where it is permitted to deal directly with  $\Delta H_N^{(n)}$  or where one should rather consider  $H_N^{(n)}$  first.

In this appendix we will concentrate on the vanishing of boundary terms. Not all arguments presented in section 5 will be spelled out in detail again. In particular, the limiting procedures are taken for granted. This explicit presentation has been shifted to this appendix because the explicit formulae are a bit nasty although the ideas are quite simple.

In order to be able to discuss  $r$ -particle states we first write down the generalization of (5.2) to any partition of  $N$  in  $r$  arbitrary integers  $N_j > 1$  ( $N = \sum_{j=1}^r N_j$ ):

$$H_N^{(n)} = H_{N_1}^{(n)} \otimes \mathbf{1} + \dots + \mathbf{1} \otimes H_{N_j}^{(n)} \otimes \mathbf{1} + \dots + \mathbf{1} \otimes H_{N_r}^{(n)} + \sum_{j=1}^r \mathcal{O}(H_{N_j}^{(n)}), \quad (\text{A.1})$$

$$T_N = (\mathbf{1} + \mathcal{O}(T_{N_r})) \dots (\mathbf{1} + \mathcal{O}(T_{N_1})) T_{N_1} \otimes \dots \otimes T_{N_r}.$$

Let  $\nu_j := \sum_{i=1}^{j-1} N_i$  ( $\nu_0 := 0$ ). Then, the boundary operators  $\mathcal{O}(H_{N_j}^{(n)})$  and  $\mathcal{O}(T_{N_j})$  are given by

$$\begin{aligned} \mathcal{O}(H_{N_j}^{(n)}) &= \lambda \sum_{k=1}^{n-1} \alpha_k \Gamma_{\nu_j + N_j}^k \left\{ \Gamma_{\nu_j + 1}^{n-k} - \Gamma_{\nu_j + N_j + 1}^{n-k} \right\}, \\ (\mathbf{1} + \mathcal{O}(T_{N_j})) &|i_1 i_2 \dots i_{\nu_j + N_j} i_{\nu_j + N_j + 1} \dots i_{\nu_j + 1 + N_j + 1} i_{\nu_j + 1 + N_j + 1 + 1} \dots i_N\rangle \\ &= |i_1 i_2 \dots i_{\nu_j + 1 + N_j + 1} i_{\nu_j + N_j + 1} \dots i_{\nu_j + N_j} i_{\nu_j + 1 + N_j + 1 + 1} \dots i_N\rangle. \end{aligned} \quad (\text{A.2})$$

It will be useful to verify first that additivity of energy and momentum holds for  $\lambda = 0$ . To this end we show that a composite particle state

$$\|N; Q_{\text{tot}}\rangle_{P_{\text{tot}}} := \|s^{Q_1}\rangle_{P_1} \otimes \dots \otimes \|s^{Q_r}\rangle_{P_r} \quad (\text{A.3})$$

has approximately energy  $E := \sum_{j=1}^r E_{Q_j}$ , momentum  $P_{\text{tot}} := \sum_{j=1}^r P_j$  and total charge  $Q_{\text{tot}} := \sum_{j=1}^r Q_j \pmod n$ . Recall that we have defined all  $\|s^{Q_j}\rangle_{P_j}$  in the complete Hilbert space  $\mathcal{H}$  but it is useful to think of them as elements of  $\mathcal{H}_{N_j}$ , i.e.  $\|s^{Q_j}\rangle_{P_j} \in \mathcal{H}_{N_j}$ . Applying (A.1) to these states one obtains for the energy:

$$\begin{aligned} \left( H_N^{(n)} - E \right) \|N; Q_{\text{tot}}\rangle_{P_{\text{tot}}} &= \sum_{j=1}^r \mathcal{O}(H_{N_j}^{(n)}) \|N; Q_{\text{tot}}\rangle_{P_{\text{tot}}} \\ &+ \sum_{j=1}^r \|s^{Q_1}\rangle_{P_1} \otimes \dots \otimes \left( H_{N_j}^{(n)} - E_{Q_j} \right) \|s^{Q_j}\rangle_{P_j} \otimes \dots \otimes \|s^{Q_r}\rangle_{P_r} \\ &= 0 \end{aligned} \quad (\text{A.4})$$

where we have used that  $\mathcal{O}(H_{N_j}^{(n)})$  vanishes at  $\lambda = 0$  (compare (A.2)). Thus, we verified that additivity of energy is exact at  $\lambda = 0$ .

Using (A.1) and (A.2) one obtains for the momentum

$$\begin{aligned}
T_N \|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} &= \prod_{j=1}^r \{ \mathbb{1} + \mathcal{O}(T_{N_j}) \} (T_{N_1} \|s^{Q_1}\rangle\rangle_{P_1}) \otimes \dots \otimes (T_{N_r} \|s^{Q_r}\rangle\rangle_{P_r}) \\
&= \prod_{j=1}^r \{ \mathbb{1} + \mathcal{O}(T_{N_j}) \} e^{iP_{\text{tot}}} \|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} \\
&= e^{iP_{\text{tot}}} \|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} + e^{iP_{\text{tot}}} \|s^{Q_1}\rangle\rangle_{P_1} \overset{\leftarrow}{\otimes} \dots \overset{\leftarrow}{\otimes} \|s^{Q_r}\rangle\rangle_{P_r} \\
&\quad - e^{iP_{\text{tot}}} \|s^{Q_1}\rangle\rangle_{P_1} \otimes \dots \otimes \|s^{Q_r}\rangle\rangle_{P_r}
\end{aligned} \tag{A.5}$$

where ' $\overset{\leftarrow}{\otimes}$ ' denotes the modifications that occur when shifting in the entire tensor product instead of acting in its individual parts. Locally, these modifications look as follows:

$$\begin{aligned}
&\|s^{Q_j}\rangle\rangle_{P_j} \overset{\leftarrow}{\otimes} \|s^{Q_{j+1}}\rangle\rangle_{P_{j+1}} - \|s^{Q_j}\rangle\rangle_{P_j} \otimes \|s^{Q_{j+1}}\rangle\rangle_{P_{j+1}} \\
&= \frac{e^{-iP_{j+1}}}{\sqrt{N_{j+1}}} \sum_{k=1}^{N_j} \frac{e^{-ikP_j}}{\sqrt{N_j}} |0 \dots \underbrace{Q_j}_{\text{position } k-1} \dots 0 Q_{j+1}\rangle \otimes |0 \dots 0\rangle \\
&\quad - \frac{e^{-iP_j}}{\sqrt{N_j}} |0 \dots 0 Q_j\rangle \otimes \|s^{Q_{j+1}}\rangle\rangle_{P_{j+1}} \\
&\rightarrow 0.
\end{aligned} \tag{A.6}$$

In (A.6) we have used the explicit form (2.9) of the single-particle states  $\|s^{Q_j}\rangle\rangle_{P_j}$ . The vanishing of the boundary terms for  $N_j \rightarrow \infty$  is ensured by the normalization factors  $N_j^{-\frac{1}{2}}$ . Thus, we have also shown that  $\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}}$  approximates an eigenstate of the translation operator to total momentum  $P_{\text{tot}}$  for  $\lambda = 0$ .

For  $\lambda > 0$  we have to consider single-particle eigenstates  $\|N_j; Q_j\rangle\rangle_{P_j}$  that are derived by perturbation series from the states  $\|s^{Q_j}\rangle\rangle_{P_j}$ . These states have the form

$$\|N_j; Q_j\rangle\rangle_{P_j} = \|s^{Q_j}\rangle\rangle_{P_j} + \sum_{\nu > 0} \lambda^\nu \sum_{\substack{i_1^{(j)} + \dots + i_{N_j}^{(j)} = Q_j \bmod n \\ \#\{i_k^{(j)} \neq 0\} \leq 2\nu + 1}} \kappa_{i_1^{(j)}, \dots, i_{N_j}^{(j)}}^{(\nu)} \|i_1^{(j)} \dots i_{N_j}^{(j)}\rangle\rangle_{P_j}. \tag{A.7}$$

It is important to note that the explicit form of the Hamiltonian (2.2) implies that at most  $2\nu + 1$  spins  $i_k$  are different from zero in the  $\nu$ th order of the perturbation expansion. In passing we mention that we do not need the explicit form of the  $\kappa_{i_1^{(j)}, \dots, i_{N_j}^{(j)}}^{(\nu)}$  and therefore the argument also applies to the more general Hamiltonian (2.1) without modification.

We should stress that the states (A.7) are in general not convergent for  $N_j \rightarrow \infty$  although the corresponding eigenvalues of  $\Delta H_{N_j}^{(n)}$  and  $T_{N_j}$  converge (compare e.g. (6.13)). However,

after re-normalizing  $\|N_j; Q_j\rangle\rangle_{P_j}$  to norm 1 we could again think of it as lying in  $\mathcal{H}$  for all  $N_j$  because the  $\nu$ th order is independent of  $N_j$  for  $\nu < N_j$ .

An  $r$ -particle state now is approximated by

$$\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} := \|N_1; Q_1\rangle\rangle_{P_1} \otimes \dots \otimes \|N_r; Q_r\rangle\rangle_{P_r}. \quad (\text{A.8})$$

Note that (A.8) cannot be directly related to (A.3) by a perturbation expansion.

First, we consider the translation operator. Thus, we have to generalize (A.5) to the states (A.7), (A.8):

$$\begin{aligned} T_N \|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} &= \prod_{j=1}^r \{ \mathbb{1} + \mathcal{O}(T_{N_j}) \} (T_{N_1} \|N_1; Q_1\rangle\rangle_{P_1}) \otimes \dots \otimes (T_{N_r} \|N_r; Q_r\rangle\rangle_{P_r}) \\ &= \prod_{j=1}^r \{ \mathbb{1} + \mathcal{O}(T_{N_j}) \} e^{iP_{\text{tot}}} \|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} \\ &= e^{iP_{\text{tot}}} \|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} + e^{iP_{\text{tot}}} \|N_1; Q_1\rangle\rangle_{P_1} \overset{\leftarrow}{\otimes} \dots \overset{\leftarrow}{\otimes} \|N_r; Q_r\rangle\rangle_{P_r} \\ &\quad - e^{iP_{\text{tot}}} \|N_1; Q_1\rangle\rangle_{P_1} \otimes \dots \otimes \|N_r; Q_r\rangle\rangle_{P_r}. \end{aligned} \quad (\text{A.9})$$

The modifications introduced by  $\overset{\leftarrow}{\otimes}$  in (A.9) are more complicated than (A.6). Locally, they look as follows:

$$\begin{aligned} &\|N_j; Q_j\rangle\rangle_{P_j} \overset{\leftarrow}{\otimes} \|N_{j+1}; Q_{j+1}\rangle\rangle_{P_{j+1}} - \|N_j; Q_j\rangle\rangle_{P_j} \otimes \|N_{j+1}; Q_{j+1}\rangle\rangle_{P_{j+1}} \\ &= \frac{1}{\sqrt{N_j} \sqrt{N_{j+1}}} \sum_{\nu, \mu \geq 0} \lambda^{\nu+\mu} \left\{ \sum_{\substack{i_1^{(j+1)} + \dots + i_{N_{j+1}}^{(j+1)} = Q_{j+1} \bmod n \\ \#\{i_k^{(j+1)} \neq 0\} \leq 2\mu+1 \\ i_{m_{j+1}}^{(j+1)} \neq i_{k_j}^{(j)}}}^* \sum_{\substack{i_1^{(j)} + \dots + i_{N_j}^{(j)} = Q_j \bmod n \\ \#\{i_k^{(j)} \neq 0\} \leq 2\nu+1}} \right. \\ &\quad e^{-ik_j P_j} \kappa_{i_1^{(j)}, \dots, i_{N_j}^{(j)}}^{(\nu)} |i_{k_j+1}^{(j)} \dots i_{N_j}^{(j)} i_1^{(j)} \dots i_{k_j-1}^{(j)} i_{m_{j+1}}^{(j+1)}\rangle \otimes \\ &\quad \left. e^{-im_{j+1} P_{j+1}} \kappa_{i_1^{(j+1)}, \dots, i_{N_{j+1}}^{(j+1)}}^{(\mu)} |i_{m_{j+1}+1}^{(j+1)} \dots\rangle \right\} \\ &\quad - \frac{1}{\sqrt{N_j}} \sum_{\nu \geq 0} \lambda^\nu \left\{ \sum_{\substack{i_1^{(j)} + \dots + i_{N_j}^{(j)} = Q_j \bmod n \\ \#\{i_k^{(j)} \neq 0\} \leq 2\nu+1 \\ i_{m_j}^{(j)} \neq 0}}^{**} \right. \\ &\quad \left. e^{-im_j P_j} \kappa_{i_1^{(j)}, \dots, i_{N_j}^{(j)}}^{(\nu)} |i_{m_j+1}^{(j)} \dots i_{N_j}^{(j)} i_{m_j}^{(j)}\rangle \right\} \otimes \|N_{j+1}; Q_{j+1}\rangle\rangle_{P_{j+1}} \\ &\rightarrow 0. \end{aligned} \quad (\text{A.10})$$

The explicit form of (A.10) may be slightly confusing. Note, however, that the vanishing of the boundary terms is guaranteed by the same argument as in (A.6). It is sufficient to

guarantee that the coefficients of  $\lambda^\nu$  in the boundary terms become small with respect to the coefficients of the eigenstates. The index set of the sums  $*$  and  $**$  has precisely this property. The sum  $*$  has at most  $2\mu + 1$  non-zero terms whereas the complete momentum decomposition has  $N_{j+1}$  terms. Thus, the complete sum  $*$  is suppressed by the factor  $N_{j+1}^{-\frac{1}{2}}$ . Also the sum  $**$  has  $2\nu + 1$  terms (or less) compared to  $N_j$  for the complete fourier transform such that it is suppressed by the normalization factor  $N_j^{-\frac{1}{2}}$ . This shows that  $\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}}$  indeed approximates an eigenstate of the translation operator with total momentum  $P_{\text{tot}}$ .

Translating these statements into weak language we conclude the following: The scalar product of (A.10) with an arbitrary momentum eigenstate tends to zero if we rescale both states such that they lie on the unit sphere. On the other hand, for any  $N$  a normalized true eigenstate of the translation operator  $T_N$  exists such that the scalar product of this state with  $\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}}$  tends to its norm for  $N \rightarrow \infty$ .

Now we have to consider the generalization of (A.4) to  $\lambda > 0$ :

$$\begin{aligned}
\left(H_N^{(n)} - E\right)\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} &= \sum_{j=1}^r \mathcal{O}(H_{N_j}^{(n)})\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} \\
&+ \sum_{j=1}^r \|N_1; Q_1\rangle\rangle_{P_1} \otimes \dots \otimes \left(H_{N_j}^{(n)} - E_{Q_j}\right)\|N_j; Q_j\rangle\rangle_{P_j} \otimes \dots \otimes \|N_r; Q_r\rangle\rangle_{P_r} \\
&= \sum_{j=1}^r \mathcal{O}(H_{N_j}^{(n)})\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}}.
\end{aligned} \tag{A.11}$$

The boundary terms explicitly read as follows:

$$\begin{aligned}
\mathcal{O}(H_{N_j}^{(n)})\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}} &= \|N_1; Q_1\rangle\rangle_{P_1} \otimes \dots \\
&\sum_{k=1}^{n-1} \alpha_k \sum_{\nu \geq 0} \lambda^{\nu+1} \sum_{\substack{i_1^{(j)} + \dots + i_{N_j}^{(j)} = Q_j \bmod n \\ \#\{i_k^{(j)} \neq 0\} \leq 2\nu+1}} \kappa_{i_1^{(j)}, \dots, i_{N_j}^{(j)}}^{(\nu)} \frac{1}{\sqrt{N_j}} \left\{ \sum_{m=0}^{N_j-1} \right. \\
&e^{-imP_j} \left| (i_m^{(j)} - k \bmod n) i_{m+1}^{(j)} \dots i_{m-2}^{(j)} (i_{m-1}^{(j)} + k \bmod n) \right\rangle \otimes \|N_{j+1}; Q_{j+1}\rangle\rangle_{P_{j+1}} \\
&\left. - e^{-imP_j} \left| i_m^{(j)} \dots i_{m-2}^{(j)} (i_{m-1}^{(j)} + k \bmod n) \right\rangle \otimes \left( \Gamma_1^{n-k} \|N_{j+1}; Q_{j+1}\rangle\rangle_{P_{j+1}} \right) \right\} \\
&\dots \otimes \|N_r; Q_r\rangle\rangle_{P_r}.
\end{aligned} \tag{A.12}$$

The crucial point in (A.12) is that the states  $|i_m^{(j)} \dots i_{m-1}^{(j)}\rangle$  do not combine to momentum eigenstates any more. More precisely, at fixed order  $\nu$  of the perturbation expansion one obtains at most  $2\nu + 3$  terms of a complete momentum eigenstate. Therefore, these states are suppressed for  $N_j$  sufficiently large by the normalization factor  $N_j^{-\frac{1}{2}}$ . In other words: If we project (A.12) at fixed order in  $\lambda$  onto any momentum eigenstate and correct by the norm of  $\|N; Q_{\text{tot}}\rangle\rangle_{P_{\text{tot}}}$  the result tends to zero. Note that we may not draw direct

conclusions for the limit of  $H_N^{(n)}$  because the single-particle energies  $E_{Q_j}$  do not converge. But for  $\Delta H_N^{(n)}$  the single-particle energies  $\Delta E_{Q_j,0}$  converge and from (A.11), (A.12) we may conclude that the states (A.8) behave precisely like  $r$ -particle states in the limit  $N \rightarrow \infty$ .

We have shown so far that the quasiparticle excitations describe a subset of the spectrum of  $\Delta H_N^{(n)}$  in the weak limit. To complete the proof we have to argue that this is already the complete spectrum. This is guaranteed by the fact that for any finite  $M$  the complete Hilbert space  $\mathcal{H}_M$  can be mapped onto a subspace of  $\mathcal{H}_N$  ( $N$  sufficiently large) that is spanned precisely by the states (A.8). One natural choice is the mapping

$$\|Q_1 \dots Q_M\rangle\rangle_{P_{\text{tot}}} \mapsto \|N_1; Q_1\rangle\rangle_{P_1} \otimes \dots \otimes \|N_M; Q_M\rangle\rangle_{P_M} \in \mathcal{H}_N. \quad (\text{A.13})$$

This completes the proof.

Let us conclude with a summary of what we have assumed and what we were able to prove. Of course, the explicit form of the Hamiltonian (2.2) played an important rôle. We needed three facts:

- 1) For some values of the parameters ( $\lambda = 0$ ) the quasiparticle spectrum is trivially guaranteed.
- 2) In the vicinity of this point ( $\lambda > 0$ ) only nearest neighbours interact.
- 3) The Hamiltonian (2.2) possesses a  $\mathbb{Z}_n$ -symmetry.

It might seem that the third property was convenient mainly for notational reasons because it straightforwardly encoded property 1). However, we also needed the explicit form of the Hamiltonian (2.2) in order to ensure the absence of further selection rules (at least at  $\lambda = 0$ ). Thus, although we did not rely heavily on property 3), we doubt that our proof of the quasiparticle spectrum can easily be generalized to models having more complicated selection rules.

We further required that

- 4) The perturbation expansions for the *single*-particle states converge.

Note that we did not assume the Hamiltonian  $H_N^{(n)}$  to be hermitean nor did we require it to be diagonalizable – only the existence of the single-particle eigenvalues is needed.

Already in section 4 we inferred from property 4) (and 2) ) that the limits  $N \rightarrow \infty$  of the single-particle eigenvalues of  $\Delta H_N^{(n)}$  exist. The proof presented in this appendix shows that under these assumptions

- a) The weak limits of the operators  $T_N$  and  $\Delta H_N^{(n)}$  exist,
- b) The weak limits can be ‘diagonalized’, i.e. the projection-valued measure of (2.16) does indeed exist, and
- c) In this limit their spectrum can be expressed in terms of quasiparticle excitations. In particular, the spectrum of the weak limit of  $\Delta H_N^{(n)}$  is explicitly known if the dispersion relations of the single-particle excitations can be calculated.

## Appendix B: Symmetries of the Hamiltonian and the oscillation length

In this appendix we first discuss the behaviour of the Hamiltonian (2.2) under parity for special values of the parameters. One finds symmetries that were observed numerically in [46] for the integrable submanifold and can be derived e.g. along the lines of appendix B of [14]. The resulting identities will subsequently be used in order to derive the values of the oscillation length given in (6.7) from the form factor expansion (6.5).

### Symmetries of the Hamiltonian:

Denote the projection of the Hamiltonian  $H_N^{(n)}$  in eq. (2.2) onto the spaces  $\mathcal{H}_N^{P,Q}$  in eq. (2.15) by ' $H_N^{(n)}(P, Q)$ '. Furthermore, introduce a parity operator  $\mathfrak{P}$  by the following action on the states (2.5):

$$r(\mathfrak{P}) |i_1 \dots i_j \dots i_N\rangle = |i_1 i_N i_{N-1} \dots i_j \dots i_2\rangle. \quad (\text{B.1})$$

Note that  $\mathfrak{P}\sigma_{1+x}\mathfrak{P} = \sigma_{1-x}$ ,  $\mathfrak{P}\Gamma_{1+x}\mathfrak{P} = \Gamma_{1-x}$ . Then one has the following identities (see also [46]):

$$\begin{aligned} \alpha_k = \alpha_{n-k} &\Rightarrow \mathfrak{P}H_N^{(n)}(P, Q)\mathfrak{P} = H_N^{(n)}(-P, Q), \\ \bar{\alpha}_k^* = \bar{\alpha}_{n-k} \text{ and } \alpha_k \in \mathbb{R} &\Rightarrow \mathfrak{P}H_N^{(n)}(P, Q)\mathfrak{P} = \left(H_N^{(n)}(-P, Q)\right)^+, \\ \bar{\alpha}_k^* = \bar{\alpha}_{n-k} \text{ and } \alpha_k^* = e^{-2\pi izk} \alpha_k &\Rightarrow \mathfrak{P}H_N^{(n)}(P_{m,Q} + P, Q)\mathfrak{P} = \left(H_N^{(n)}(P_{m,Q} - P, Q)\right)^+ \end{aligned} \quad (\text{B.2})$$

where the symmetry of the last line holds for those  $P_{m,Q}$  satisfying  $P_{m,Q}Q^{-1} + \pi z \equiv 0 \pmod{\pi}$  as well as  $e^{i2P_{m,Q}}$  being an  $n$ th root of unity. Note that with the parametrization (2.10) the cases covered by (B.2) are precisely those covered by (6.7) with  $z = \frac{2}{n}$ . In this case,  $P_{m,Q} = \pi(1 - \frac{2Q}{n})$  is a solution to  $P_{m,Q}Q^{-1} + \frac{2\pi}{n} = 0$  lying in the interval  $[-\pi, \pi]$  – the other solution is shifted by  $\pi$ . The solution  $P_{m,Q} = \pi(1 - \frac{2Q}{n})$  corresponds to the minimum in the dispersion relation of the single-particle state in this charge sector (see [32][46]).

The first two lines of (B.2) follow immediately by looking at  $\mathfrak{P}H_N^{(n)}\mathfrak{P}$ , keeping in mind that the translation operator defined in (2.8) satisfies  $\mathfrak{P}T_N\mathfrak{P} = T_N^{-1} = T_N^+$ . The derivation of the third line of (B.2) is more complicated. For  $Q$  invertible in  $\mathbb{Z}_n$  it can be shown choosing a suitable basis (see appendix B of [44]). For  $z = \frac{2}{n}$  and  $N \equiv 0 \pmod{n}$  one can follow the lines of eqs. (B.12) – (B.16) in appendix B of [14] to elegantly prove the third line of (B.2).

In the case  $z = \frac{2}{n}$  and  $N \equiv 0 \pmod{n}$  we introduce an operator  $U$  following [14] by

$$U := \mathfrak{P} \left( \prod_{x=1}^N \sigma_x^{-2x} \right). \quad (\text{B.3})$$

Now, observing that

$$U\sigma_{1+x}U^{-1} = \sigma_{1-x}, \quad U\Gamma_{1+x}U^{-1} = \omega^{2x}\Gamma_{1-x} \quad (\text{B.4})$$

one concludes that  $U H_N^{(n)} U^{-1} = \left(H_N^{(n)}\right)^+$  for  $\alpha_k^* = \omega^{-2k} \alpha_k$ . Finally one verifies that

$$T_N U \|i_1 \dots i_N\rangle\rangle_P = \mathfrak{P} T_N^{-1} \left( \prod_{x=1}^N \sigma_x^{-2x} \right) \|i_1 \dots i_N\rangle\rangle_P = U \hat{Q}^{-2} e^{-iP} \|i_1 \dots i_N\rangle\rangle_P \quad (\text{B.5})$$

where  $\hat{Q}$  is the charge operator given by eq. (2.7). Eq. (B.5) implies that the operator  $U$  maps a state of charge  $Q$  and momentum  $P$  to a state of charge  $Q$  and momentum  $-\frac{4\pi Q}{n} - P$ . After putting things together one obtains the desired result.

### Oscillation length from symmetries of the Hamiltonian:

Assume that the Hamiltonian  $H(P, Q)$  projected onto momentum and charge eigenspaces with eigenvalues  $P$  and  $Q$  has one of the following symmetries:

$$\mathfrak{P} H(P_{m,Q} + P, Q) \mathfrak{P} = H(P_{m,Q} - P, Q) \quad \text{or} \quad \mathfrak{P} H(P_{m,Q} + P, Q) \mathfrak{P} = (H(P_{m,Q} - P, Q))^+ \quad (\text{B.6})$$

with some  $P_{m,Q}$  depending on the charge sector  $Q$ . Assume furthermore that  $P|v\rangle = 0$  and that  $\Xi_1 |v\rangle$  has charge  $Q$ . Then the oscillation length  $L$  of the correlation function  $C_{\Xi}(x)$  satisfies

$$L P_{m,Q} = 2\pi. \quad (\text{B.7})$$

Note that this is true for more general Hamiltonians  $H(P, Q)$ , but it covers in particular the case (B.2) for the  $\mathbb{Z}_n$ -chiral Potts model.

For a proof of (B.7) we start from the form factor expansion (6.5) which in the present case becomes

$$C_{\Xi}(x) = \sum_r \int_0^{2\pi} dP e^{iPx} \frac{|\langle P, Q; r | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} \quad (\text{B.8})$$

where we have only written the quantum numbers  $P$  and  $Q$  explicitly and comprised the other ones in the label ' $r$ '. First we observe that  $\mathfrak{P} \Xi_1 \mathfrak{P} = \Xi_1$ . If the Hamiltonian satisfies  $\mathfrak{P} H(P_{m,Q} + P, Q) \mathfrak{P} = H(P_{m,Q} - P, Q)$ , then eigenstates of momentum  $P_{m,Q} + P$  are mapped under parity to eigenstates of momentum  $P_{m,Q} - P$ . This means that  $\langle (P_{m,Q} + P), Q; r | \Xi_1 | v \rangle = \langle (P_{m,Q} - P), Q; r | \Xi_1 | v \rangle$ . If the symmetry involves the adjoint of the Hamiltonian one finds  $\langle (P_{m,Q} + P), Q; r | \Xi_1 | v \rangle = \langle (P_{m,Q} - P), Q; r | \Xi_1 | v \rangle^*$ . Thus, the following identity is valid in both cases:

$$|\langle (P_{m,Q} + P), Q; r | \Xi_1 | v \rangle|^2 = |\langle (P_{m,Q} - P), Q; r | \Xi_1 | v \rangle|^2. \quad (\text{B.9})$$

Now we return to the form factor expansion (B.8):

$$\begin{aligned} C_{\Xi}(x) &= \sum_r \left\{ \int_{P_{m,Q}}^{P_{m,Q} + \pi} dP e^{iPx} \frac{|\langle P, Q; r | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} + \int_{P_{m,Q} - \pi}^{P_{m,Q}} dP e^{iPx} \frac{|\langle P, Q; r | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} \right\} \\ &= \sum_r \int_0^{\pi} dP \left\{ e^{i(P_{m,Q} + P)x} \frac{|\langle (P_{m,Q} + P), Q; r | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} \right. \\ &\quad \left. + e^{i(P_{m,Q} - P)x} \frac{|\langle (P_{m,Q} - P), Q; r | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} \right\} \\ &= e^{iP_{m,Q}x} \sum_r \int_0^{\pi} dP 2 \cos(Px) \frac{|\langle (P_{m,Q} + P), Q; r | \Xi_1 | v \rangle|^2}{\langle v | v \rangle} \end{aligned} \quad (\text{B.10})$$

where the last equality follows from (B.9). This shows that  $C_{\Xi}(x)$  is of the form

$$C_{\Xi}(x) = e^{\frac{2\pi i x}{L}} f(x) \tag{B.11}$$

with  $L$  satisfying (B.7) and  $f(x)$  is given by the remaining integral in (B.10) which is clearly real.

Note that if the Hamiltonian has several different  $P_{m,Q}$  such that (B.6) holds (which applies to (B.2)) one obtains different expressions for  $C_{\Xi}(x)$  involving different  $L$  and  $f(x)$ . The suitable one among them can be singled out by demanding e.g.  $f(x) > 0$  for all  $x$ . Our explicit computations indicate that this requirement (which means that the oscillations are exclusively encoded in the phase factor) indeed leads to the oscillations lengths presented in (6.7).



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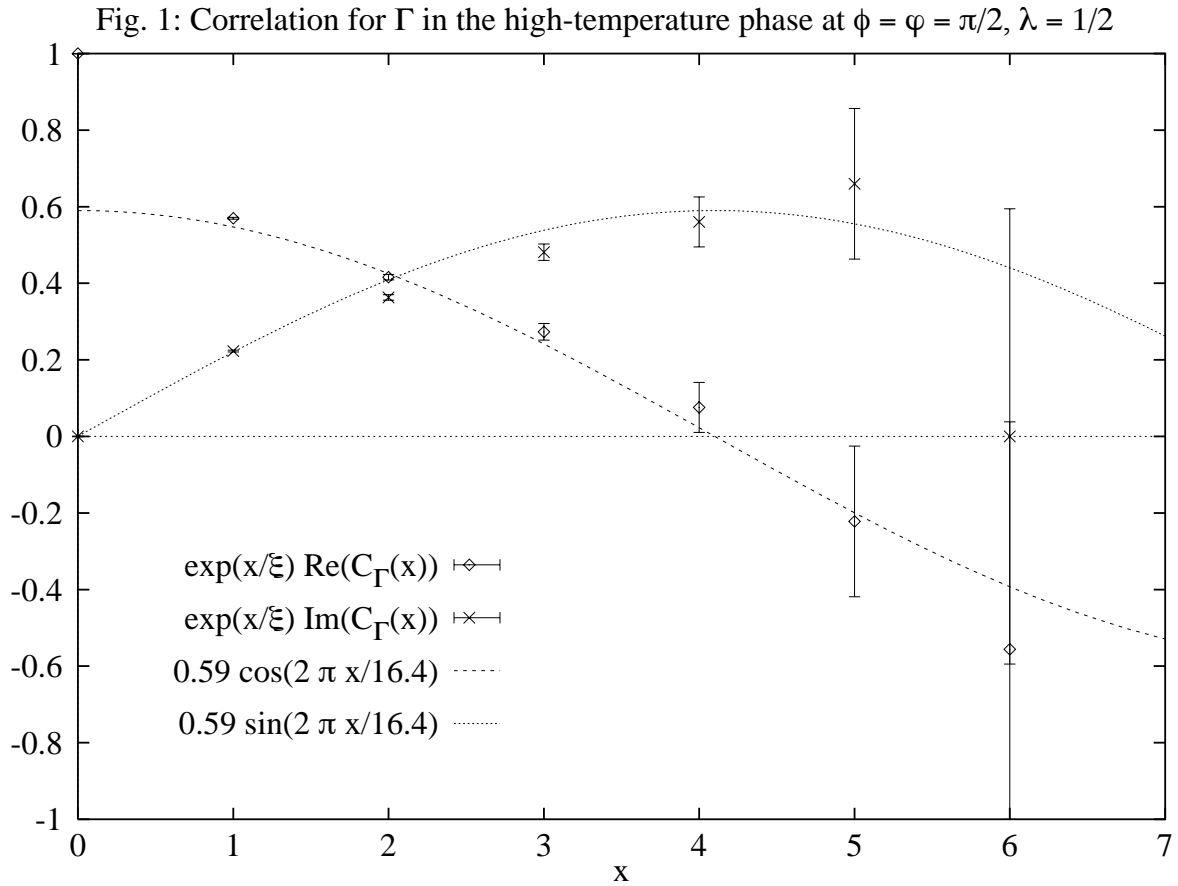


Fig. 1: Correlation function  $C_{\Gamma}(x)$  stretched by  $e^{\frac{x}{\xi_{\Gamma}}}$  in comparison to the fits (6.18a) at  $\phi = \varphi = \frac{\pi}{2}, \lambda = \frac{1}{2}$ . The ‘error bars’ are given by  $ae^{\frac{x-6}{\xi_{\Gamma}}}$  which conveys an idea how much the values have actually been stretched. The oscillatory contribution to  $C_{\Gamma}(x)$  is clearly visible.

Fig. 2: Correlation length for  $C_\Gamma(x)$  in the high-temperature phase at  $\phi = \varphi = 0$

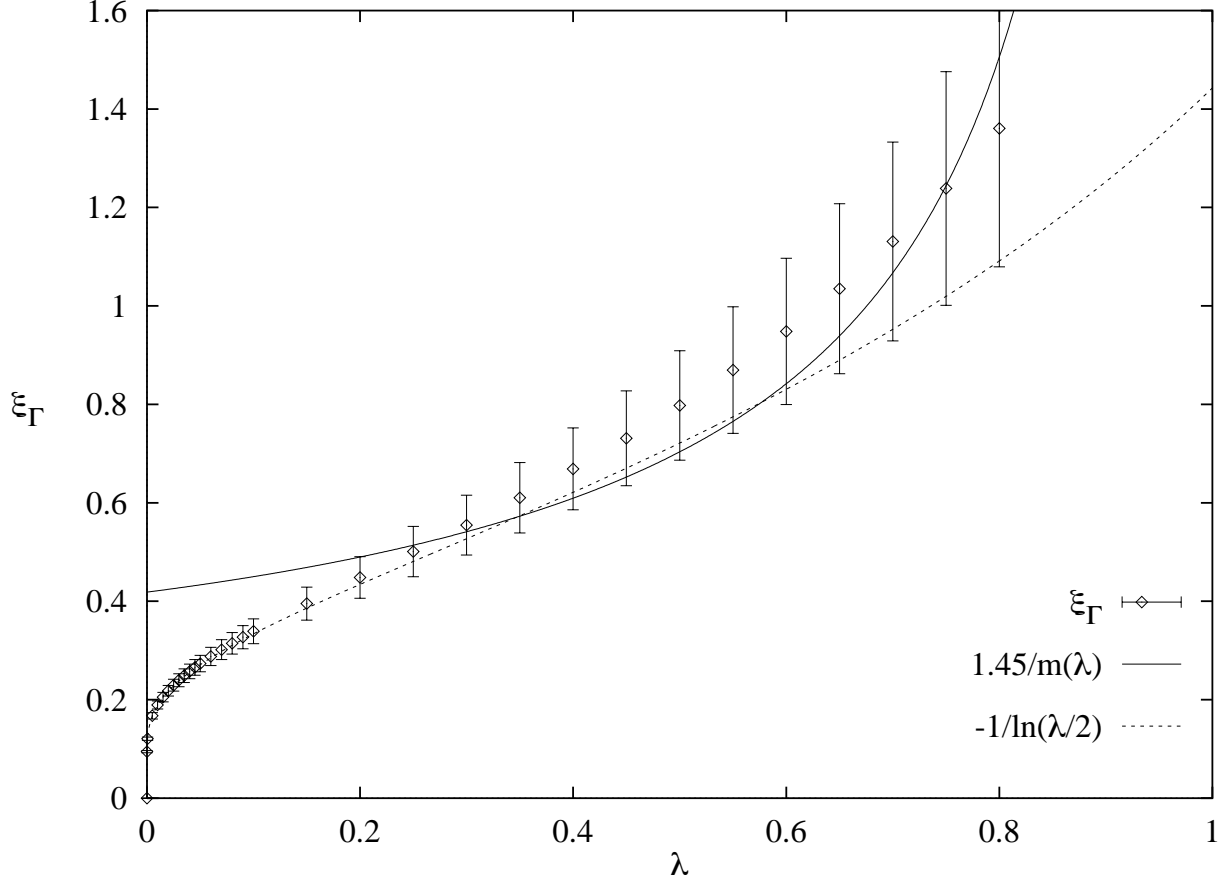


Fig. 2: Correlation length for  $C_\Gamma(x)$  in the massive high-temperature phase on the parity conserving line  $\phi = \varphi = 0$ . The points indicate estimates obtained from a perturbative evaluation of  $C_\Gamma(x)$ . The lines indicate the approximation (6.21) for  $\xi_\Gamma$  and the properly normalized inverse mass gap  $m(\lambda)^{-1}$ .

Fig. 3: Radii of convergence for  $n = 3$  and hermitian Hamiltonian

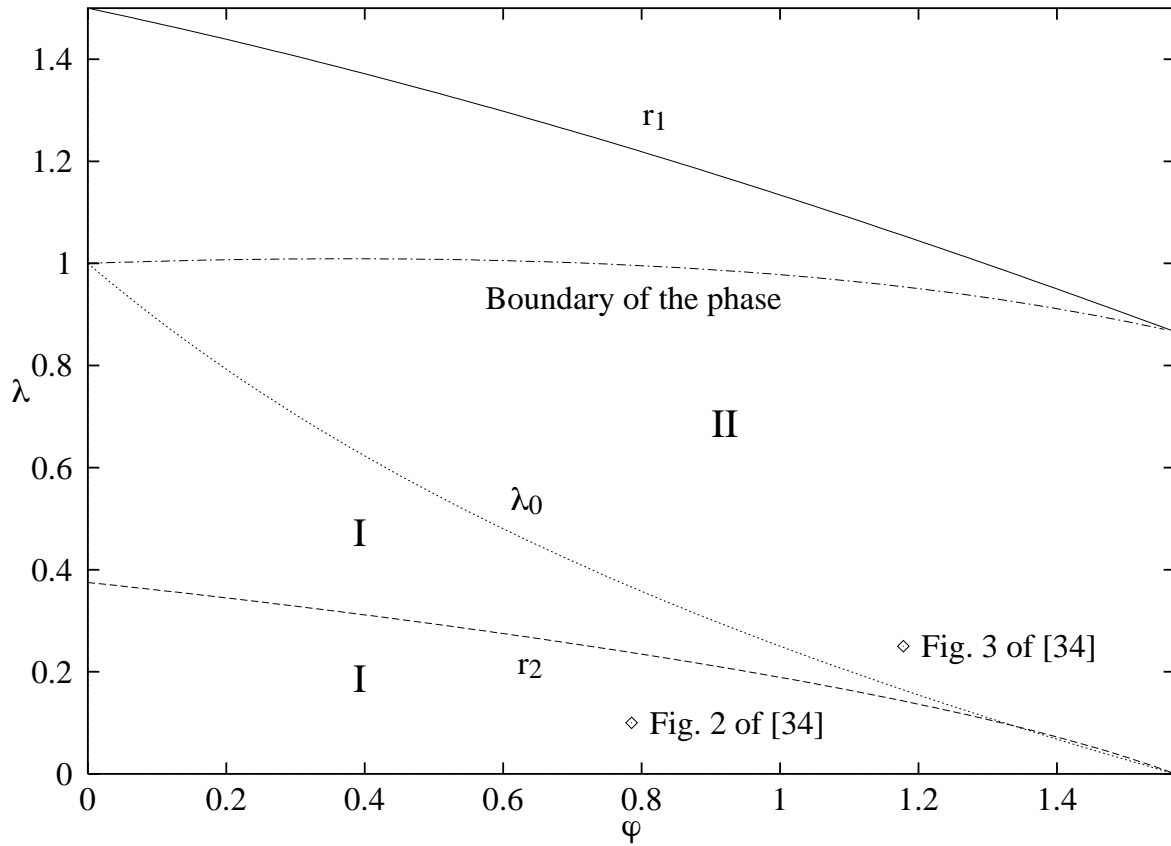


Fig. 3: Radii of convergence and boundary of the massive high-temperature phase for the hermitean  $\mathbb{Z}_3$ -chain.  $r_1$  is an estimate ensuring convergence if *no level crossings between point and continuous spectrum occur*. The estimate  $r_2$  also ensures the absence of level crossings. The perturbation series are definitely convergent for  $\lambda < r_2$  although the true radius of convergence is larger. It extends until the value  $\lambda_0$  where the first level crossings between fundamental quasiparticles and scattering states occur.  $\lambda_0$  has been approximated using a first order perturbation expansion which is surprisingly accurate.

The boundary of the massive high-temperature phase close to  $\lambda = 1$  has been approximated using a second order perturbation expansion.

Note that  $r_1$  and  $r_2$  are independent of  $\phi$  up to the order calculated whereas for  $\lambda_0$  we put  $\varphi = \phi$ .