

UNIVERSITÄT BONN Physikalisches Institut

Automorphisms of W -Algebras and Extended Rational Conformal Field Theories

by

A. Honecker

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Abstract

Many extended conformal algebras with one generator in addition to the Virasoro field as well as Casimir algebras have non-trivial outer automorphisms which enables one to impose 'twisted' boundary conditions on the chiral fields. We study their effect on the highest weight representations. We give formulae for the enlarged rational conformal field theories in both series of \mathcal{W} -algebras with two generators and conjecture a general formula for the additional models in the minimal series of Casimir algebras. A third series of \mathcal{W} -algebras with two generators which includes the spin three algebra at $c = -2$ also has finitely many additional fields in the twisted sector although the model itself is apparently not rational. The additional fields in the twisted sector have applications in statistical mechanics as we demonstrate for \mathbb{Z}_n -quantum spin chains with a particular type of boundary conditions.

Post address:
Nußallee 12
W-5300 Bonn 1
Germany
e-mail:
unp06b@ibm.rhrz.uni-bonn.de



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1. Introduction

Rational conformal field theories (RCFTs) have attracted much attention after they were introduced in the seminal work of Belavin et al. [1]. While the classification of all RCFTs is still an open question there has been some progress studying extended conformal algebras, so-called \mathcal{W} -algebras (for a recent review see e.g. [2]). With the help of explicit formulae for local chiral algebras [3][4] many new \mathcal{W} -algebras with two and three generators were constructed [5][6]. The study of their highest weight representations (HWRs) revealed new rational models [7 – 11]. By now, there is a good chance that all rational models belonging to \mathcal{W} -algebras with one additional generator are classified – although there is no proof of this fact yet.

Already some time ago, Q. Ho-Kim and H.B. Zheng [12] have noticed that Zamolodchikov’s $\mathcal{W}(2, 3)$ [13] along with other Casimir algebras have non-trivial outer automorphisms and therefore admit ‘twisted’ boundary conditions. As we will show in this paper, this is valid also for certain extended conformal algebras with two generators (one in addition to the Virasoro field). We will show that these twists lead to additional HWRs and thus enlarge the RCFTs. One could also take a different point of view and project the \mathcal{W} -algebra onto the invariant subspace. This is called ‘orbifolding’. We will say more about the precise connection between these two approaches later on.

The outline of this paper is as follows: In the next two subsections we will briefly summarize our approach to \mathcal{W} -symmetry and introduce twisted boundary conditions. Section 2 focusses on the well known example $\mathcal{W}(2, 3)$ and illustrates our methods. In section 3 we discuss the three series of bosonic $\mathcal{W}(2, \delta)$ -algebras that admit twists. Section 4 contains a discussion of twists of Casimir algebras. Finally, in section 5 we shall show that some of the additional representations can indeed be realized in statistical mechanics models. This shall be demonstrated in the case of \mathbb{Z}_n -spin quantum chains with a special type of boundary conditions.

1.1. Theorems about \mathcal{W} -algebras

Before starting the main issues, we would like to summarize briefly the notions which we will need in this paper. For precise definitions and explicit formulae, however, we refer to [6][8].

Let \mathcal{F} be a local chiral conformal field theory. On \mathcal{F} there are three important operations: A commutator, a normal ordered product N and the usual derivative ∂ . Any field ϕ in \mathcal{F} can be written as

$$\phi(z) = \sum_{n-d(\phi) \in \mathbb{Z}} z^{n-d(\phi)} \phi_n. \quad (1.1.1)$$

$d(\phi)$ is called the ‘conformal dimension’ of ϕ and ϕ_n the ‘modes’ of ϕ . The modes of the energy-momentum tensor L in \mathcal{F} are well known to satisfy the Virasoro algebra (the explicit form is given in the first line of (2.1)). If the commutator of a field ϕ with the Virasoro algebra yields only the field ϕ itself, ϕ is called ‘primary’. If this holds only for the $SU(1, 1)$ -subalgebra spanned by L_{-1} , L_0 and L_1 , the field ϕ is called ‘quasiprimary’.

There is a general formula for the commutator of two quasiprimary local ¹⁾ chiral fields [3]. The Lie bracket structure in \mathcal{F} is fixed by universal polynomials p_{ijk} depending exclusively on the conformal dimensions, and a few structure constants which are basically given by the two- and three-point-functions. Since the usual normal ordered product $N(\phi, \partial^n \psi)$ of two quasiprimary fields ϕ and ψ is not quasiprimary any more, we use a quasiprimary normal ordering prescription $\mathcal{N}(\phi, \partial^n \psi)$ [3]. The explicit formula is somewhat lengthy (see e.g. [6]) but the basic result is rather simple: From $N(\phi, \partial^n \psi)$ all fields turning up in the commutator of ϕ and ψ have to be subtracted with dimension-dependent factors. From any finite set of fields the operations ∂ and \mathcal{N} will generate infinitely many fields. It is therefore convenient to define ‘simple’ fields which are non-composite and non-derivative. The algebra generated by simple fields $\phi_1 \dots \phi_n$ is called a ‘ $\mathcal{W}(d(\phi_1), \dots, d(\phi_n))$ ’.

In this paper we will only consider \mathcal{W} -algebras where the zero modes of the simple fields commute and thus may be considered as the Cartan subalgebra. For these \mathcal{W} -algebras a highest weight representation may be defined [8] via the existence of a cyclic vector $|h, w\rangle$ which is an eigenvector of the zero modes of all simple bosonic fields (i.e. the fields with integer dimension) and satisfies

$$\phi_n |h, w\rangle = 0 \quad \forall \phi, \forall n < 0. \quad (1.1.2)$$

For a \mathcal{W} -algebra with two generators we will denote the additional simple field by W and the corresponding eigenvalues of the energy-momentum tensor L and the field W by h and w :

$$L_0 |h, w\rangle = h |h, w\rangle, \quad W_0 |h, w\rangle = w |h, w\rangle. \quad (1.1.3)$$

In order to define correlation functions one introduces a linear form $\langle h, w|$ dual to $|h, w\rangle$ [8].

There are two main approaches to the explicit study of the HWRs of \mathcal{W} -algebras. The first one is based on the fact that only those HWRs are physically relevant which vanish identically under the application of null fields (fields with zero two point functions in the vacuum representation). Thus, writing down states containing null fields and demanding that they should vanish yields conditions on $|h, w\rangle$. We shall give an example for this approach in section 2. The second approach is based on the observation of R. Varnhagen [7] that correlation functions in HWRs are not automatically associative, even if the algebra is. Basically, one studies special Jacobi identities in order to check whether the commutator $[\phi_n, \psi_m]$ is represented by $\phi_n \psi_m - \psi_m \phi_n$. For more details see [8].

1.2. Automorphisms of \mathcal{W} -algebras and boundary conditions

In this paper we shall be interested in non-trivial outer automorphisms of \mathcal{W} -algebras and their effect on the HWRs. An automorphism ρ of a \mathcal{W} -algebra is a bijective map of the algebra that is compatible with the Lie bracket structure and the normal ordering prescription. ρ is called an ‘outer’ automorphism if it is not generated by the \mathcal{W} -algebra

¹⁾ fields with dimension in $\frac{\mathbb{Z}_+}{2}$.

itself. Each such automorphism enables one to impose non-trivial boundary conditions on the fields ϕ_j in the algebra:

$$\phi_j(e^{2\pi i}z) = \rho(\phi_j(z)). \quad (1.2.1)$$

This type of boundary condition will be called a ‘twist’.

The free fermion ψ with commutation relations

$$[\psi_m, \psi_n]_+ = \delta_{n,-m} \quad (1.2.2)$$

provides us with a simple example for such an automorphism: $\rho(\psi) = -\psi$. More generally, for any \mathcal{W} -algebra that contains fermionic fields, there is an automorphism of the following type:

$$\begin{aligned} \rho(\phi_j) &= \phi_j & \forall \phi_j : d(\phi_j) \in \mathbb{Z} \\ \rho(\psi_j) &= -\psi_j & \forall \psi_j : d(\psi_j) \in \mathbb{Z} + \frac{1}{2}. \end{aligned} \quad (1.2.3)$$

Note that the Lie bracket structure as well as the normal ordered product respect the \mathbb{Z}_2 -grading of the \mathcal{W} -algebra into bosonic and fermionic fields. This shows that (1.2.3) indeed is an automorphism of the \mathcal{W} -algebra. Since for any field with $\phi_j(e^{2\pi i}z) = -\phi_j(z)$ the Laurent expansion reads

$$\phi_j(z) = \sum_{n-d(\phi_j) \in \mathbb{Z} + \frac{1}{2}} z^{n-d(\phi_j)} \phi_{j,n} \quad (1.2.4)$$

this leads to the Ramond-sector of a fermionic \mathcal{W} -algebra. For the algebra $\mathcal{W}(2,3)$ the reflection of the additional bosonic field W with dimension three $\rho(W) = -W$ is an automorphism of the algebra and leads to half-integral modes of the field W [12]. In the next section we will illustrate the effect of the boundary conditions in this well known example.

For general bosonic $\mathcal{W}(2, \delta)$ -algebras an outer automorphism of this kind exists iff the self coupling constant vanishes. We recall that there are exactly three series that admit an automorphism ρ with:

$$\rho(L) = L, \quad \rho(W) = -W \quad (1.2.5)$$

and therefore admit half-integral modes for the additional bosonic field:

$$W(z) = \sum_{n-\delta \in \mathbb{Z} + \frac{1}{2}} z^{n-\delta} W_n. \quad (1.2.6)$$

The first of these series is related to Virasoro minimal models [7], a second so-called ‘parabolic’ series exists for $c = 1 - 8\delta$ [9] and a third series exists for $c = c_{1,k}$ [14]. They will be discussed in three subsequent subsections of section 3.

It is well known that one can also project the \mathcal{W} -algebra onto the invariant subspace and then study the representations of the orbifold (for a detailed discussion of orbifolding see e.g. [15]). However, there is a close connection between both approaches. We prefer to study the original algebra with twisted boundary conditions because it is no trivial

question to find out for a given \mathcal{W} -algebra to which algebra an orbifold construction leads. Furthermore, the orbifolds tend to have more generators (with higher dimensions) than the original \mathcal{W} -algebra and thus it is far more difficult to study their representations explicitly.

2. $\mathcal{W}(2, 3)$: The method and results

$\mathcal{W}(2, 3)$ has been written down as early as 1985 by A.B. Zamolodchikov [13]. Although this algebra is well known we shall use it to illustrate our notations and methods. In our notation $\mathcal{W}(2, 3)$ is given by the following commutation relations of the simple fields:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \\ [L_m, W_n] &= (n - 2m)W_{m+n} \\ [W_m, W_n] &= C_{WW}^L p_{332}(m, n)L_{m+n} + C_{WW}^\Lambda p_{334}(m, n)\Lambda_{m+n} + \frac{c}{3} \binom{n+2}{5} \delta_{n,-m} , \end{aligned} \tag{2.1}$$

where

$$\Lambda = \mathcal{N}(L, L) = N(L, L) - \frac{3}{10}\partial^2 L \tag{2.2a}$$

$$C_{WW}^L = 2, \quad C_{WW}^\Lambda = \frac{32}{5c + 22} \tag{2.2b}$$

$$p_{334}(m, n) = \frac{n - m}{2} , \quad p_{332}(m, n) = \frac{n - m}{60}(2m^2 - mn + 2n^2 - 8) . \tag{2.2c}$$

The representation theory of $\mathcal{W}(2, 3)$ is well studied (see e.g. [16]): Coset constructions using $su(3)$ [17][18] and even more general affine Lie algebras [19] are known in the literature. Still, one can derive interesting new results for this algebra studying null fields as we shall do now.

First, we consider the case $c = \frac{4}{5}$. Here, one can calculate that the fields

$$\begin{aligned} \mathcal{A} &:= \mathcal{N}(W, W) - \frac{95}{117}\mathcal{N}(\Lambda, L) + \frac{11}{6}\mathcal{N}(L, \partial^2 L), \\ \mathcal{B} &:= \mathcal{N}(W, \partial^2 L) - \frac{315}{1196}\mathcal{N}(\mathcal{N}(W, L), L) \end{aligned} \tag{2.3}$$

have vanishing two point functions and thus are null fields. A quite effective set of null

states is given by:

$$\begin{aligned}
\mathcal{A}_0 |h, w\rangle &= -\frac{1}{585} \left((95h - 7)(5h - 2)h - 585w^2 \right) |h, w\rangle \\
\mathcal{B}_0 |h, w\rangle &= -\frac{7}{1196} (15h - 1)(3h - 2)w |h, w\rangle \\
\mathcal{B}_{-1}\mathcal{B}_1 |h, w\rangle &= \frac{49}{836793360} \left(1170(225h^2 - 160h + 14)(90h - 71)w^2 \right. \\
&\quad \left. + (80h + 3)(27h - 7)^2(5h - 2)^2h \right) |h, w\rangle \\
\mathcal{B}_{-2}\mathcal{B}_2 |h, w\rangle &= \frac{49}{209198340} \left(585(20250h^3 + 6075h^2 - 2605h - 5734)w^2 \right. \\
&\quad \left. + 2(72900h^4 + 610605h^3 - 570491h^2 + 38684h - 2184)(5h - 2)h \right) |h, w\rangle.
\end{aligned} \tag{2.4}$$

Thus, for $c = \frac{4}{5}$, we necessarily have either $w = 0$ or $h \in \{\frac{1}{15}, \frac{2}{3}\}$. For $w = 0$ we additionally have $h \in \{0, \frac{2}{5}\}$ whereas for the two other h -values (2.4) also fixes w^2 . We conclude that at most four HWRs are physically relevant for $c = \frac{4}{5}$ (see also table below).

From (2.1) we explicitly see that $\mathcal{W}(2, 3)$ is invariant under $W \mapsto -W$ for generic value of c . Thus we can twist the field W as described in section 1.2. Having done this, there will be no zero mode of W left, such that we have to fix the eigenvalue of L_0 only and a single condition might be sufficient (compared to (2.4))

Let us now return to the case $c = \frac{4}{5}$. Like for Super- \mathcal{W} -algebras [20], it is not possible to evaluate the normal ordered product of two twisted fields using the standard formula due to the non-local effect of the boundary conditions. Thus, the part $\mathcal{N}(W, W)$ of \mathcal{A} cannot be evaluated easily and we only consider the field \mathcal{B} . In order to derive an equation for the eigenvalue of L_0 we have to consider the product of two modes of \mathcal{B} with vanishing total grade. One evaluates e.g.:

$$\begin{aligned}
\mathcal{B}_{-\frac{1}{2}}\mathcal{B}_{\frac{1}{2}} |h, w\rangle &= p_1(h)(40h - 1)^2(8h - 1) |h, w\rangle \\
\mathcal{B}_{-\frac{3}{2}}\mathcal{B}_{\frac{3}{2}} |h, w\rangle &= p_2(h)(40h - 1)(8h - 1) |h, w\rangle
\end{aligned} \tag{2.5}$$

with p_1 and p_2 two coprime polynomials in h . Thus, we find physically relevant representations in the twisted sector of $\mathcal{W}(2, 3)$ at $c = \frac{4}{5}$ only for $h \in \{\frac{1}{40}, \frac{1}{8}\}$.

In principle, one can study all representations of $\mathcal{W}(2, 3)$ examining null fields. Among the fields with dimension not larger than seven, we find at least two null fields for $c \in \{\frac{4}{5}, -2, -23, -\frac{114}{7}\}$. It should be clear to the reader that the restriction to these cases is a technical one rather than a principal one. Going through the same steps as before we find rational theories for $c = -23$ and $c = -\frac{114}{7}$ in addition to $c = \frac{4}{5}$. The values of h for these theories are contained in the following table:

$\mathcal{W}(2, 3)$						
$c = c_{4,5}^{\mathcal{A}_2} = \frac{4}{5}$		$c = c_{3,8}^{\mathcal{A}_2} = -23$		$c = c_{3,7}^{\mathcal{A}_2} = -\frac{114}{7}$		$c = -2$
<i>untwisted</i>	<i>twisted</i>	<i>untwisted</i>	<i>twisted</i>	<i>untwisted</i>	<i>twisted</i>	<i>twisted</i>
0	$\frac{1}{8}$	0	$-\frac{31}{32}$	0	$-\frac{5}{8}$	$-\frac{3}{32}$
$\frac{2}{3}$	$\frac{1}{40}$	$-\frac{1}{2}$	$-\frac{23}{32}$	$-\frac{3}{7}$	$-\frac{19}{56}$	$\frac{5}{32}$
$\frac{2}{5}$		$-\frac{3}{4}$	$-\frac{15}{16}$	$-\frac{4}{7}$	$-\frac{39}{56}$	
$\frac{1}{15}$		$-\frac{7}{8}$		$-\frac{5}{7}$		
		-1				

We have also included the h -values of the twisted sector for $c = -2$ in this table although in the untwisted sector we were not able to exclude any pair (h, w) satisfying the following relation:

$$w^2 = \frac{2}{27}(8h + 1)h^2. \quad (2.6)$$

In fact, $c = -2$ does not belong to the minimal series of $\mathcal{W}(2, 3)$ given by

$$c_{p,q}^{\mathcal{A}_2} = 2 \left(1 - 12 \frac{(p-q)^2}{pq} \right)$$

$$h_{p,q;r_1,s_1,r_2,s_2}^{\mathcal{A}_2} = \frac{3 \left(q(r_1 + r_2) - p(s_1 + s_2) \right)^2 + \left(q(r_1 - r_2) - p(s_1 - s_2) \right)^2}{12pq} + \frac{c_{p,q}^{\mathcal{A}_2} - 2}{24} \quad (2.7)$$

with coprime $q > p$, $p > 2$ as well as $r_1 + r_2 < p$ and $s_1 + s_2 < q$. The parametrization of c in the minimal series was well known (see e.g. [21][22]) whereas the values of h were known only for the unitary case $q = p + 1$ (see e.g. [23]). With the above data we guessed that the formula for the h -values can be extended to the non-unitary case simply by replacing $p + 1$ with q . Recently, (2.7) has been proved using quantized Drinfeld-Sokolov reduction [24]. Note that (2.7) correctly reproduces the representations of the untwisted sector in the table above.

In the case of twisted $\mathcal{W}(2, 3)$ up to now only a formula for the minimal unitary series was known [25][26]. Rewriting this result as in (2.7), i.e. identifying the contribution of the central charge to the conformal dimensions and replacing $p + 1$ by q , we conjecture that the conformal dimensions in the complete minimal series of $\mathcal{W}(2, 3)$ are given by

$$h_{p,q;r,s}^{\mathcal{A}_2^{(2)}} = \Omega^2 \frac{(sp - rq)^2}{2pq} + \frac{c_{p,q}^{\mathcal{A}_2} - 2}{24} + \tilde{h}. \quad (2.8)$$

In the unitary minimal series we have $\Omega^2 = \frac{1}{2}$ and the dimension of the twisted field is $\tilde{h} = \frac{1}{16}$ [25][26]. Indeed, with this choice the first values for $r = 1$ and $s \geq 1$ correctly reproduce the h -values for $c = \frac{4}{5}$ and $c = -23$ in the table above. However, for $c = -\frac{114}{7}$

we need $\Omega^2 = 1$ and $\tilde{h} = \frac{1}{24}$. Note that these two cases differ in $p + q$ being odd or even, respectively, though we do not know whether this already covers all possible cases.

Although we can parametrize $-2 = c_{2,3}^{A_2}$, this model is not minimal because the condition $p > 2$ is violated. In fact, this model belongs to the series of non-minimal models discussed in section 3.3.

3. Explicit results

In the following three subsections we will give formulae that correctly reproduce the HWRs in the twisted sector of all bosonic $\mathcal{W}(2, \delta)$ -algebras with $3 < \delta \leq 8$. The explicit calculations were performed on a computer using a special purpose program [27] supported by the computer algebra system REDUCE. In all cases with $\delta > 3$ both methods mentioned in section 1 lead to identical results. The values of h and w , however, will not be presented explicitly here. The interested reader may look them up in [28].

Although we expect our formulae to be valid for all members of the three series to be discussed below, we will not prove this rigorously. With respect to the classification problem we should mention that there is good reason to believe that these three series include all $\mathcal{W}(2, \delta)$ -algebras with vanishing self coupling constant and $\delta > 3$ but there is no proof of this fact yet.

3.1. \mathcal{W} -algebras related to Virasoro minimal models

As already pointed out in [6] many $\mathcal{W}(2, \delta)$ -algebras are related to Virasoro-minimal models. The central charge and conformal dimensions of the primary fields in Virasoro-minimal models are given for any coprime p, q by the following expressions:

$$c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}, \quad (3.1.1a)$$

$$h_{p,q;r,s} = \frac{(pr - qs)^2 - (p-q)^2}{4pq}, \quad 1 \leq r \leq q-1, \quad 1 \leq s \leq p-1. \quad (3.1.1b)$$

The representation theory of the corresponding \mathcal{W} -algebras can be reduced to the representation theory of the Virasoro algebra. This has been noticed and worked out for fermionic $\mathcal{W}(2, \delta)$ -algebras in [7]. The formulae for the untwisted sector of bosonic \mathcal{W} -algebras have been presented in [8].

In the ADE-classification of Cappelli et al. [29] there are non-diagonal partition functions in terms of Virasoro-minimal models. Now one can extend the symmetry algebra by those fields where the corresponding characters turn up in the same summand as the character for $h = 0$ which corresponds to the field L . Additional arguments that this procedure of extending the symmetry algebra does indeed yield a closed algebra can be inferred from fusion rule considerations [6]. In this case, the characters χ^W of the \mathcal{W} -algebra can be written as finite sums of characters χ of the Virasoro algebra such that the characters χ^W diagonalize the partition function. For these models, the central charge and the conformal dimensions can be parametrized according to (3.1.1). The modular transformations of the

characters χ^W and – via the Verlinde formula [30] – the fusion rules can be derived from those of the corresponding Virasoro-minimal models. For fermionic $\mathcal{W}(2, \delta)$ -algebras this has been worked out in detail in [7].

The characters χ and χ^W are defined as a trace over the representation module V with formal powers in q :

$$\chi := \text{tr}_V \left(q^{(L_0 - \frac{c}{24})} \right). \quad (3.1.2)$$

If we can write a character χ^W as a sum of Virasoro characters, (3.1.2) shows that the value of h for the HWR of the \mathcal{W} -algebra is the smallest of the corresponding Virasoro HWRs.

The case we shall be interested in here is described by the proposition in section 4 of [6] and is closely related to the (A_{q-1}, D_{2n}) -series of modular invariant partition functions. This series includes exactly those bosonic $\mathcal{W}(2, \delta)$ -algebras which are related to Virasoro minimal models and have vanishing self coupling constant. For these $\mathcal{W}(2, \delta)$ -algebras $\delta = (q - 2)(n - 1)$ and $c = c_{(4n-2), q}$ holds. In particular, the effective central charge $\tilde{c} = c - 24h_{min}$ always satisfies $\tilde{c} < 1$.

We claim that the characters χ^W are given by:

$$\begin{aligned} \chi_{i,j}^W &= \chi_{i,j} + \chi_{q-i,j}, & 1 \leq i \leq \frac{q}{2}, & \quad 1 \leq j < \frac{p}{2}, & \quad i, j \in \mathbb{Z}, j \in \mathcal{I} \\ \chi_{i, \frac{p}{2}}^W &= \chi_{i, \frac{p}{2}}, & 1 \leq i \leq \frac{q}{2}, & \quad i \in \mathbb{Z}, \frac{p}{2} \in \mathcal{I}. \end{aligned} \quad (3.1.3)$$

Consequently, the values of h are given by:

$$h_{i,j}^W = h_{i,j}, \quad 1 \leq i \leq \frac{q}{2}, \quad 1 \leq j \leq \frac{p}{2}, \quad i \in \mathbb{Z}, j \in \mathcal{I}. \quad (3.1.4)$$

For fermionic algebras we obtain for $\mathcal{I} = 2\mathbb{Z} + 1$ the Neveu-Schwarz-sector and for $\mathcal{I} = 2\mathbb{Z}$ the Ramond-sector [7]. For bosonic algebras $\mathcal{I} = 2\mathbb{Z} + 1$ yields the untwisted sector of the algebra which has already been discussed in [8]. Our explicit results for bosonic $\mathcal{W}(2, \delta)$ -algebras show that with $\mathcal{I} = 2\mathbb{Z}$ one obtains the twisted sectors of these algebras. Thus, adding a twisted sector, the bosonic algebras become similar to the fermionic ones.

Let us give a further argument that (3.1.3) gives the correct characters also in the twisted sector of the bosonic algebras. If we extend the symmetry algebra by a field W we can use the modes of this field for a mapping of two different Virasoro representation modules V_1 and V_2 . Thus, the representation module for the \mathcal{W} -algebra is given by $V_1 \oplus V_2$ and the corresponding characters have to be added. Now, the difference of the h -values of the two Virasoro HWRs has to equal some mode of the field W and therefore has to be integral in the untwisted sector and half-integral in the twisted sector. Generically, the only characters meeting this requirement are those given by (3.1.3).

However, the action of the modular group on the characters of bosonic algebras is quite different from the fermionic case because one has to impose different boundary conditions in order to obtain a modular invariant partition function. It is well known that the untwisted

sector of a bosonic $\mathcal{W}(2, \delta)$ -algebra is invariant under the full modular group generated by S and T . The corresponding partition function is non-diagonal in terms of Virasoro-characters. The twisted sector is in contrast only invariant under T^2 and TST (in the fermionic case TST intertwines Neveu-Schwarz- and Ramond-sector). If we want to act with the full modular group on it we have to add further characters $\tilde{\chi}^W$:

$$\tilde{\chi}_{i,j}^W = \chi_{i,j} - \chi_{q-i,j}, \quad 1 \leq i \leq \frac{q}{2}, \quad 1 \leq j < \frac{p}{2}, \quad i, j \in \mathbb{Z}, \quad j \in \mathcal{I}. \quad (3.1.5)$$

Now we can use all characters χ^W and $\tilde{\chi}^W$ to write down a new diagonal modular invariant partition function. If one expresses this ‘new’ partition function in terms of Virasoro characters one reobtains the original diagonal partition function, i.e. the (A_{q-1}, A_{4n-3}) modular invariant partition function which is diagonal in terms of Virasoro characters ¹⁾. Still, the additional fields with dimensions given by (3.1.4) are physically relevant as we shall show for $\mathcal{W}(2, 3)$ at $c = \frac{4}{5}$ in section 5.

The series of \mathcal{W} -algebras related to Virasoro minimal models provides us with a good example to understand the precise connection between boundary conditions and orbifold constructions. It is a general feature of \mathcal{W} -algebras with a \mathbb{Z}_2 -automorphism $\rho^2 = \mathbf{1}$ that one has two partition functions, one where only the characters χ^W of the untwisted sector enter, and one where the characters χ^W and $\tilde{\chi}^W$ of both sectors enter. The latter can be identified with the partition function Z of the orbifold ²⁾:

$$\begin{aligned} Z &= \sum_{k: \text{untwisted}} (\chi_k^W)^* \chi_k^W + (\tilde{\chi}_k^W)^* \tilde{\chi}_k^W + \sum_{k: \text{twisted}} (\chi_k^W)^* \chi_k^W + (\tilde{\chi}_k^W)^* \tilde{\chi}_k^W \\ &= 2 \sum_{\substack{k: \text{untwisted} \\ \text{and twisted}}} \frac{1}{2} (\chi_k^W + \tilde{\chi}_k^W)^* \frac{1}{2} (\chi_k^W + \tilde{\chi}_k^W) + \frac{1}{2} (\chi_k^W - \tilde{\chi}_k^W)^* \frac{1}{2} (\chi_k^W - \tilde{\chi}_k^W). \end{aligned} \quad (3.1.6)$$

This implies that the characters of the orbifold \mathcal{W} -algebra are given by $\frac{1}{2}(\chi_k^W + \tilde{\chi}_k^W)$ and $\frac{1}{2}(\chi_k^W - \tilde{\chi}_k^W)$. We conclude that the h -values for the HWRs of the orbifold are those of the original \mathcal{W} -algebra in both sectors in addition to some which differ by (half-) integers. It is a special feature of the \mathcal{W} -algebras discussed in this section that orbifolding yields just the Virasoro algebra. In general, the orbifold will also be non-linear and will have more generators than the original algebra. The identification of partition functions (3.1.6) will be valid for all \mathcal{W} -algebras with a \mathbb{Z}_2 -automorphism.

3.2. Parabolic \mathcal{W} -algebras

There is another series of $\mathcal{W}(2, \delta)$ -algebras with vanishing self coupling constant that leads to rational models [8]. Here, the relation $c = 1 - 8\delta$ holds [6] and the effective central charge satisfies $\tilde{c} = 1$ [8] (therefore these models are called ‘parabolic’). The bosonic members of this series that we have studied explicitly are $\mathcal{W}(2, 3)$ at $c = -23$ and $\mathcal{W}(2, 6)$ at $c = -47$.

¹⁾ For $\mathcal{W}(2, 3)$ at $c = \frac{4}{5}$ this has already been pointed out in [12].

²⁾ We simplify notation by absorbing multiplicities of characters into the index set.

Define m by $c = 1 - 12m$. Then our result is that the relevant HWRs can be parametrized by

$$h_{c; \frac{n}{2m}, \frac{n}{2m}} = \frac{n^2}{8m} - \frac{m}{2} \quad (3.2.1a)$$

$$h_{c; \frac{n}{2m+4}, -\frac{n}{2m+4}} = \frac{n^2}{8m+16} - \frac{m}{2} \quad (3.2.1b)$$

with $n \in \mathbb{Z}_+$, $n \leq m$ and $n = 2m$. For the bosonic algebras even n yields the untwisted sector of the algebra while odd n leads to the twisted sector. This parallels the structure of the fermionic members of this series ($\mathcal{W}(2, \frac{9}{2})$ at $c = -35$ and $\mathcal{W}(2, \frac{15}{2})$ at $c = -59$) where even and odd n yield the Neveu-Schwarz- respectively Ramond-sector. As soon as the twisted sector of bosonic \mathcal{W} -algebras is taken into account all members of this series are described by the same formula (3.2.1) and no different cases (as in [8]) have to be considered.

These explicit results being present, the characters of the twisted sector of these algebras also have been realized in terms of Jacobi-Riemann-Theta functions and general arguments have been given that (3.2.1) actually describes a complete series of $\mathcal{W}(2, \delta)$ -algebras [9].

3.3. \mathcal{W} -algebras related to non-minimal $(1, k)$ -models

In this section we want to discuss the last series of bosonic \mathcal{W} -algebras that have vanishing self coupling constant. Here we can evaluate $c = c_{1,k}$ and $\delta = h_{1,k;1,3}$ from (3.1.1) (we use the parametrization of the central charge as a name for this series). It was shown by H.G. Kausch [14] that the series indeed exists and that it can be realized in terms of free fields. Explicit examples for this series are $\mathcal{W}(2, 3)$ at $c = -2$, $\mathcal{W}(2, 5)$ at $c = -7$ and $\mathcal{W}(2, 7)$ with $c = -\frac{25}{2}$. For the untwisted sector of these algebras two one-parameter families of representations were obtained in [8]. This was a first hint that these models are not rational. Since we know the vacuum character for these algebras [31]:

$$\chi_0^W = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} \text{sign}(n) q^{\frac{(2kn+k-1)^2}{4k}} \quad (3.3.1)$$

we could in principle calculate the orbit of χ_0^W under the modular group and check if it is finite. However, if one wants to rewrite (3.3.1) in terms of quadratic forms one is lead almost immediately to infinite sums and it is not very plausible that this orbit is finite. This provides us with a second argument that these models are not rational.

The fact that we can still use (3.1.1) for the parametrization of c and δ might seem strange for the conformal grid degenerates to a void set at $p = 1$. However, it does make sense to restrict to $r = 1$ in the approach of Felder [32]. In [8] it was pointed out that the representations of the untwisted sector of these algebras satisfy a relation of the form

$$w^2 = \alpha_k \prod_{\substack{1 \leq r \leq 2k-1 \\ r \in \mathbb{Z}}} (h - h_{1,k;r,1}) \quad (3.3.2)$$

where the explicit values of α_k have been given in [8]. However, this can be rewritten as

$$w^2 = \alpha_k \prod_{\substack{3-2k \leq n \leq 2k+1 \\ n \in 2\mathbb{Z}}} (h - h_{1,k;1, \frac{n}{2k}}). \quad (3.3.3)$$

If we now study the HWRs of the twisted sector of these algebras explicitly we obtain from every condition in both approaches *finitely* many HWRs. The results can be summarized in the following parametrization of the L_0 -eigenvalues:

$$h \in \{h_{1,k;1, \frac{n}{2k}} \mid n \in 2\mathbb{Z} + 1, 3 - 2k \leq n \leq 2k + 1\}. \quad (3.3.4)$$

As these models are not rational, it is surprising that their twisted sector is finite.

Obviously, (3.3.3) and (3.3.4) should generalize to all members of this series.

Recently, rational models have been discovered for $c_{1,k}$ [33]. The results of [33] confirm our observation that there are no rational models at $c_{1,k}$ with a $\mathcal{W}(2, \delta)$ symmetry algebra because the model becomes rational only if two currents are included in the symmetry algebra. The algebras we discussed here may be considered as subalgebras of the algebra containing currents. In this language, the energy-momentum tensor L and the additional simple field W are presumably composite. A further interesting property of the rational models at $c_{1,k}$ is that they are effective $c = 1$ theories [33].

4. Generalizations to Casimir algebras

So far we have been discussing bosonic $\mathcal{W}(2, \delta)$ -algebras. The next step is to consider other, more general \mathcal{W} -algebras. Most attention is attracted by so-called ‘Casimir’ algebras $\mathcal{W}\mathcal{L}_n$ where the dimension of the simple fields equals the order of the Casimir invariants in a simple Lie algebra \mathcal{L}_n [17][18]. One common approach to their study is Toda field theory [34][35][36]. Their unitary minimal series can also be studied via GKO-constructions [37]. Q. Ho-Kim and H.B. Zheng have noticed that in this approach outer automorphisms of the Lie algebra give rise to automorphisms of the \mathcal{W} -algebra and argued that there no further ones [12][25][26]. Owing to their work twists of the unitary minimal series of Casimir algebras are well understood. Still, we would like to comment on Casimir algebras from the point of view of extended conformal algebras, especially in their non-unitary regime.

We have already stated that the Casimir algebras $\mathcal{W}(2, 3) \cong \mathcal{W}\mathcal{A}_2$ with generic c and $\mathcal{W}(2, 6) \cong \mathcal{W}\mathcal{G}_2$ at $c = -\frac{516}{13}$ and $c = -47$ have exactly one outer automorphism while the algebras $\mathcal{W}(2, 4) \cong \mathcal{W}\mathcal{B}_2 \cong \mathcal{W}\mathcal{C}_2$ as well as $\mathcal{W}(2, 6) \cong \mathcal{W}\mathcal{G}_2$ with generic c have no outer automorphism. The next simple example is $\mathcal{W}(2, 3, 4) \cong \mathcal{W}\mathcal{A}_3$. This algebra has been explicitly constructed by R. Blumenhagen et al. [6]. The explicit structure of the algebra (especially the vanishing of some coupling constants) shows that this \mathcal{W} -algebra possesses exactly one outer automorphism which is given by $V \mapsto -V$ where V is the simple field of dimension 3. The algebra contains one null field of dimension 6 and one of dimension 7 at $c = 1$ and $c = -\frac{116}{3}$. This enables us to study the representation theory at these values of the central charge explicitly. For both values of the central charge there are only finitely

many values of h (with specific eigenvalues of the zero modes of the additional bosonic fields which we omit here) for which these two null fields do indeed vanish. They are listed in the following table:

$\mathcal{W}(2, 3, 4) \cong \mathcal{WA}_3$						
$c = c_{4,9}^{\mathcal{A}_3} = -\frac{116}{3}$				$c = c_{5,6}^{\mathcal{A}_3} = 1$		
<i>untwisted</i>		<i>twisted</i>		<i>untwisted</i>	<i>twisted</i>	
0	$-\frac{13}{9}$	$-\frac{5}{4}$	$-\frac{59}{36}$	0	$\frac{9}{16}$	$\frac{1}{16}$
$-\frac{4}{3}$	$-\frac{14}{9}$	$-\frac{17}{12}$		1	$\frac{1}{16}$	$\frac{3}{16}$
$-\frac{5}{3}$	$-\frac{2}{3}$	$-\frac{19}{12}$		$\frac{1}{3}$		$\frac{1}{48}$
$-\frac{8}{9}$	-1	$-\frac{29}{36}$		$\frac{1}{12}$		$\frac{25}{48}$
$-\frac{11}{9}$		$-\frac{53}{36}$		$\frac{3}{4}$		

We have not included the solution $h = \frac{17}{32}$ in the twisted sector at $c = 1$ into the above table because we believe it to be a remnant that would vanish if further conditions were studied.

The next algebra in the \mathcal{WA}_n -series is $\mathcal{W}(2, 3, 4, 5) \cong \mathcal{WA}_4$. It has been shown in [38] that there are two solutions for a \mathcal{W} -algebra with additional simple fields of dimension 3, 4 and 5 which we denote by U , V and S . For both solutions $C_{US}^V \neq 0$ holds. Thus, both solutions – in particular the one corresponding to \mathcal{WA}_4 – have exactly one outer automorphism which is given by $U \mapsto -U$ and $S \mapsto -S$.

From [12][25][26] we know that for the unitary minimal series of Casimir algebras the number of automorphisms of the \mathcal{W} -algebra coincides with that of the corresponding Lie algebra. This is also clear in the Fateev-Lykyanov-construction [39] of these algebras. If one demands the additional fields to be primary one has to add correction terms to the pure Casimir invariant terms. It is highly non-trivial that these respect the automorphisms of the underlying Lie algebra. However, we explicitly observed that \mathcal{WA}_2 , \mathcal{WA}_3 and \mathcal{WA}_4 have exactly one outer automorphism for generic value of the central charge c while for \mathcal{WA}_1 , $\mathcal{WB}_2 \cong \mathcal{WC}_2$ and \mathcal{WG}_2 there are none. For these cases the number of automorphisms of \mathcal{WL}_n coincides with those of \mathcal{L}_n for generic value of the central charge c . Consequently, we expect that \mathcal{WL}_n generically has as many automorphisms as \mathcal{L}_n has. This observation is equivalent to a covariance property of a free field construction which has up to now never been used although one automorphism reduces the number of unknowns by one half in such a construction.

Thus, there will be no outer automorphisms for \mathcal{WA}_1 , \mathcal{WB}_n , \mathcal{WC}_n , \mathcal{WE}_7 , \mathcal{WE}_8 , \mathcal{WF}_4 and \mathcal{WG}_2 . The algebras \mathcal{WA}_n for $n > 1$, \mathcal{WD}_n for $n > 4$ and \mathcal{WE}_6 should have exactly one outer automorphism and correspondingly exactly one twisted sector in addition to the untwisted one. For these algebras our explicit data shows how the formulae for the

h -values in [25][26] might generalize:

$$\begin{aligned} c_{p,q}^{\mathcal{L}_n} &= n - 12\rho^2 \frac{(p-q)^2}{pq}, \\ h_{p,q;\lambda,\mu}^{\mathcal{L}_n^{(2,3)}} &= \frac{(p\lambda - q\mu)^2}{2pq} + \frac{c_{p,q}^{\mathcal{L}_n} - n}{24} + \tilde{h}n_1, \end{aligned} \quad (4.1)$$

where

$$\rho = \sum_{i=1}^n \tilde{\Omega}_i, \quad \lambda = \sum_{i=1}^{n_0} r_i \Omega_i, \quad \mu = \sum_{i=1}^{n_0} s_i \Omega_i, \quad (4.2)$$

and n_0, n_1 are the dimensions of the invariant subalgebra $\hat{\mathcal{L}}_{n_0}$ respectively twisted subalgebra of \mathcal{L}_n ; $\tilde{\Omega}_i$ are the fundamental weights of \mathcal{L}_n ; Ω_i the fundamental weights of $\hat{\mathcal{L}}_{n_0}$; \tilde{h} the conformal dimension of the twisted field and r_i, s_i positive integers subject to certain constraints. For the case of \mathcal{A}_n the invariant subalgebra is $\mathcal{C}_{[\frac{n+1}{2}]}$. In the unitary minimal series of $\mathcal{WA}_n, \mathcal{WD}_n$ ($n > 4$) and \mathcal{WE}_6 one has $\tilde{h} = \frac{1}{16}$. We observe that (4.1) indeed reproduces our explicit data for $\mathcal{W}(2, 3, 4)$ if we use the weights of \mathcal{C}_2 (which is the invariant subalgebra) and $\tilde{h} = \frac{1}{16}$ not only in the unitary case but also in the non-unitary case. It should be possible to prove (4.1) with (4.2) rigorously applying quantized Drinfeld-Sokolov reduction to the characters of $\mathcal{L}_n^{(2)}$ and $\mathcal{D}_4^{(3)}$, thus generalizing the work of [24] on $\mathcal{L}_n^{(1)}$.

The exceptional cases \mathcal{WE}_6 and \mathcal{WD}_4 are particularly interesting. Especially for $\mathcal{WD}_4 \cong \mathcal{W}(2, 4, 4, 6)$ the group of outer automorphisms should be \mathcal{S}_3 for generic c . This case is up to now the only case where more than just one outer automorphism is known. In [6] and [5] the algebra $\mathcal{W}(2, 4, 4)$ has been shown to be consistent for $c = 1$ and $c = -\frac{656}{11}$. It has then been conjectured in [14] that the Casimir algebra of \mathcal{D}_4 should reduce to $\mathcal{W}(2, 4, 4)$ for these two values of the central charge. For $\mathcal{W}(2, 4, 4)$ one structure constant remains free such that one can define an operation of $O(2)$ on the two additional fields under which the structure constants transform covariant. From the explicit form of the structure constants given in [14] one sees that the algebra is invariant under the natural embedding of \mathcal{S}_3 into the group $O(2)$ operating on the fields if one chooses the self coupling constants of the two additional simple fields to be equal. In fact, this is also true for $\mathcal{W}(2, 4, 4, 6)$ [40].

Denote the primary fields of dimension 4 in $\mathcal{W}(2, 4, 4, 6)$ by $V(z)$ and $W(z)$. Then the \mathcal{S}_3 -symmetry of this algebra translates into the following type of boundary conditions for the given choice of coupling constants:

$$\begin{aligned} V(e^{2\pi i} z) &= \cos(\alpha)V(z) - \sin(\alpha)W(z) \\ W(e^{2\pi i} z) &= \sin(\alpha)V(z) + \cos(\alpha)W(z) \end{aligned} \quad (4.3)$$

or

$$\begin{aligned} V(e^{2\pi i} z) &= \cos(\alpha)W(z) - \sin(\alpha)V(z) \\ W(e^{2\pi i} z) &= \sin(\alpha)W(z) + \cos(\alpha)V(z) \end{aligned} \quad (4.4)$$

with $\alpha \in \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}$. The three different boundary conditions given by (4.3) correspond to those elements of \mathcal{S}_3 which under the embedding yield elements of $SO(2)$. The boundary

conditions (4.4) correspond to the three elements of \mathcal{S}_3 that are mapped to elements in $O(2)$ with determinant -1 .

The h -values in the unitary minimal series (4.1) of \mathcal{WD}_4 have been calculated in [25] without having to consider the boundary conditions of the additional simple fields which look quite strange at first sight. Note that for \mathcal{D}_4 the invariant subalgebra is the exceptional algebra \mathcal{G}_2 and here the dimension of the twisted field is $\tilde{h} = \frac{1}{18}$.

In [2] it has already been stated that the \mathcal{S}_3 -symmetry should lead to modes in $\frac{\mathbb{Z}}{3}$. We shall show now that this is indeed correct, and we will discuss how the modes have to be chosen precisely. Let us first focus on (4.3). Set $U^{(1)}(z) := V(z) + iW(z)$ and $U^{(2)}(z) := V(z) - iW(z)$. Then (4.3) turns into $U^{(1)}(e^{2\pi i}z) = e^{i\alpha}U^{(1)}(z)$ and $U^{(2)}(e^{2\pi i}z) = e^{-i\alpha}U^{(2)}(z)$ which can be satisfied by choosing modes in $\mathbb{Z} + \frac{\alpha}{2\pi}$ for $U^{(1)}$ and those for $U^{(2)}$ in $\mathbb{Z} - \frac{\alpha}{2\pi}$. Consider now (4.4). For this case set $Y^{(1)}(z) := \cos(\alpha)V(z) + (\sin(\alpha) + 1)W(z)$ and $Y^{(2)}(z) := \cos(\alpha)V(z) + (\sin(\alpha) - 1)W(z)$. Now (4.4) turns into $Y^{(1)}(e^{2\pi i}z) = Y^{(1)}(z)$ and $Y^{(2)}(e^{2\pi i}z) = -Y^{(2)}(z)$. This can be satisfied by choosing the modes for $Y^{(1)}$ in \mathbb{Z} and those for $Y^{(2)}$ in $\mathbb{Z} + \frac{1}{2}$.

However, this procedure shows that outer automorphisms of a \mathcal{W} -algebra can lead to even more complicated structures than discussed in this paper. Nevertheless, for the case of \mathcal{WA}_n the outer automorphism will operate as reflection on the space of primary fields with odd dimension. Therefore, in the twisted sector of \mathcal{WA}_n all primary fields with odd dimension will obtain half-integral modes and the representation theory will follow the lines of this paper.

As we have seen for $\mathcal{W}(2, 6) \cong \mathcal{WG}_2$ there may be specific values of the central charge where some structure constants vanish and thus enlarge the automorphism group. These phenomena are very interesting but in order to discuss them generally one would have to know the structure constants of all Casimir algebras which goes beyond current knowledge.

It would be interesting to generalize the observations of this chapter to the supersymmetric case. Of special interest is $osp(4 | 4)$ which is the supersymmetric analogon of \mathcal{D}_4 . The corresponding Casimir algebra is a $\mathcal{SW}(\frac{3}{2}, 2, 2, \frac{7}{2})$. In [41] the algebra $\mathcal{SW}(\frac{3}{2}, 2, 2)$ has been shown to be consistent only for $c = \frac{3}{2}$ and one may expect this algebra to coincide with the Casimir algebra of $osp(4 | 4)$ for this specific value of the central charge. As in the case of $\mathcal{W}(2, 4, 4)$ one structure constant remains free. For $\mathcal{SW}(\frac{3}{2}, 2, 2)$ it has been noticed in [41] that this algebra is invariant under the natural embedding of \mathcal{S}_3 into the group $O(2)$ operating on the fields if one chooses the self coupling constants of the two additional simple fields to be equal. Thus, $\mathcal{SW}(\frac{3}{2}, 2, 2, \frac{7}{2})$ should also admit boundary conditions similar to (4.3), (4.4).

5. Applications in statistical mechanics

As already pointed out in [12] there is a close connection of the partition function of $\mathcal{W}(2, 3)$ at $c = \frac{4}{5}$ including the twisted sector and the three states Potts model. In fact, the different boundary conditions of $\mathcal{W}(2, 3)$ correspond to the different boundary conditions of the Potts quantum spin chain at critical temperature. Choosing the spin shift operator at the end of the chain to be equal to the spin shift operator at the first site yields the

field content of the ‘untwisted’ sector of $\mathcal{W}(2,3)$ (see e.g. [23][42] and [43] for numerical verification). We recall the remarkable fact [12] that the twisted sector of $\mathcal{W}(2,3)$ yields additional representations which can be identified with fields in the thermodynamic limit of the three states Potts quantum spin chain if the spin shift operator at the end of the chain is chosen to equal the adjoint of the one at the first site [44]. This has been verified by Cardy using the inversion identity method. In [44] he called this type of boundary conditions also for the statistical mechanics model ‘twisted’.

It would be interesting to know if this observation generalizes to *all* \mathbf{Z}_n . For twisted boundary conditions only partial results are available in the literature (see e.g. [45]). We shall therefore present an explicit verification of this statement in the case of \mathbf{Z}_4 . We will follow the approach of [43] and study the spectrum of the following hamiltonian numerically:

$$H_N^{(n)} = -\frac{1}{n} \sum_{j=1}^N \sum_{k=1}^{n-1} \frac{1}{\sin \frac{\pi k}{n}} \left(\sigma_j^k + \lambda \Gamma_j^k \Gamma_{j+1}^{n-k} \right), \quad (5.1)$$

where σ_j and Γ_j freely generate a finite dimensional associative algebra by the following relations ($1 \leq j, l \leq N$):

$$\begin{aligned} \sigma_j \Gamma_l &= \Gamma_l \sigma_j \omega^{\delta_{j,l}}, & \sigma_j^n &= \Gamma_j^n = \mathbf{1}, \\ \sigma_j \sigma_l &= \sigma_l \sigma_j, & \Gamma_j \Gamma_l &= \Gamma_l \Gamma_j, \end{aligned} \quad (5.2)$$

with $\omega = e^{\frac{2\pi i}{n}}$. One can impose different types of boundary conditions for $H_N^{(n)}$. We will follow Cardy [44] and call the boundary condition $\Gamma_{N+1} = \Gamma_1$ ‘periodic’ and $\Gamma_{N+1} = \omega^{-R} \Gamma_1$ ($R \neq 0$) ‘cyclic’. Our main interest is in ‘twisted’ boundary conditions which are given by $\Gamma_{N+1} = \Gamma_1^+$. Finally, the spin quantum chain (5.1) also admits boundary conditions of the form $\Gamma_{N+1} = \omega^{-R} \Gamma_1^+$ ($R \neq 0$).

It is convenient to use an irreducible representation of the algebra (5.2) in $\otimes^N \mathbf{C}^n$ where σ_j and Γ_j are represented in terms of diagonal respective shift matrices.

Let $E_{N,i}$ be the eigenvalues of $H_N^{(n)}$ with periodic boundary conditions in ascending order and $\tilde{E}_{N,i}$ those with twisted boundary conditions. Then the relevant scaling functions are given by [46][47]:

$$\begin{aligned} \xi_{N,i} &:= \frac{N}{2\pi} (\tilde{E}_{N,i} - E_{N,0}) \\ \xi_i &:= \lim_{N \rightarrow \infty} \xi_{N,i}. \end{aligned} \quad (5.3)$$

In the case of periodic and cyclic boundary conditions, the eigenvalue ω^Q ($Q = 0, \dots, n-1$) of the charge operator $\hat{Q} = \prod_{j=1}^N \sigma_j$ and momentum are good quantum numbers. In the case of twisted boundary conditions neither charge nor momentum are conserved any more and one does not have any obvious conserved quantities. At least for even n the charge $Q \bmod 2$ is conserved.

Assume that (5.1) exhibits conformal invariance at $\lambda = 1$ and denote the dimensions of the fields in the left chiral part by h and of those in the right chiral part by \bar{h} . For periodic

and twisted boundary conditions the field theory is diagonal, i.e. the fields $\phi(z, \bar{z})$ with dimension $h + \bar{h}$ satisfy $h = \bar{h}$ and thus have vanishing spin $h - \bar{h}$. Therefore, the modes of the fields $\phi(z, \bar{z})$ yield levels in the spectrum with $\xi = h + \bar{h} + r$ where $r \in \mathbb{Z}_+$ for periodic boundary conditions and $r \in \frac{\mathbb{Z}_+}{2}$ for twisted boundary conditions.

In order to test this method we shall first study the well known three states Potts model. For \mathbb{Z}_3 we have studied 3 to 9 sites. Thus, it was necessary to partially diagonalize matrices of dimension $3^9 = 19683$. The limits $N \rightarrow \infty$ of the lowest gaps ξ_i are given in the following table:

\mathbb{Z}_3		
i	ξ_i	$h + \bar{h} + r$
0	0.050000(2)	$\frac{1}{40} + \frac{1}{40}$
1	0.250005(5)	$\frac{1}{8} + \frac{1}{8}$
2	0.5500(5)	$\frac{1}{40} + \frac{1}{40} + \frac{1}{2}$
3	1.05(4)	$\frac{1}{40} + \frac{1}{40} + 1$

The numbers in brackets indicate the estimated error in the last given digit. For details on the extrapolation procedures and error estimation see e.g. [48]. We do not give more than four levels because the errors of the next levels make an accurate identification impossible. Note that we can nicely identify the dimensions $\frac{1}{40}$ and $\frac{1}{8}$ of the chiral fields – as expected.

Let us now turn to \mathbb{Z}_4 . The spectrum of the \mathbb{Z}_4 chain has already been derived in [49] also for twisted boundary condition. This was done applying numerical and Kac-Moody algebra techniques. Nonetheless, we will present results of a direct calculation here because we would like to demonstrate the correspondence between boundary conditions in statistical mechanics and conformal field theory. Note that the \mathbb{Z}_4 -version of (5.1) is a special case of the Ashkin-Teller quantum chain which was introduced in [50] setting the parameter $h = \frac{1}{3}$ (in the notations of [51]).

For \mathbb{Z}_4 we have at least a splitting of the spectrum into two sectors of $Q \bmod 2$. We have studied 4 to 8 sites, implying the partial diagonalization of matrices in dimensions up to $\frac{4^8}{2} = 32768$.

\mathbb{Z}_4					
$Q \bmod 2 = 0$			$Q \bmod 2 = 1$		
i	ξ_i	$h + \bar{h} + r$	i	ξ_i	$h + \bar{h} + r$
0	0.04167(2)	$\frac{1}{48} + \frac{1}{48}$	0	0.1254(1)	$\frac{1}{16} + \frac{1}{16}$
1	0.375(2)	$\frac{3}{16} + \frac{3}{16}$	1	0.6231(3)	$\frac{1}{16} + \frac{1}{16} + \frac{1}{2}$
2	1.040(2)	$\frac{25}{48} + \frac{25}{48}$	2	0.623(7)	$\frac{1}{16} + \frac{1}{16} + \frac{1}{2}$
			3	1.12(1)	$\frac{1}{16} + \frac{1}{16} + 1$

The dimensions $\frac{1}{48}$, $\frac{1}{16}$, $\frac{3}{16}$ and $\frac{25}{48}$ of the chiral field theory can be nicely seen in these explicit results.

It is well known that the \mathbb{Z}_n -models (5.1) with periodic boundary conditions at their second order phase transition $\lambda = 1$ exhibit a $\mathcal{WA}_{n-1} \cong \mathcal{W}(2, \dots, n)$ -symmetry [39][52]. The field content of this model is given by the first unitary representation of \mathcal{WA}_{n-1} , i.e. by (4.1) with $\hat{\mathcal{L}}_{n_0-1} = \mathcal{L}_{n-1} = \mathcal{A}_{n-1}$ and $p = n + 1$, $q = n + 2$. In particular, the central charge for a \mathbb{Z}_n -model (5.1) equals $c = \frac{2(n-1)}{n+2}$. In this section we have explicitly verified for $n = 3$ and 4 that the representations of the twisted sector of \mathcal{WA}_{n-1} correspond to the spectrum of the \mathbb{Z}_n -model with twisted boundary conditions. In fact, this is also true for the \mathbb{Z}_5 -version of (5.1) [53]. These explicit results are in agreement with the statement that the field content of the spin quantum chain (5.1) at $\lambda = 1$ with twisted boundary conditions $\Gamma_{N+1} = \Gamma_1^+$ can be described by a representation of a twisted \mathcal{WA}_{n-1} for all n . Thus, it is possible to calculate the spectrum of the twisted \mathbb{Z}_n -quantum chain by (4.1) using $\mathcal{L}_{n-1} = \mathcal{A}_{n-1}$, $\hat{\mathcal{L}}_{n_0} = \mathcal{C}_{[\frac{n}{2}]}$ and $p = n + 1$, $q = n + 2$.

For cyclic boundary conditions ($\Gamma_{N+1} = \omega^{-R}\Gamma_1$, $0 < R < n$) the diagonal symmetry of the statistical mechanics model is known to be broken such that the spin $h - \bar{h}$ takes on rational values. The dimensions of the chiral fields, however, are unaffected by this change of boundary conditions. This has been verified in [44] and [43] for the case $n = 3$, in [49] for $n = 4$ and more abstractly for general n in [54]. In [44] a similar result has been obtained for $\Gamma_{N+1} = \omega^{-R}\Gamma_1^+$, $0 < R < 3$, $n = 3$ and the only effect of a factor ω^{-R} for all n should be to combine the left- and right-chiral parts in a non-diagonal way.

Finally, we should stress that also the $(1, k)$ -models may have important applications in statistical mechanics. In particular, $c_{1,2} = -2$ turns up in the context of two-dimensional polymers [55][56]. In the dense phase of two-dimensional polymers, the surface exponents correspond to the conformal dimensions of the fields in the complete chiral algebra whereas the representations of this algebra can be identified with the bulk exponents [55]. In particular, the twisted representations we discussed in this paper turn up naturally when the bulk exponents are evaluated [56].

6. Conclusion

We have shown that for all bosonic $\mathcal{W}(2, \delta)$ -algebras with vanishing self coupling constant one can impose anti-periodic boundary conditions on the additional field. From the point of view of representations this leads to an additional twisted sector of these algebras. The two sectors of these algebras parallel much the Neveu-Schwarz- and Ramond-sector of fermionic $\mathcal{W}(2, \delta)$ -algebras (which always have vanishing self coupling constant). Specifically, for the bosonic \mathcal{W} -algebras coming from the ADE-classification as well as for those with $c = 1 - 8\delta$ the representation theory of both sectors is much alike that of the fermionic theory. For the algebras with $c = 1 - 8\delta$ this has recently been well understood in the work of M. Flohr [9].

Unfortunately, we have not been able to obtain any new modular invariant partition function although the rational models enlarge when adding the twisted sector.

The third and last series of bosonic $\mathcal{W}(2, \delta)$ -algebras with a twisted sector is not rational. Here $c = c_{1,k}$ and $h = h_{1,k;1,3}$ holds and H.G. Kausch has shown that a free field construction for these algebras is possible [14]. We have shown that the twisted sector of these

algebras is finite. Recently, rational models have been discovered at $c_{1,k}$ where the symmetry algebras contains currents [33]. The \mathcal{W} -algebras we considered here can be interpreted as subalgebras of these larger symmetry algebras. In particular, the representations of the twisted $\mathcal{W}(2, \delta)$ -algebras in this series should lead to chiral fields in the rational models.

At least the representations of the first member of this series – $\mathcal{W}(2, 3)$ at $c = -2$ – seem to be closely related to two-dimensional polymer physics [55][56], and in particular the representations of the twisted sector turn up naturally in this context.

We have argued und verified in the first cases that Casimir algebras $\mathcal{W}\mathcal{L}_n$ have generically exactly as many outer automorphisms as the underlying Lie algebra \mathcal{L}_n . In particular, all algebras of type $\mathcal{W}\mathcal{A}_n$ ($n > 1$) possess exactly one twisted sector. For the unitary minimal series this observation is in agreement with [25][26]. On the basis of explicit results we were able to conjecture a general formula (4.1) for the h -values in their complete minimal series. We have pointed out that the algebra $\mathcal{W}\mathcal{D}_4 \cong \mathcal{W}(2, 4, 4, 6)$ should be particularly interesting because the group of outer automorphisms is \mathcal{S}_3 , leading to the possibility to choose modes in $\frac{\mathbb{Z}}{3}$.

Finally, we have shown that the representations of the twisted sector of these algebras have applications in statistical mechanics. Generally, the first unitary minimal model of $\mathcal{W}\mathcal{A}_n$ is closely related related to a second order phase transition in a \mathbb{Z}_{n+1} -model. The twist of the symmetry algebra corresponds to twisted boundary conditions in the \mathbb{Z}_{n+1} -model. This has been explicitly verified for the three states Potts model which is described by $\mathcal{W}(2, 3)$ at $c = \frac{4}{5}$ and the Ashkin-Teller model at a special parameter value which corresponds to $\mathcal{W}(2, 3, 4)$ at $c = 1$.

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