

Unifying \mathcal{W} -Algebras

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Abstract

We show that quantum Casimir \mathcal{W} -algebras truncate at degenerate values of the central charge c to a smaller algebra if the rank is high enough: Choosing a suitable parametrization of the central charge in terms of the rank of the underlying simple Lie algebra, the field content does not change with the rank of the Casimir algebra any more. This leads to identifications between the Casimir algebras themselves but also gives rise to new, ‘unifying’ \mathcal{W} -algebras. For example, the k th unitary minimal model of \mathcal{WA}_n has a unifying \mathcal{W} -algebra of type $\mathcal{W}(2, 3, \dots, k^2 + 3k + 1)$. These unifying \mathcal{W} -algebras are non-freely generated on the quantum level and belong to a recently discovered class of \mathcal{W} -algebras with infinitely, non-freely generated classical counterparts. Some of the identifications are indicated by level-rank-duality leading to a coset realization of these unifying \mathcal{W} -algebras. Other unifying \mathcal{W} -algebras are new, including e.g. algebras of type \mathcal{WD}_{-n} . We point out that all unifying quantum \mathcal{W} -algebras are *finitely*, but non-freely generated.

0. Introduction

Quantum \mathcal{W} -algebras play a fundamental rôle in two dimensional conformal field theory (for a recent review see e.g. [1]). In this letter we shall focus on the so-called ‘generic’ (or ‘deformable’) \mathcal{W} -algebras where the structure constants are continuous functions (with isolated singularities) of the Virasoro centre c for a given set of generating fields. There is also the class of ‘non-deformable’ \mathcal{W} -algebras that exist only for finitely many isolated values of the central charge c . Non-deformable \mathcal{W} -algebras with few generators and their representations have been extensively studied in [2 – 6]. On the basis of these results it is now generally believed that non-deformable \mathcal{W} -algebras can be regarded as extensions or truncations of deformable ones ¹⁾ – with known exceptions [7, 8].

The finitely generated generic quantum \mathcal{W} -algebras fall at least into two classes. The class that consists of freely generated \mathcal{W} -algebras is already quite well understood. There are indications [9 – 11] that all algebras in this class can be obtained by quantized [12] Drinfeld-Sokolov reduction (see e.g. [10] and references therein). In particular, quantized Drinfeld-Sokolov reduction for a principal $sl(2)$ embedding into a simple Lie algebra \mathcal{L}_n of rank n leads to the so-called ‘Casimir algebras’ [13] which we denote by ‘ $\mathcal{W}\mathcal{L}_n$ ’ (we will also call n the ‘rank of $\mathcal{W}\mathcal{L}_n$ ’). Note that the classical counterparts of the \mathcal{W} -algebras in this class are *finitely, freely* generated.

It was recently shown in [14] that there is another class of \mathcal{W} -algebras with *infinitely, non-freely* generated classical counterparts. Due to cancellations between generators and relations upon normal ordering, the quantized versions of these \mathcal{W} -algebras become finitely, non-freely generated in all known examples. This class of \mathcal{W} -algebras contains –among others– orbifolds and cosets ²⁾. Two unexpected examples of finitely, but non-freely generated quantum \mathcal{W} -algebras had been found earlier: Particular extensions of the Virasoro algebra with additional fields of conformal dimensions 4, 6 [2] and 3, 4, 5 [15], respectively. In particular, the origin of the former one (which we denote by ‘ $\mathcal{W}(2, 4, 6)$ ’) was mysterious for some time [6] although it was proposed in [16] to identify this algebra formally with ‘ \mathcal{WD}_{-1} ’. In [14, 17] these $\mathcal{W}(2, 4, 6)$ and $\mathcal{W}(2, 3, 4, 5)$ were explained in terms of the cosets $\widehat{sl(2, \mathbb{R})}_k \oplus \widehat{sl(2, \mathbb{R})}_{-\frac{1}{2}} / \widehat{sl(2, \mathbb{R})}_{k-\frac{1}{2}}$ and $\widehat{sl(2, \mathbb{R})}_k / \widehat{U(1)}$.

In this letter we will show that this second class contains ‘unifying \mathcal{W} -algebras’ which interpolate the rank n of Casimir algebras $\mathcal{W}\mathcal{L}_n$ at particular values of the central charge. For example, the $\mathcal{W}(2, 3, 4, 5)$ unifies the first unitary minimal models of \mathcal{WA}_{n-1} whereas the $\mathcal{W}(2, 4, 6)$ corresponds to certain minimal models of \mathcal{WC}_n . Using the coset realization of these algebras, the unifying \mathcal{W} -algebras can be regarded as a generalization of level-rank-duality [18, 19]. These identifications between a priori different \mathcal{W} -algebras for particular values of the central charge c are closely related to the fact that for certain values of the central charge c some generators become null fields leading to a ‘truncation’ of the

¹⁾ Note that in [6] there is a misleading remark on this question because it was not recognized that $\mathcal{W}(2, 8)$ at $c = -\frac{712}{7}$ and $c = -\frac{3164}{23}$ arises as a truncation of \mathcal{WE}_8 and \mathcal{WE}_7 respectively.

²⁾ On the classical level all cosets belong to this class. However, particular quantum coset constructions (e.g. of Casimir algebras) yield finitely, *freely* generated \mathcal{W} -algebras.

\mathcal{W} -algebra. We will further show from inspection of the Kac determinant for Casimir \mathcal{W} -algebras that truncations of them are in fact a very general phenomenon. These truncations of Casimir \mathcal{W} -algebras imply truncations of the various linear \mathcal{W}_∞ -algebras to a finitely generated algebra which can indeed be verified.

Unifying \mathcal{W} -algebras

We introduce the notion of ‘unifying \mathcal{W} -algebras’ before establishing their existence. A unifying \mathcal{W} -algebra is a finitely generated deformable quantum \mathcal{W} -algebra (i.e. existing for generic c) with the following properties:

- There exists an integer n_0 and simple Lie algebras \mathcal{L}_n such that for all $n \geq n_0$ and $c = c_{\mathcal{L}_n}(p(n), q(n))$ it is isomorphic to the Casimir algebra $\mathcal{W}\mathcal{L}_n$.
- \mathcal{L}_n is either of type \mathcal{A} , \mathcal{B} , \mathcal{C} or \mathcal{D} independent of n .

This implies that for all $n > n_0$ the Casimir algebras $\mathcal{W}\mathcal{L}_n$ at central charge $c_{\mathcal{L}_n}(p(n), q(n))$ truncate to the unifying \mathcal{W} -algebra whose spin content is independent of n .

Since unifying \mathcal{W} -algebras exist for generic c , they can be considered as continuations to real values of the rank n of Casimir algebras $\mathcal{W}\mathcal{L}_n$ at $c_{\mathcal{L}_n}(p(n), q(n))$. Some Casimir algebras play the rôle of unifying \mathcal{W} -algebras. In contrast to them, the ‘true’ unifying \mathcal{W} -algebras are non-freely generated and belong to the second class of quantum \mathcal{W} -algebras.

1. Truncations and identifications from structure constants

The first unitary minimal model of \mathcal{WA}_{k-1} has the symmetry algebra $\mathcal{W}(2, 3, 4, 5) \cong \widehat{sl(2, \mathbb{R})}_k / \widehat{U(1)}$ independent of k [14, 17]. This means that in the algebra \mathcal{WA}_{k-1} all simple fields of dimension ≥ 6 become null fields and the algebra \mathcal{WA}_{k-1} truncates at $c_{\mathcal{A}_{k-1}}(k+1, k+2) = \frac{2(k-1)}{k+2}$ to the algebra $\mathcal{W}(2, 3, 4, 5)$ which is a *unifying* algebra for the first unitary model of all \mathcal{WA}_{k-1} . In the following we will investigate more general cases such as the n th unitary model of \mathcal{WA}_{k-1} . Similarly, the coset $\mathcal{W}_3^{(2)} / \widehat{U(1)} \cong \mathcal{W}(2, 3, 4, 5, 6, 7)$ which will be discussed in [17] ($\mathcal{W}_3^{(2)}$ denotes the algebra of Polyakov and Bershadsky [20, 21]) is a unifying algebra for the non-unitary minimal models $c_{\mathcal{A}_{k-1}}(k+1, k+3)$ of \mathcal{WA}_{k-1} .

In order to study these truncations and identifications in a systematic way we make use of the observation [16] that deformable \mathcal{W} -algebras can be parametrized with ‘universal’ formulae in which only the conformal dimension of a certain representation enters as parameter. More precisely, the structure constants of many \mathcal{W} -algebras fall into different classes, so that in each class the structure constants can be described by universal formulae $C_{nm}^l(c, h(c))$. The parametrizing function $h(c)$ varies for each of these algebras, being in fact any one of the two conformal dimensions which correspond to representations with a particularly low-lying null-state. Usually, there is also a third parametrizing function $h(c)$ that does not correspond to degenerate representations.

One of these classes are \mathcal{W} -algebras of type $\mathcal{W}(2, 3, 4, \dots)$, that is \mathcal{W} -algebras with primary fields of dimensions 3 and 4 and any number of fields with dimension ≥ 5 . Therefore, all \mathcal{WA}_{n-1} -algebras and all truncations of them including in particular the special algebras $\mathcal{W}(2, 3, 4, 5) \cong \widehat{sl(2, \mathbb{R})}_k / \widehat{U(1)}$ and $\mathcal{W}(2, 3, 4, 5, 6, 7) \cong \mathcal{W}_3^{(2)} / \widehat{U(1)}$ belong to this class. Another class is formed of some \mathcal{W} -algebras of type $\mathcal{W}(2, 4, \dots)$ which includes the algebras \mathcal{WB}_n and \mathcal{WC}_n as well as the orbifolds of \mathcal{WD}_n and $\mathcal{WB}(0, n)$. In this approach, the

structure constants of the algebras \mathcal{WD}_n can be continued to negative values of the rank such that the algebra $\mathcal{W}(2, 4, 6) \cong \mathcal{WD}_{-1}$ belongs to this class [16]. In [17] we shall give a realization of \mathcal{WD}_{-n} in terms of the coset $\widehat{sp(2n)}_k \oplus \widehat{sp(2n)}_{-\frac{1}{2}} / \widehat{sp(2n)}_{k-\frac{1}{2}}$ and show that they are of type $\mathcal{W}(2, 4, \dots, 2n(n+2))$.

The fact that the structure constants in each class can be described solely by the set of values of the three parameters for $h(c)$, $H = \{h_1(c), h_2(c), h_3(c)\}$, has the immediate consequence that the structure constants for two algebras X and V , and hence the algebras, become identical, whenever the sets H_X and H_V are equal. In general, the identification will lead to a truncation of the algebra with the higher number of simple fields, say X , to an algebra V with less fields at a *finite* set of values for the central charge. We shall denote this truncation by the symbol ' \triangleright ', i.e.

$$X \triangleright V \quad \text{if} \quad H_X = H_V. \quad (1.1)$$

However, we remark that the two sets H_X and H_V are also equal if V is a subalgebra of X (at some fixed c -values) and no truncation takes place. There are indeed special cases known where this occurs. Thus, one has to prove the existence of null fields in order to confirm the truncation expected from this argument. Indications for the presence of null fields will be presented in later sections.

Note that (1.1) is usually automatically satisfied if one $h_i \in H_X$ is equal to one $h_j \in H_V$. The exceptions are related to singularities that occur e.g. at $c = 0$ or at $c = -2$ (see (1.2)).

Identifications of \mathcal{W} -algebras of type $\mathcal{W}(2, 3, 4, \dots)$

The first few structure constants for these algebras are given in [16]. It is immediately clear from the structure constants that any \mathcal{W} -algebra of type $\mathcal{W}(2, 3, 4, \dots)$ (which can be described by these structure constants) truncates to the algebra $\mathcal{W}(2, 3)$ at $c = -2$:

$$\left. \begin{array}{lcl} \mathcal{WA}_{n-1} & \triangleright & \mathcal{W}(2, 3) \\ \mathcal{W}(2, 3, 4, 5) & \triangleright & \mathcal{W}(2, 3) \\ \mathcal{W}(2, 3, 4, 5, 6, 7) & \triangleright & \mathcal{W}(2, 3) \end{array} \right\} \quad \text{at } c = -2. \quad (1.2)$$

These truncations are independent of the values of h whence in these cases it is not necessary that the corresponding sets H agree. We will omit such cases in the following and restrict our discussion to cases where the sets H agree.

For the three algebras mentioned the set $H = \{h_1, h_2, h_3\}$ consists of the following values:

$$\begin{aligned} \mathcal{WA}_{n-1} : \quad & 4n^2 h_{1,2}^2 + 2(c - (n-1)(2n+1))h_{1,2} + c(n-1) = 0; \quad h_3 = \frac{c(n+1)}{2(c+1-n)} \\ \mathcal{W}(2, 3, 4, 5) : \quad & h_1 = \frac{3c}{2(c+1)}; \quad h_2 = \frac{c(2-c)}{8(c+1)}; \quad h_3 = -\frac{c+4}{2(c+1)} \\ \mathcal{W}(2, 3, 4, 5, 6, 7) : \quad & h_1 = \frac{3(k+1)}{2k+3}; \quad h_2 = \frac{3(k+1)(k+2)}{2(2k+3)}; \quad h_3 = -\frac{(k+1)^2}{(k+3)(2k+3)} \\ & \text{with } c = -6 \frac{(k+1)^2}{k+3}. \end{aligned} \quad (1.3)$$

By identifying the sets $H_{\mathcal{WA}_{m-1}}$ and $H_{\mathcal{WA}_{n-1}}$ we find a solution for the central charge where \mathcal{WA}_{m-1} truncates to \mathcal{WA}_{n-1} ($m > n$):

$$\mathcal{WA}_{m-1} \triangleright \mathcal{WA}_{n-1} \quad \text{at} \quad c_m(n) = -\frac{(m-1)(n-1)(m+n+mn)}{m+n}. \quad (1.4)$$

The identification (1.4) has already been proposed in [22] where equality of characters was shown using the coset realization of \mathcal{WA}_{n-1} . Let us now investigate the truncations of \mathcal{WA}_{n-1} to $\mathcal{W}(2, 3, 4, 5)$. From eq. (1.3) we find the solutions ³⁾

$$\mathcal{WA}_{n-1} \triangleleft \mathcal{W}(2, 3, 4, 5) \cong \frac{sl(\widehat{2, \mathbb{R}})_n}{\widehat{U(1)}} \quad \text{at} \quad \begin{aligned} c(n) &= \frac{2(n-1)}{n+2} \\ c(n) &= -1 - 3n \\ c(n) &= \frac{2(1-2n)}{n-2}. \end{aligned} \quad (1.5)$$

The first truncation is the well-known identification of the coset $sl(\widehat{2, \mathbb{R}})_n/\widehat{U(1)}$ with the \mathcal{WA}_{n-1} -algebra for the first unitary minimal model. The other two truncations do not correspond to minimal models.

We proceed in the same way for truncations of \mathcal{WA}_{n-1} to $\mathcal{W}(2, 3, 4, 5, 6, 7)$:

$$\mathcal{WA}_{n-1} \triangleleft \mathcal{W}(2, 3, 4, 5, 6, 7) \cong \frac{\mathcal{W}_3^{(2)}}{\widehat{U(1)}} \quad \text{at} \quad \begin{aligned} k(n) &= \frac{-3(n+1)}{n+2} & c(n) &= \frac{-2(2n+1)^2}{n+2} \\ k(n) &= \frac{n-3}{2} & c(n) &= \frac{-3(n-1)^2}{n+3} \\ k(n) &= \frac{3-2n}{n-2} & c(n) &= \frac{6(n-1)^2}{(2-n)(n-3)}. \end{aligned} \quad (1.6)$$

Here, we denote the level entering $\mathcal{W}_3^{(2)}$ by ‘ k ’ [17, 20, 21]. Note that the second line of (1.6) corresponds to minimal models of $\mathcal{W}_3^{(2)}$.

Identification of \mathcal{W} -algebras of type $\mathcal{W}(2, 4, \dots)$

We will now look for identifications between algebras of type $\mathcal{W}(2, 4, \dots)$. Since the bosonic projections of the superalgebras $\mathcal{WB}(0, m)$ are described by the formulae for the orbifold of \mathcal{WD}_n with $n = m + \frac{1}{2}$ half-integer, we shall not consider these bosonic projections separately, but use the notation of the orbifold of \mathcal{WD}_n for both of them, having in mind that we treat the superalgebra $\mathcal{WB}(0, n - \frac{1}{2})$ whenever n is half-integer. A further argument for this notation is that both algebras are realized in terms of diagonal $so(k)$ -cosets with even and odd $k = 2n$ respectively (see e.g. [1]). The orbifold of $\mathcal{WD}_{\frac{3}{2}}$ for example is the bosonic projection of the $N = 1$ Super Virasoro algebra – a non-freely generated algebra of type $\mathcal{W}(2, 4, 6)$ (see e.g. [23, 14]). Moreover, also negative values of n are allowed for \mathcal{WD}_n .

The first structure constants for this class of \mathcal{W} -algebras have been presented in [16]. We

³⁾ For $n \leq 4$ $\mathcal{W}(2, 3, 4, 5)$ truncates to \mathcal{WA}_{n-1} , for $n = 5$ the two algebras are isomorphic for the given values of c . We use therefore the symbol ‘ \triangleleft ’.

give only the classifying sets H :

$$\begin{aligned}
\text{Orb}(\mathcal{WD}_n) : \quad & 4n(n-1)h_{1,2}^2 + (c+n(3-4n))h_{1,2} + c(n-1) = 0; \quad h_3 = n; \\
\mathcal{WB}_n, \mathcal{WC}_n : \quad & 8n(1+n)h_1^2 + 2(c-n(3+2n))h_1 + c(2n-1) = 0; \\
& (2n-1)(2n+1)h_2^2 + (c-n(6n+1))h_2 + cn + 2n^2 = 0; \\
& 2(c+2n)h_3^2 - (n(2n+3) + c(4n+3))h_3 + c(n+1)(2n+1) = 0; \\
& \text{where one of the solutions is for } \mathcal{WB}, \text{ the other one for } \mathcal{WC}.
\end{aligned} \tag{1.7}$$

For this class of algebras we find the following truncations with generic $m > n$:

$$\text{Orb}(\mathcal{WD}_m) \triangleright \text{Orb}(\mathcal{WD}_n) \quad \text{at} \quad c = -\frac{mn(3-4m-4n+4mn)}{m+n-1}, \tag{1.8}$$

$$\mathcal{WC}_m \triangleright \mathcal{WC}_n \quad \text{at} \quad c = \frac{mn(3+2m+2n-4mn)}{1+m+n}, \tag{1.9}$$

$$\mathcal{WB}_m \triangleright \mathcal{WB}_n \quad \text{at} \quad c = -\frac{(2m+n+2mn)(m+2n+2mn)}{m+n}, \tag{1.10}$$

$$\mathcal{WB}_m \triangleright \mathcal{WC}_n \quad \text{at} \quad c = \frac{4mn(3+2m+2n)}{(1+2m-2n)(1-2m+2n)}. \tag{1.11}$$

Finally, for generic m and n , one has the following identifications where the larger algebra truncates to the smaller one:

$$\mathcal{WB}_m \Leftrightarrow \mathcal{WC}_n \quad \text{at} \quad c = -\frac{2mn(3+2m+2n+4mn)}{1+2m+2n}, \tag{1.12}$$

$$\text{Orb}(\mathcal{WD}_m) \Leftrightarrow \mathcal{WB}_n \quad \text{at} \quad c = \frac{2mn(-3+4m-2n+4mn)}{1-2m-2n}, \tag{1.13}$$

$$\text{Orb}(\mathcal{WD}_m) \Leftrightarrow \mathcal{WB}_n \quad \text{at} \quad c = \frac{(n-m-2mn)(2mn-m-2n)}{m+n}, \tag{1.14}$$

$$\text{Orb}(\mathcal{WD}_m) \Leftrightarrow \mathcal{WC}_n \quad \text{at} \quad c = \frac{mn(3-4m+2n)}{(1-m+n)(1-2m+2n)}. \tag{1.15}$$

For special values of m and n one might find additional solutions.

Let us now discuss some interesting consequences of these identifications. Eq. (1.14) can be parametrized by the formula for the minimal models of \mathcal{WB}_n as $c_{\mathcal{B}_n}(2n, 2n+2(m-\frac{1}{2})+1)$. Thus, for half-integer $m = k + \frac{1}{2}$ \mathcal{WB}_n truncates at $c_{\mathcal{B}_n}(2n, 2n+2k+1)$ to $\text{Orb}(\mathcal{WD}_m) = \text{Orb}(\mathcal{WB}(0, k))$. We can interpret $\text{Orb}(\mathcal{WB}(0, k))$ as the unifying algebra of the \mathcal{WB}_n models at $c_{\mathcal{B}_n}(2n, 2n+2k+1)$. Eq. (1.13) can be parametrized as $c_{\mathcal{B}_n}(2n-1+2m, 2n+1)$. We infer that \mathcal{WB}_n truncates at these values to $\text{Orb}(\mathcal{WD}_m)$ which serves as a unifying algebra for these \mathcal{WB}_n minimal models. Using the realization of \mathcal{WD}_m and $\text{Orb}(\mathcal{WB}(0, m))$ in terms of $so(k)$ -cosets these identifications can be summarized as follows:

$$\begin{aligned}
\mathcal{WB}_n \cong & \begin{cases} \frac{\widehat{so(k)}_\mu \oplus \widehat{so(k)}_1}{\widehat{so(k)}_{\mu+1}} \cong \text{Orb}(\mathcal{WB}(0, \frac{k-1}{2})), & k \text{ odd} \\ \text{Orb}\left(\frac{\widehat{so(k)}_\mu \oplus \widehat{so(k)}_1}{\widehat{so(k)}_{\mu+1}}\right) \cong \text{Orb}\left(\mathcal{WD}_{\frac{k}{2}}\right), & k \text{ even} \end{cases} \\
& \text{at} \quad \begin{cases} c = \frac{kn(3+2n-2k-2kn)}{k+2n-1}, & \mu = -\frac{2n(k-2)}{2n+1}, \\ c = -\frac{(2kn+k-2n)(2kn-k-4n)}{2(k+2n)}, & \mu = -\frac{(2n-1)k-4n}{2n}. \end{cases}
\end{aligned} \tag{1.16}$$

Note also that when replacing m by $-m$, eq. (1.15) is parametrized by $c_{\mathcal{C}_n}(m+n+1, 2m+2n+1)$ which can be interpreted as a truncation to the algebra $\text{Orb}(\mathcal{WD}_{-m})$ [17].

Interesting consequences follow from the fact that also the algebra $\mathcal{W}(2, 4, 6, 8, 10)$ which arises as the orbifold of the coset $\widehat{sl(2, \mathbb{R})}_k / \widehat{U(1)}$ [17] (hence the orbifold of the first unitary model of \mathcal{WA}_{k-1}) belongs to the class of \mathcal{W} -algebras of type $\mathcal{W}(2, 4, \dots)$ with

$$h_1 = \frac{2-c}{3}; \quad h_2 = \frac{c+1}{2(2-c)}; \quad h_3 = \frac{3c}{2(c+1)}. \quad (1.17)$$

Note that in contrast to this the orbifold of \mathcal{WA}_{n-1} for c generic does not belong to this class, i.e. cannot be described by the general structure constants. Proceeding in the same way as for the previous truncations we find

$$\text{Orb}(\mathcal{WD}_m) \triangleright \mathcal{W}(2, 4, 6, 8, 10) \cong \text{Orb}\left(\frac{\widehat{sl(2, \mathbb{R})}_k}{\widehat{U(1)}}\right) \quad \text{at} \quad \begin{aligned} c(m) &= 2-3m \\ c(m) &= \frac{4m-1}{2m+1} \\ c(m) &= -\frac{2m}{2m-3}, \end{aligned} \quad (1.18)$$

$$\mathcal{WB}_m \triangleright \mathcal{W}(2, 4, 6, 8, 10) \cong \text{Orb}\left(\frac{\widehat{sl(2, \mathbb{R})}_k}{\widehat{U(1)}}\right) \quad \text{at} \quad \begin{aligned} c(m) &= \frac{4m}{2m+3} \\ c(m) &= -2m \\ c(m) &= -\frac{m+2}{m-1}, \end{aligned} \quad (1.19)$$

$$\mathcal{WC}_m \triangleright \mathcal{W}(2, 4, 6, 8, 10) \cong \text{Orb}\left(\frac{\widehat{sl(2, \mathbb{R})}_k}{\widehat{U(1)}}\right) \quad \text{at} \quad c(m) = -1 - 6m. \quad (1.20)$$

These truncations are compatible with the truncation of \mathcal{WA}_{n-1} to $\mathcal{W}(2, 3, 4, 5)$ (see eq. (1.5)) exactly at the c -values for the first unitary model of \mathcal{WA}_{n-1} . In particular, we find that the orbifold of the first unitary model of \mathcal{WA}_{2n-1} is equal to the orbifold of the second unitary model of $\mathcal{WD}_{\frac{n}{2}}$ (this can be confirmed using level-rank duality [17]), and that the orbifold of the first unitary model of \mathcal{WA}_{2n} is equivalent to a unitary model of \mathcal{WB}_n . For the example $n = 2$, $c = \frac{8}{7}$ a detailed verification of this truncation $\text{Orb}(\mathcal{W}(2, 3, 4, 5)) \triangleright \mathcal{W}(2, 4)$ is possible [17]. Note that this model is probably the only unitary minimal model of $\mathcal{W}(2, 4) \cong \mathcal{WB}_2$ [24].

2. Null fields in linear \mathcal{W}_∞ -algebras

In the previous section we have found identifications between \mathcal{W} -algebras. They imply truncations for which we will provide further support in this section by verifying that in the limit $n \rightarrow \infty$ one does indeed obtain truncations of the various linear \mathcal{W}_∞ -algebras. There are four types of linear \mathcal{W}_∞ -algebras, the usual \mathcal{W}_∞ , $\mathcal{W}_{1+\infty}$ (see e.g. [25–27]) and their subalgebras \mathcal{WB}_∞ and \mathcal{WC}_∞ formed by the even-dimensional fields. The results in the previous section predict the following truncations:

$$\begin{aligned} (1.2) &\Rightarrow \mathcal{W}_\infty \triangleright \mathcal{W}(2, 3) && \text{at } c = -2 \\ (1.5) &\Rightarrow \mathcal{W}_\infty \triangleright \mathcal{W}(2, 3, 4, 5) && \text{at } c = -4 \quad \text{and } c = 2 \\ (1.6) &\Rightarrow \mathcal{W}_\infty \triangleright \mathcal{W}(2, 3, 4, 5, 6, 7) && \text{at } c = -6 \\ (1.11) &\Rightarrow \mathcal{WB}_\infty \triangleright \mathcal{WC}_n && \text{at } c = -2n \\ (1.15) &\Rightarrow \mathcal{WC}_\infty \triangleright \text{Orb}(\mathcal{WD}_n) && \text{at } c = n \\ (1.19) &\Rightarrow \mathcal{WB}_\infty \triangleright \mathcal{W}(2, 4, 6, 8, 10) && \text{at } c = -1 \quad \text{and } c = 2. \end{aligned} \quad (2.1)$$

The OPEs for the \mathcal{W}_∞ -algebras are particularly simple [25, 26] and one can calculate the first null fields explicitly. In our calculations we have restricted to the first null field which indicates but does not really prove the truncation of a \mathcal{W}_∞ -algebra to a finitely generated \mathcal{W} -algebra.

For \mathcal{W}_∞ and \mathcal{WB}_∞ we have checked the presence of null fields up to scale dimension 12, for \mathcal{WC}_∞ up to dimension 18. One does indeed find null fields for those values of c and the scale dimension predicted by (2.1). In addition to the confirmation of (2.1) we find further truncations. For \mathcal{W}_∞ we find that $\mathcal{W}_\infty \supset \mathcal{W}(2, 3, \dots, 9)$ at $c = -8$ and $\mathcal{W}_\infty \supset \mathcal{W}(2, 3, \dots, 11)$ at $c = -10$ and $c = 4$. On the basis of these truncations we conjecture that for any integer r a generic algebra of type $\mathcal{W}(2, 3, 4, \dots, 2r + 1)$ exists which is a unifying algebra for \mathcal{WA}_{n-1} at $c_{\mathcal{A}_{n-1}}(n - r, n - r + 1)$ (the conjecture for the central charge is based on the cases $1 \leq r \leq 3$). The truncation $\mathcal{W}_\infty \supset \mathcal{W}(2, 3, \dots, 11)$ at $c = 4$ corresponds to the unifying \mathcal{W} -algebra for the second unitary minimal models of \mathcal{WA}_{n-1} (see below and [17]).

Also for \mathcal{WB}_∞ we find one additional truncation to a $\mathcal{W}(2, 4)$ at $c = 1$ which plays the same rôle as the truncation of algebras of type $\mathcal{W}(2, 3, 4, \dots)$ to $\mathcal{W}(2, 3)$ at $c = -2$: The truncation to a $\mathcal{W}(2, 4)$ or $\mathcal{W}(2, 4, n)$ is a general feature for algebras of type $\mathcal{W}(2, 4, \dots)$. Note that because of the presence of the algebras \mathcal{WD}_{-n} eq. (2.1) also contains the truncations $\mathcal{WC}_\infty \supset \mathcal{W}(2, 4, \dots, 2n(n + 2))$ at $c = -n$. Furthermore, we find the unexpected truncation $\mathcal{WC}_\infty \supset \mathcal{W}(2, 4, \dots, 14)$ at $c = -\frac{1}{2}$.

In the case of $\mathcal{W}_{1+\infty}$ we have investigated null fields up to dimension 7. We find that $\mathcal{W}_{1+\infty} \supset \mathcal{W}(1, \dots, n)$ at $c = n$, i.e. for the n th unitary minimal model [28], and $\mathcal{W}_{1+\infty} \supset \mathcal{W}(1, 2, 3)$ at $c = -1$. This result is consistent with the determinant formulae of $\mathcal{W}_{1+\infty}$ presented recently [29]. It is interesting to note that truncations take place for all unitary quasi-finite representations [28]. Thus, the existence of non-trivial unitary quasi-finite modules seems to imply the truncation of $\mathcal{W}_{1+\infty}$ to a finitely generated algebra.

3. Truncations of Casimir \mathcal{W} -algebras and the Kac determinant

In the two previous sections we have obtained indications for truncations of Casimir \mathcal{W} -algebras. We will now provide further support for these truncations by inspection of the Kac determinant.

Let \mathcal{L}_k be a simple Lie algebra of rank k over \mathbb{C} , Δ the set of its roots, ρ (ρ^\vee) the sum of its (dual) fundamental weights, h (h^\vee) its (dual) Coxeter number and \mathcal{WL}_k the corresponding Casimir \mathcal{W} -algebra.

Then the Kac determinant of the vacuum Verma module \mathcal{M}_N at level N related to \mathcal{WL}_k is given by [1]:

$$\det \mathcal{M}_N \sim \prod_{\beta \in \Delta} \prod_{\substack{mn \leq N \\ m, n \in \mathbb{N}}} ((\alpha_+ \rho + \alpha_- \rho^\vee, \beta) + (\frac{1}{2}(\beta, \beta)m\alpha_+ + n\alpha_-))^{p_k(N-mn)} \quad (3.1)$$

where $p_k(x)$ is the number of partitions of x into k colours and the α_\pm are related to the central charge by

$$\alpha_+ \alpha_- = -1, \quad c = k - 12(\alpha_+ \rho + \alpha_- \rho^\vee)^2. \quad (3.2)$$

At least one \mathcal{W} -algebra singular vector has to exist if a Casimir \mathcal{W} -algebra truncates at a certain value of the central charge. Therefore, the Kac determinant of the vacuum Verma module has to vanish at the level corresponding to the singular vector.

Let us now concentrate on degenerate values of the central charge of the Casimir \mathcal{W} -algebra where one has $\alpha_+ = \frac{q}{\sqrt{pq}}$ and $\alpha_- = -\frac{p}{\sqrt{pq}}$ with $p, q \in \mathbb{N}$ such that

$$c_{\mathcal{L}_k}(p, q) = k - 12 \frac{(q\rho - p\rho^\vee)^2}{pq}. \quad (3.3)$$

The condition that a singular vector occurs at level N implies that one of the non-embedded factors of the Kac determinant with $N = mn$ (m, n positive) is zero for a certain root β :

$$(\alpha_+\rho + \alpha_-\rho^\vee, \beta) + \left(\frac{1}{2}(\beta, \beta)m\alpha_+ + n\alpha_-\right) = 0. \quad (3.4)$$

Using the expressions for α_\pm this implies

$$\frac{p+q}{p-q} = \frac{\frac{1}{2}(\beta, \beta)m + n + (\rho, \beta) + (\rho^\vee, \beta)}{\frac{1}{2}(\beta, \beta)m - n + (\rho, \beta) - (\rho^\vee, \beta)}. \quad (3.5)$$

For given p and q the root leading to the lowest singular vector is the highest root of length 2 for all simple Lie algebras besides \mathcal{C}_k where one has to take the sum of the simple roots (a table of these roots can be found in [30]). Choosing this root for β and parametrizing p and q as $p = h^\vee - 1 + r$, $q = h - 1 + s$ we obtain

$$\frac{r+s+h+h^\vee-2}{r-s-h+h^\vee} = \frac{m+n+h+h^\vee-2}{m-n-h+h^\vee}. \quad (3.6)$$

Obviously, $m = r$ and $n = s$ are solutions of this equation as long as

$$r + h^\vee \neq s + h. \quad (3.7)$$

If $r, s > 0$ this implies the existence of a \mathcal{W} -algebra singular vector at level $N = rs$ for central charge $c_{\mathcal{L}_k}(h^\vee - 1 + r, h - 1 + s)$. Note that for a given value of the central charge one has to choose minimal p, q such that $p \geq h^\vee$, $q \geq h$. This choice of not necessarily coprime integers ensures $r, s > 0$.

A truncation takes place if the singular vector at level $N = rs$ corresponds to a non-composite field and if with the vanishing of the simple field of dimension N also all other simple fields with dimension greater than N become null fields. A singular vector at $N = rs$ with N less than the maximal spin of the generating simple fields indicates a truncation of the Casimir algebra. We will now discuss examples where truncations can be obtained from this argument.

The Casimir algebra of type \mathcal{WA}_n truncates for $c = c_{\mathcal{A}_n}(n+r, n+s)$ to an algebra of type $\mathcal{W}(2, 3, \dots, rs-1)$. This supports the identification (1.4) which corresponds to $r = 1$. Furthermore, for the k th unitary minimal model of \mathcal{WA}_n this predicts a truncation to a $\mathcal{W}(2, 3, \dots, k^2 + 3k + 1)$ ($r = k+1$, $s = k+2$) which will be established in [17]. From the free field realization [31] it follows that all non-composite fields of dimension greater than

N vanish if the non-composite field of dimension N is a null field. Thus, for \mathcal{WA}_n the only assumption which is not proven yet is that the \mathcal{W} -algebra singular vector at level $N = rs$ corresponds to a non-composite field.

The Casimir algebras related to the simple Lie algebras of type \mathcal{B}_n and \mathcal{C}_n truncate for $r \cdot s$ even to \mathcal{W} -algebras of type $\mathcal{W}(2, 4, \dots, rs - 2)$ for $c = c_{\mathcal{B}_n}(2n - 2 + r, 2n - 1 + s)$ and $c = c_{\mathcal{C}_n}(n + r, 2n - 1 + s)$, respectively. This verifies eqs. (1.8)-(1.15). Furthermore, it implies the truncation of \mathcal{WC}_n at $c_{\mathcal{C}_n}(m + n + 1, 2m + 2n + 1)$ to an algebra of type $\mathcal{W}(2, 4, \dots, 2m(m + 2))$. These unifying objects are in fact the algebras \mathcal{WD}_{-m} as will be shown in [17]

The \mathbb{Z}_2 orbifold of the \mathcal{WD}_n Casimir algebra [17] truncates for $c = c_{\mathcal{D}_n}(2n - 3 + r, 2n - 3 + s)$ for $r \cdot s$ even to an algebra of type $\mathcal{W}(2, 4, \dots, rs - 2)$. For the k th unitary minimal model of $\text{Orb}(\mathcal{WD}_n)$ ($s = r + 1 = k + 2$) this predicts a unifying \mathcal{W} -algebra of type $\mathcal{W}(2, 4, \dots, k^2 + 3k)$. This identification can be verified applying a character argument to level-rank-duality [17]. The case $k = 2$ is the second line of (1.18).

The truncations of Casimir \mathcal{W} -algebras indicated by the Kac determinant are summarized in table 1.

<i>Casimir algebra</i>	<i>c</i>	<i>truncated algebra</i>
\mathcal{WA}_n	$c_{\mathcal{A}_n}(n + r, n + s)$	$\mathcal{W}(2, \dots, rs - 1)$
\mathcal{WB}_n	$c_{\mathcal{B}_n}(2n - 2 + r, 2n - 1 + s)$	$\mathcal{W}(2, 4, \dots, rs - 2)$ for $r \cdot s$ even
\mathcal{WC}_n	$c_{\mathcal{C}_n}(n + r, 2n - 1 + s)$	$\mathcal{W}(2, 4, \dots, rs - 2)$ for $r \cdot s$ even
$\text{Orb}(\mathcal{WD}_n)$	$c_{\mathcal{D}_n}(2n - 3 + r, 2n - 3 + s)$	$\mathcal{W}(2, 4, \dots, rs - 2)$ for $r \cdot s$ even

Table 1: Truncations of Casimir \mathcal{W} -algebras

4. Conclusion and outlook

In this letter we have shown the existence of many identifications between \mathcal{W} -algebras using a particular parametrization of the structure constants. The identifications are closely related to truncations of \mathcal{W} -algebras. In particular, we predicted various truncations of Casimir \mathcal{W} -algebras and the linear \mathcal{W}_∞ -algebras. We confirmed and generalized these truncations by inspection of the Kac determinant.

Bits and pieces of this picture have been scattered over the literature including a large number of misleading statements which we did not refer to.

A unifying \mathcal{W} -algebra exists at generic c for each truncation in table 1 and fixed positive integers r, s satisfying (3.7). This can be established e.g. by regarding the structure constants of these algebras which have fixed field content as continuous functions of the level n using the parametrizations (1.3) and (1.7). Another argument is that for a given set of generating fields the Jacobi identities are satisfied for all c because of their polynomial character once they are satisfied for infinitely many values of the central charge c . Note that each Casimir \mathcal{W} -algebra truncates only for finitely many values of the central charge c . Nevertheless one obtains infinitely many unifying \mathcal{W} -algebras (for each r, s) which start at a rank that increases as quickly as $r \cdot s$ does.

Some of these unifying \mathcal{W} -algebras can be realized in terms of cosets generalizing the notion of level-rank-duality [18, 19]. Note that a coset realization automatically ensures the existence of the \mathcal{W} -algebra for generic c . We already presented a few of these realizations in this letter, further ones will be established in [17]. These realizations are summarized in table 2.

<i>Casimir algebra</i>	<i>central charge c</i>	<i>coset realization of unifying algebra</i>	<i>dimensions of simple fields</i>	<i>dimension of first null field</i>
\mathcal{WA}_{n-1}	$c_{\mathcal{A}_{n-1}}(n+k, n+k+1)$	$\frac{\widehat{su(k+1)}_n}{\widehat{su(k)}_n \oplus U(1)}$	$2, 3, \dots, k^2 + 3k + 1$	$k^2 + 3k + 4$
\mathcal{WA}_{n-1}	$c_{\mathcal{A}_{n-1}}(n+1, n+3)$	$\frac{\mathcal{W}_3^{(2)}}{U(1)}$	$2, 3, 4, 5, 6, 7$	10
$\text{Orb}(\mathcal{WD}_n)$	$c_{\mathcal{D}_n}(n+k+1, n+k+2)$	$\text{Orb}\left(\frac{\widehat{so(k+1)}_{2n}}{\widehat{so(k)}_{2n}}\right)$	$2, 4, \dots, k^2 + 3k$	$k^2 + 3k + 4$
\mathcal{WB}_n	$c_{\mathcal{B}_n}(2n+k-1, 2n+1)$ $c_{\mathcal{B}_n}(2n, 2n+k)$	$(\text{Orb})\left(\frac{\widehat{so(k)}_\mu \oplus \widehat{so(k)}_1}{\widehat{so(k)}_{\mu+1}}\right)$	$2, 4, \dots, 2k$	$2k + 4$
\mathcal{WC}_n	$c_{\mathcal{C}_n}(n+k+1, 2n+2k+1)$	$\frac{\widehat{sp(2k)}_n \oplus \widehat{sp(2k)}_{-\frac{1}{2}}}{\widehat{sp(2k)}_{n-\frac{1}{2}}}$	$2, 4, \dots, 2k^2 + 4k$	$2k^2 + 4k + 5$

Table 2: Coset realization of unifying \mathcal{W} -algebras

The identifications between different \mathcal{W} -algebras occur not only for rational but also for degenerate models. There are indications [17] that all minimal models of the unifying \mathcal{W} -algebras arise from identifications with minimal models of Casimir \mathcal{W} -algebras. Thus, these new unifying \mathcal{W} -algebras do probably not give rise to new minimal models and may therefore be irrelevant for the classification of rational conformal field theories.

Looking for rational models, it is intriguing that at least for $\mathcal{W}_{1+\infty}$ the existence of non-trivial unitary quasi-finite modules seems to imply the truncation to a finitely generated algebra.

Still, these unifying structures provide us with new insights into conformal field theory. For example, this gives a unified approach to the conformally invariant second order phase transition of \mathbb{Z}_n spin quantum chains. They all share the same $\mathcal{W}(2, 3, 4, 5)$ symmetry algebra and the growth of the number of states with energy is always bounded by the number of partitions into two colours. Finally, we would like to mention that these unifying \mathcal{W} -algebras have supersymmetric generalizations (see [32] for examples in the case $N = 2$).

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References

- [1] P. Bouwknegt, K. Schoutens, *W-Symmetry in Conformal Field Theory*, Phys. Rep. **223** (1993) p. 183
- [2] H.G. Kausch, G.M.T. Watts, *A Study of W-Algebras Using Jacobi Identities*, Nucl. Phys. **B354** (1991) p. 740
- [3] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel, R. Varnhagen, *W-Algebras with Two and Three Generators*, Nucl. Phys. **B361** (1991) p. 255
- [4] R. Varnhagen, *Characters and Representations of New Fermionic W-Algebras*, Phys. Lett. **B275** (1992) p. 87
- [5] W. Eholzer, M. Flohr, A. Honecker, R. Hübel, W. Nahm, R. Varnhagen, *Representations of W-Algebras with Two Generators and New Rational Models*, Nucl. Phys. **B383** (1992) p. 249
- [6] W. Eholzer, A. Honecker, R. Hübel, *How Complete is the Classification of W-Symmetries ?*, Phys. Lett. **B308** (1993) p. 42
- [7] H.G. Kausch, *Extended Conformal Algebras Generated by a Multiplet of Primary Fields*, Phys. Lett. **B259** (1991) p. 448
- [8] M. Flohr, *W-Algebras, New Rational Models and Completeness of the $c = 1$ Classification*, Commun. Math. Phys. **157** (1993) p. 179
- [9] P. Bowcock, G.M.T. Watts, *On the Classification of Quantum W-Algebras*, Nucl. Phys. **B379** (1992) p. 63
- [10] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui, *On the Completeness of the Set of Classical W-Algebras Obtained from DS Reductions*, preprint BONN-HE-93-14 (1993), DIAS-STP-93-02, hep-th/9304125, to appear in Commun. Math. Phys.
- [11] L. Fehér, L. O’Raifeartaigh, I. Tsutsui, *The Vacuum Preserving Lie Algebra of a Classical W-Algebra*, Phys. Lett. **B316** (1993) p. 275
- [12] J. de Boer, T. Tjin, *The Relation between Quantum W Algebras and Lie Algebras*, Commun. Math. Phys. **160** (1994) p. 317
- [13] F.A. Bais, P. Bouwknegt, M. Surridge, K. Schoutens, *Extensions of the Virasoro Algebra Constructed from Kac-Moody Algebras Using Higher Order Casimir Invariants; Coset Construction for Extended Virasoro Algebras*, Nucl. Phys. **B304** (1988) p. 348; p. 371
- [14] J. de Boer, L. Fehér, A. Honecker, *A Class of W-Algebras with Infinitely Generated Classical Limit*, BONN-HE-93-49, ITP-SB-93-84, hep-th/9312049, to appear in Nucl. Phys. **B**
- [15] K. Hornfeck, *W-Algebras with Set of Primary Fields of Dimensions $(3, 4, 5)$ and $(3, 4, 5, 6)$* , Nucl. Phys. **B407** (1993) p. 237
- [16] K. Hornfeck, *Classification of Structure Constants for W-Algebras from Highest Weights*, Nucl. Phys. **B411** (1994) p. 307
- [17] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. Hübel, *Coset Realization of Unifying W-Algebras*, in preparation
- [18] P. Bowcock, P. Goddard, *Coset Constructions and Extended Conformal Algebras*, Nucl. Phys. **B305** (1988) p. 685
- [19] D. Altschuler, *Quantum Equivalence of Coset Space Models*, Nucl. Phys. **B313** (1989) p. 293

- [20] A.M. Polyakov, *Gauge Transformations and Diffeomorphisms*, Int. Jour. of Mod. Phys. **A5** (1990) p. 833
- [21] M. Bershadsky, *Conformal Field Theories via Hamiltonian Reduction*, Commun. Math. Phys. **139** (1991) p. 71
- [22] D. Altschuler, M. Bauer, H. Saleur, *Level-Rank Duality in Non-Unitary Coset Theories*, J. Phys. A: Math. Gen. **23** (1990) p. L789
- [23] P. Bouwknegt, *Extended Conformal Algebras from Kac-Moody Algebras*, Proceedings of the meeting 'Infinite dimensional Lie algebras and Groups' CIRM, Luminy, Marseille (1988) p. 527
- [24] A. Honecker, *Darstellungstheorie von \mathcal{W} -Algebren und Rationale Modelle in der Konformen Feldtheorie*, Diplomarbeit BONN-IR-92-09 (1992)
- [25] C.N. Pope, L.J. Romans, X. Shen, *\mathcal{W}_∞ and the Racah-Wigner Algebra*, Nucl. Phys. **B339** (1990) p. 191
- [26] C.N. Pope, L.J. Romans, X. Shen, *A New Higher-Spin Algebra and the Lone-Star Product*, Phys. Lett. **B242** (1990) p. 401
- [27] I. Bakas, *The Structure of the \mathcal{W}_∞ Algebra*, Commun. Math. Phys. **134** (1990) p. 487
- [28] V. Kac, A. Radul, *Quasifinite Highest Weight Modules over the Lie algebra of Differential Operators on the Circle*, Commun. Math. Phys. **157** (1993) p. 429
- [29] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, *Determinant Formulae of Quasi-Finite Representations of $\mathcal{W}_{1+\infty}$ Algebra at Lower Levels*, preprint YITP/K-1054, UT-669, SULDP-1994-1, hep-th/9402001
- [30] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, Heidelberg, Berlin (1972)
- [31] V.A. Fateev, S.L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with \mathbb{Z}_n Symmetry*, Int. Jour. of Mod. Phys. **A3** (1988) p. 507
- [32] R. Blumenhagen, A. Wißkirchen, work in progress